QCD critical point, fluctuations and hydrodynamics

M. Stephanov
Cagniard de la Tour (1822): discovered continuous transition from liquid to vapour by heating alcohol, water, etc. in a gun barrel, glass tubes.
Faraday (1844) – liquefying gases:

“Cagniard de la Tour made an experiment some years ago which gave me occasion to want a new word.”

Mendeleev (1860) – measured vanishing of liquid-vapour surface tension: “Absolute boiling temperature”.

Andrews (1869) – systematic studies of many substances established continuity of vapour-liquid phases. Coined the name “critical point”.
van der Waals (1879) – in “On the continuity of the gas and liquid state” (PhD thesis) wrote e.o.s. with a critical point.

Smoluchowski, Einstein (1908, 1910) – explained critical opalescence.

Landau – classical theory of critical phenomena

Fisher, Kadanoff, Wilson – scaling, full fluctuation theory based on RG.
Critical opalescence
Critical point is a ubiquitous phenomenon

<table>
<thead>
<tr>
<th>Substance</th>
<th>Critical temperature</th>
<th>Critical pressure (absolute)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argon</td>
<td>-122.4 °C (150.8 K)</td>
<td>48.1 atm (4,870 kPa)</td>
</tr>
<tr>
<td>Ammonia</td>
<td>132.4 °C (405.5 K)</td>
<td>111.3 atm (11,280 kPa)</td>
</tr>
<tr>
<td>Bromine</td>
<td>310.8 °C (584.0 K)</td>
<td>102 atm (10,300 kPa)</td>
</tr>
<tr>
<td>Caesium</td>
<td>1,664.85 °C (1,938.00 K)</td>
<td>94 atm (9,500 kPa)</td>
</tr>
<tr>
<td>Chlorine</td>
<td>143.8 °C (416.9 K)</td>
<td>76.0 atm (7,700 kPa)</td>
</tr>
<tr>
<td>Ethanol</td>
<td>241 °C (514 K)</td>
<td>62.18 atm (6,300 kPa)</td>
</tr>
<tr>
<td>Fluorine</td>
<td>-128.85 °C (144.30 K)</td>
<td>51.5 atm (5,220 kPa)</td>
</tr>
<tr>
<td>Helium</td>
<td>-267.96 °C (5.19 K)</td>
<td>2.24 atm (227 kPa)</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>-239.95 °C (33.20 K)</td>
<td>12.8 atm (1,300 kPa)</td>
</tr>
<tr>
<td>Krypton</td>
<td>-63.8 °C (209.3 K)</td>
<td>54.3 atm (5,500 kPa)</td>
</tr>
<tr>
<td>CH₄ (methane)</td>
<td>-82.3 °C (190.8 K)</td>
<td>45.79 atm (4,640 kPa)</td>
</tr>
<tr>
<td>Neon</td>
<td>-228.75 °C (44.40 K)</td>
<td>27.2 atm (2,760 kPa)</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>-146.9 °C (126.2 K)</td>
<td>33.5 atm (3,390 kPa)</td>
</tr>
<tr>
<td>Oxygen</td>
<td>-118.6 °C (154.6 K)</td>
<td>49.8 atm (5,050 kPa)</td>
</tr>
<tr>
<td>CO₂</td>
<td>31.04 °C (304.19 K)</td>
<td>72.8 atm (7,380 kPa)</td>
</tr>
<tr>
<td>N₂O</td>
<td>36.4 °C (309.5 K)</td>
<td>71.5 atm (7,240 kPa)</td>
</tr>
<tr>
<td>H₂SO₄</td>
<td>654 °C (927 K)</td>
<td>45.4 atm (4,600 kPa)</td>
</tr>
<tr>
<td>Xenon</td>
<td>16.6 °C (289.8 K)</td>
<td>57.6 atm (5,840 kPa)</td>
</tr>
<tr>
<td>Lithium</td>
<td>2,950 °C (3,220 K)</td>
<td>652 atm (66,100 kPa)</td>
</tr>
<tr>
<td>Mercury</td>
<td>1,476.9 °C (1,750.1 K)</td>
<td>1,720 atm (174,000 kPa)</td>
</tr>
<tr>
<td>Sulfur</td>
<td>1,040.85 °C (1,314.00 K)</td>
<td>207 atm (21,000 kPa)</td>
</tr>
<tr>
<td>Iron</td>
<td>8,227 °C (8,500 K)</td>
<td></td>
</tr>
<tr>
<td>Gold</td>
<td>6,977 °C (7,250 K)</td>
<td>5,000 atm (510,000 kPa)</td>
</tr>
<tr>
<td>Water</td>
<td>373.946 °C (647.096 K)</td>
<td>217.7 atm (22.06 MPa)</td>
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Critical point between the QGP and hadron gas phases?
QCD is a relativistic theory of a fundamental force.
CP is a singularity of EOS, anchors the 1st order transition.
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QCD is a relativistic theory of a fundamental force. CP is a singularity of EOS, anchors the 1st order transition.

