# Fractional quantum Hall effect and duality

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Canterbury Tales of hot QFTs, Oxford
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# Plan

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- General prologue: Fractional Quantum Hall Effect (FQHE)

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- Composite fermions

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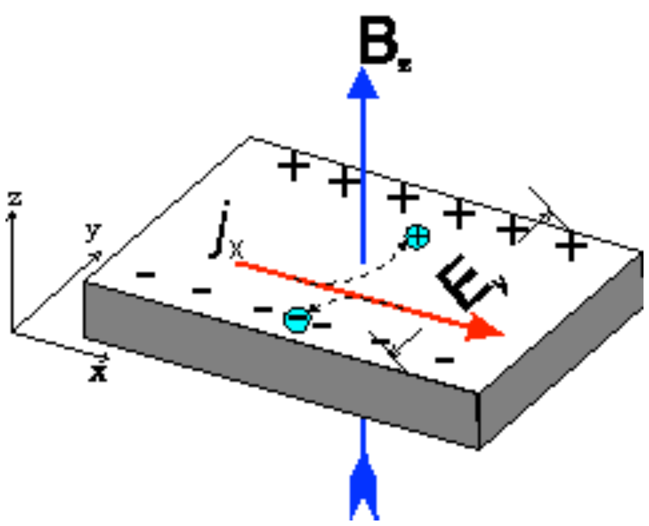
- Composite fermions
- The puzzle of particle-hole symmetry
- Dirac composite fermions

# General Prologue

- QCD is a prime example of a strongly coupled theory
- The particle excitations of the vacuum are very different from the microscopic degree of freedom
- A very similar situation in FQHE

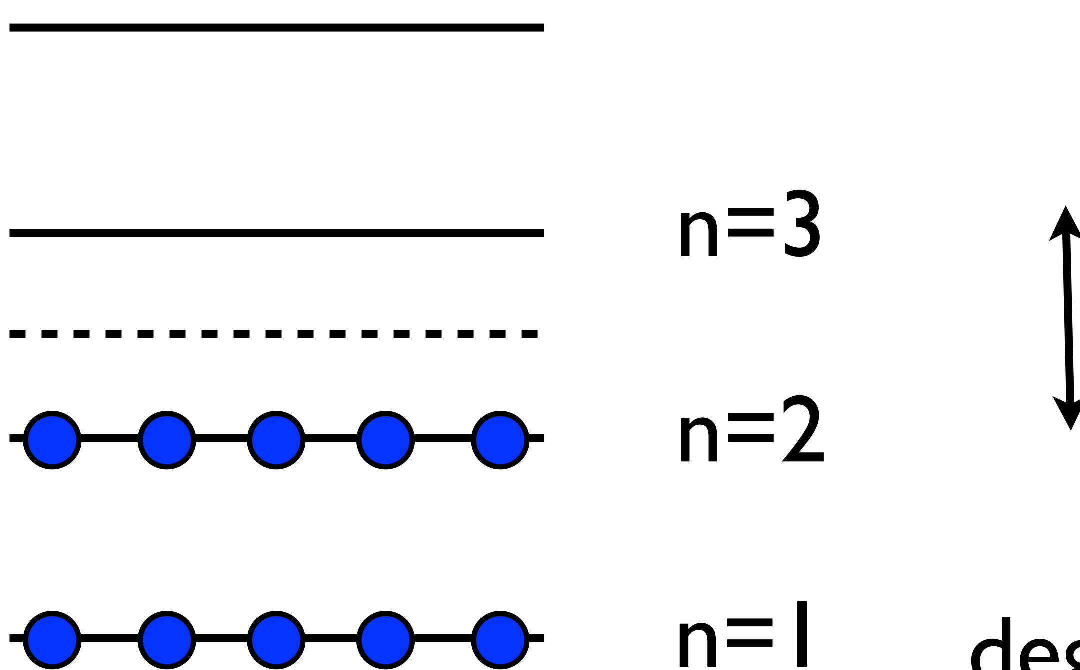
# The setup

$$H = \sum_a \frac{(\mathbf{p}_a + e\mathbf{A}_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$