Lattice QCD at $\mu_B \lesssim 2T$ – a crossover.

C.P. is ubiquitous in models (NJL, RM, Holog., Strong coupl. LQCD, …)
Essentially two approaches to discovering the QCD critical point. Each with its own challenges.

- **Lattice simulations.**

  The *sign problem* restricts reliable lattice calculations to $\mu_B = 0$.

  Under different assumptions one can estimate the position of the critical point, assuming it exists, by extrapolation from $\mu = 0$.

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- **Heavy-ion collisions. Non-equilibrium.**
Outline

- Equilibrium
- Non-equilibrium
Why fluctuations are large at a critical point?

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CLT? \( \sigma \) is not a sum of \( \infty \) many \textit{uncorrelated} contributions: \( \xi \to \infty \)
Fluctuations of order parameter and $\xi$

- Fluctuations at CP – conformal field theory. Parameter-free $\rightarrow$ universality. Only one scale $\xi = m_\sigma^{-1} < \infty$,

$$P[\sigma] \sim \exp \left\{-\Omega[\sigma]/T\right\},$$

$$\Omega = \int d^3 x \left[ \frac{1}{2} (\nabla \sigma)^2 + \frac{m_\sigma^2}{2} \sigma^2 + \frac{\lambda_3}{3} \sigma^3 + \frac{\lambda_4}{4} \sigma^4 + \ldots \right].$$

- Width/shape of $P(\sigma_0 \equiv \int_x \sigma)$ best expressed via cumulants:

- Higher cumulants (shape of $P(\sigma_0)$) depend stronger on $\xi$. Universal:

$$\left\langle \sigma_0^k \right\rangle_c \sim V \xi^p,$$

$p = k(3 - [\sigma]) - 3$, $[\sigma] = \beta/\nu \approx 1/2$.

E.g., $p \approx 2$ for $k = 2$, but $p \approx 7$ for $k = 4$. 

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Higher moments also depend on which side of the CP we are

\[ \kappa_3[\sigma] = 2VT^{3/2} \tilde{\lambda}_3 \xi^{4.5}; \quad \kappa_4[\sigma] = 6VT^2 [2(\tilde{\lambda}_3)^2 - \tilde{\lambda}_4] \xi^7. \]

This dependence is also universal.

2 relevant directions/parameters. Using Ising model variables:
Experiments do not measure $\sigma$. 
Observed fluctuations are not the same as $\sigma$, but related:

Think of a collective mode described by field $\sigma$ such that $m = m(\sigma)$:

$$
\delta n_p = \delta n_p^{\text{free}} + \frac{\partial \langle n_p \rangle}{\partial \sigma} \times \delta \sigma
$$

The cumulants of multiplicity $M \equiv \int_p n_p$:

$$
\kappa_4[M] = \langle M \rangle + \kappa_4[\sigma] \times g^4 \left( \underbrace{\text{baseline}}_{\sim M^4} \right)^4 + \ldots ,
$$

$g$ – coupling of the critical mode ($g = dm/d\sigma$).
Mapping to QCD and experimental observables

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NB: Sensitivity to $M_{\text{accepted}}$: \((\kappa_4)_{\sigma} \sim M^4\) (number of 4-tets).
Mapping Ising to QCD phase diagram

$T \text{ vs } \mu_B$:

In QCD $(t, H) \rightarrow (\mu - \mu_{\text{CP}}, T - T_{\text{CP}})$
Mapping Ising to QCD phase diagram

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$\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$
Beam Energy Scan

intriguing hint (2015 LRPNS)

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“intriguing hint” (2015 LRPNS)
QM2017 update: another intriguing hint

Preliminary, but very interesting:

- Non-monotonous $\sqrt{s}$ dependence with max near 19 GeV.
- Charge/isospin blind.
- $\Delta \phi$ (in)dependence is as expected from critical correlations.
- Width $\Delta \eta$ suggests soft thermal pions – but $p_T$ dependence need to be checked.
- But: no signal in $R_2$ for $K$ or $p$. 