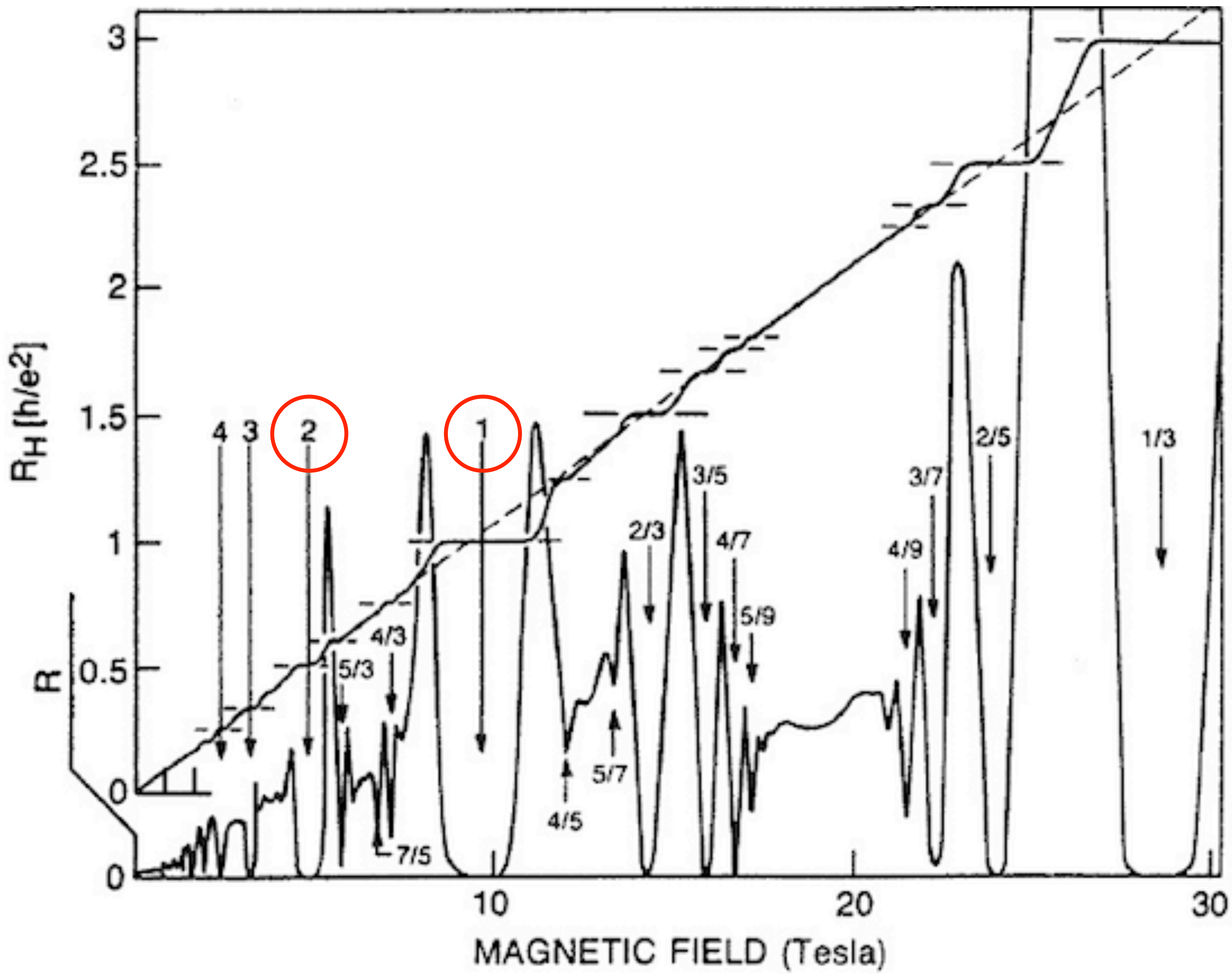
# Integer quantum Hall effect

- Ignore Coulomb interactions
- When electrons moving in 2D in a magnetic field, energy is quantized: Landau level
- IQHE: electrons filling n Landau levels




$$\Delta = \frac{B}{m}$$

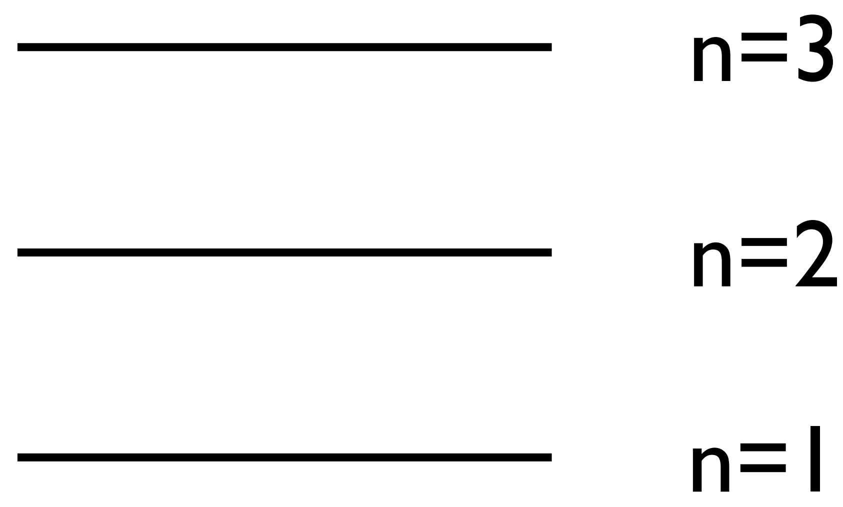
degeneracy $\frac{BA}{2\pi}$




Plateaux require energy gap

# Fractional QHE

Assume we have less particles than states on LLL




In the approximation of noninteracting electrons:
exponential degeneracy of states