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Non-equilibrium physics is essential near the critical point.

The goal for **BEST Collaboration**
Why $\xi$ is finite

System expands and is *out of equilibrium*

Kibble-Zurek mechanism:

Critical slowing down means $\tau_{\text{relax}} \sim \xi^z$.

Given $\tau_{\text{relax}} \lesssim \tau$ (expansion time scale):

$$\xi \lesssim \tau^{1/z},$$

$$z \approx 3 \text{ (universal)}.$$
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$z \approx 3$ (universal).

Estimates: $\xi \sim 2 - 3$ fm (Berdnikov-Rajagopal)

KZ scaling for $\xi(t)$ and cumulants (Mukherjee-Venugopalan-Yin)
\[ \kappa_n \sim \xi^p \quad \text{and} \quad \xi_{\text{max}} \sim \tau^{1/z} \]

Therefore, the magnitude of fluctuation signals is determined by non-equilibrium physics.
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Logic so far:

Equilibrium fluctuations + a non-equilibrium effect (finite $\xi$)  
$\rightarrow$ Observable critical fluctuations
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- Equilibrium fluctuations + a non-equilibrium effect (finite \( \xi \))

\[ \longrightarrow \] Observable critical fluctuations

Can we get critical fluctuations from hydrodynamics directly?
Mukherjee-Venugopalan-Yin

Relaxation to equilibrium

\[
\frac{dP(\sigma_0)}{d\tau} = F[P(\sigma_0)]
\]

\[\Downarrow\]

\[
\frac{d\kappa_n}{d\tau} = L[\kappa_n, \kappa_{n-1}, \ldots]
\]

Signs of cumulants also depend on off-equilibrium dynamics.
Time evolution of cumulants (memory)

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\[+\quad -\]

\[\kappa_3\]

\[\kappa_3\quad \kappa_4\]

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Hydrodynamics breaks down at CP

\[ T^{\mu\nu} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} + \tilde{T}_{\text{visc}}^{\mu\nu} \]
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\[ \tilde{T}^{\mu\nu}_{\text{visc}} = -\zeta \Delta^{\mu\nu} (\nabla \cdot u) + \ldots \]

Near CP gradient terms are dominated by \( \zeta \sim \xi^3 \to \infty \)
\( (z - \alpha/\nu \approx 3). \)
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Near CP gradient terms are dominated by \( \zeta \sim \xi^3 \rightarrow \infty \) \( (z - \alpha/\nu \approx 3). \)

When \( k \sim \xi^{-3} \) hydrodynamics breaks down, i.e., while \( k \ll \xi^{-1} \) still.

(For simplicity, measure dim-ful quantities in units of \( T \), i.e., \( k \sim T(T\xi)^{-3}. \))

Why does hydro break at so small \( k \)?
Critical slowing down and bulk viscosity

Bulk viscosity is the effect of system taking time to adjust to local equilibrium (Khalatnikov-Landau).

\[ p_{\text{hydro}} = p_{\text{equilibrium}} - \zeta \nabla \cdot v \]

\( \nabla \cdot v \) – expansion rate

\[ \zeta \sim \tau_{\text{relaxation}} \sim \xi^3 \]
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Hydrodynamics breaks down because of large relaxation time (critical slowing down).

Similar to breakdown of an effective theory due to a low-energy mode which should not have been integrated out.
There is a critically slow mode $\phi$ with relaxation time $\tau_\phi \sim \xi^3$. 
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*(MS-Yin 1704.07396, in preparation)*
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“Hydro+” has two competing limits, $k \to 0$ and $\xi \to \infty$;

or competing rates $\Gamma_\phi \sim \xi^{-3} \to 0$ and $\Gamma_{\text{hydro}} \sim k \to 0$. 

Regime I: $\Gamma_\phi \gg \Gamma_{\text{hydro}}$ – ordinary hydro ($\zeta \sim \xi^3 \to \infty$ at CP).

Crossover occurs when $\Gamma_{\text{hydro}} \sim \Gamma_\phi$, or $k \sim \xi^{-3}$.

Regime II: $k > \xi^{-3}$ – “Hydro+” regime.
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Advantages/motivation of Hydro+

- Extends the range of validity of “vanilla” hydro near CP to length/time scales shorter than $O(\xi^3)$. 

(Since noise $\sim \xi$, also the noise is not large.)
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- Extends the range of validity of “vanilla” hydro near CP to length/time scales shorter than $O(\xi^3)$.

- No kinetic coefficients diverging as $\xi^3$.
  (Since noise $\sim \zeta$, also the noise is not large.)
Ingredients of “Hydro+”

- Nonequilibrium entropy, or quasistatic EOS:

\[ s^*(\varepsilon, n, \phi) \]

Equilibrium entropy is the maximum of \( s^* \):

\[ s(\varepsilon, n) = \max_{\phi} s^*(\varepsilon, n, \phi) \]
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The 6th equation (constrained by 2nd law):

\[ (u \cdot \partial)\phi = -\gamma_\phi \pi - G_\phi (\partial \cdot u), \quad \text{where} \quad \pi = \frac{\partial s^*}{\partial \phi} \]

Another example: relaxation of axial charge.
Linearized Hydro+ has 4 longitudinal modes (sound × 2 + density + φ). In addition to the usual $c_s, D$, etc. Hydro+ has two more parameters $\Delta c^2 = c_s^2 - c_s^2$ and $\Gamma = \Gamma_\phi$.

The sound velocities are different in Regime I ($c_s k \ll \Gamma$) and II:

$$c_s^2 = \left( \frac{\partial p}{\partial \epsilon} \right)_{s/n, \pi=0}$$

and

$$c_\ast^2 = \left( \frac{\partial p^*}{\partial \epsilon} \right)_{s/n, \phi}$$

The bulk viscosity receives large contribution from the slow mode given by Landau-Khalatnikov formula

$$\Delta \zeta = w \Delta c^2 / \Gamma$$
Modes

\[ \begin{align*}
\text{I} & \quad \frac{\Gamma}{c_s} \quad \sim \quad \xi^{-3} \\
\text{II} & \quad \Gamma^* \quad \sim \quad \xi^{-1} \\
\text{III} & \quad \sqrt{\frac{\Gamma}{D}} \\
\end{align*} \]
Microscopic origins of Hydro+

Understanding the microscopic origin of the slow mode:

The fluctuations around equilibrium are controlled by the entropy functional $P \sim e^S$.

Near the critical point convenient to “rotate” the basis of variables to “Ising”-like critical variables $\mathcal{E}$ and $\mathcal{M}$. $\mathcal{M} \sim s/n - (s/n)_{CP}$.

$$\delta S[\delta \mathcal{E}, \delta \mathcal{M}] = \left[ \frac{1}{2} a_M (\delta \mathcal{M})^2 + \frac{1}{2} a_\mathcal{E} (\delta \mathcal{E})^2 + b \delta \mathcal{E} \delta \mathcal{M}^2 + \ldots \right].$$

Since $a_M \ll a_\mathcal{E}$ fluctuations of $\mathcal{M}$ are large and are slow to equilibrate.

Their magnitude is related to the slow relaxation mode $\phi$. 
Hydro + mode distribution

Separate “hard” $k > \xi^{-1}$ and “soft” $k \ll \xi^{-1}$ modes.

The new variable, “mode distribution function”:

$$
\phi(t, x, Q) = \int_y \langle \delta M(t, x + y/2) \delta M(t, x - y/2) \rangle e^{-iQ \cdot y}
$$

The additional mode distribution function relaxation equation:

$$
(u \cdot \partial)\phi(t, x, Q) = 2\Gamma_M(Q) \left[ a_M^{-1} - \phi(t, x, Q) \right]
$$

where $\Gamma_M(Q)$ is known from model H (Kawasaki).
Relaxation of slow mode(s).
A fundamental question for Heavy-Ion collision experiments:

Is there a critical point on the boundary between QGP and hadron gas phases?

Theoretical framework is needed – the goal for .

Large (non-gaussian) fluctuations – universal signature of a critical point.

In H.I.C., the magnitude of the signatures is controlled by dynamical non-equilibrium effects. The physics of the interplay of critical and dynamical phenomena can be captured by hydrodynamics with a critically slow mode(s) – Hydro+.