# Fractional QHE

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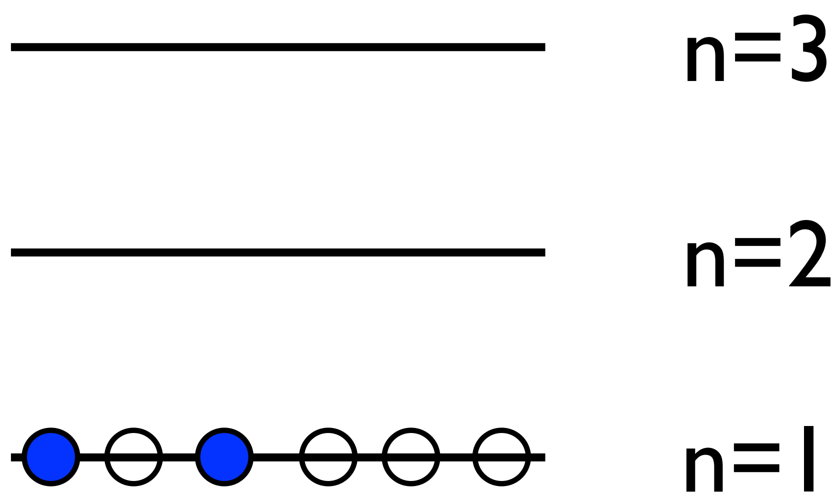
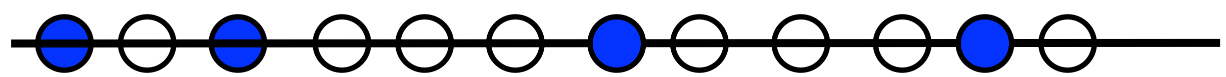



In the approximation of noninteracting electrons: exponential degeneracy of states

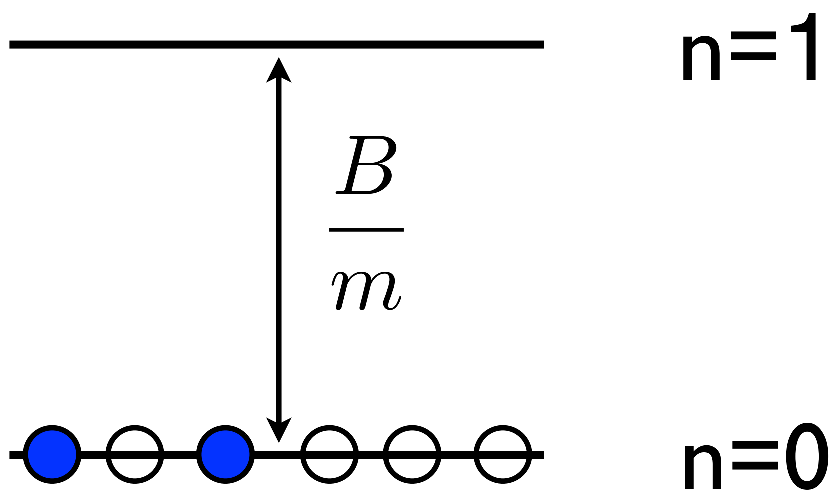
# Why the FQH problem is hard



- degenerate perturbation theory
- Starting point: exponentially large number of degenerate states
- Any small perturbation lifts the degeneracy
- no small parameter

# Lowest Landau level limit

$$H = \sum_a \frac{(\mathbf{p}_a + e\mathbf{A}_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$



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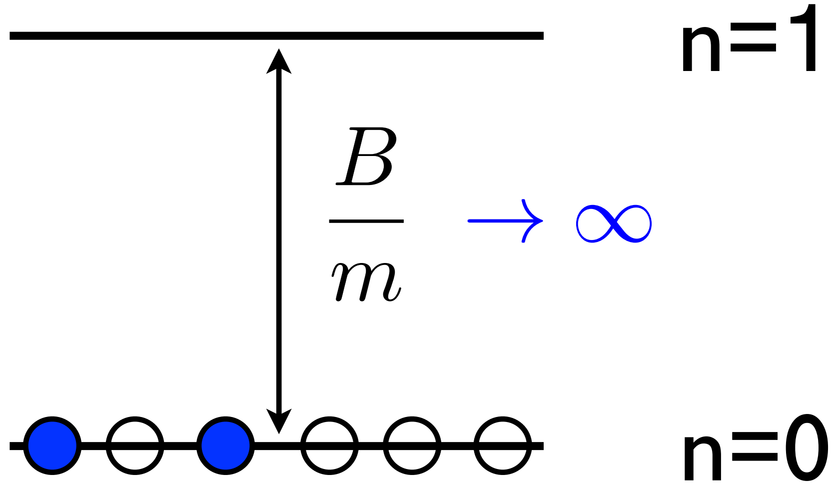
$m \to 0$



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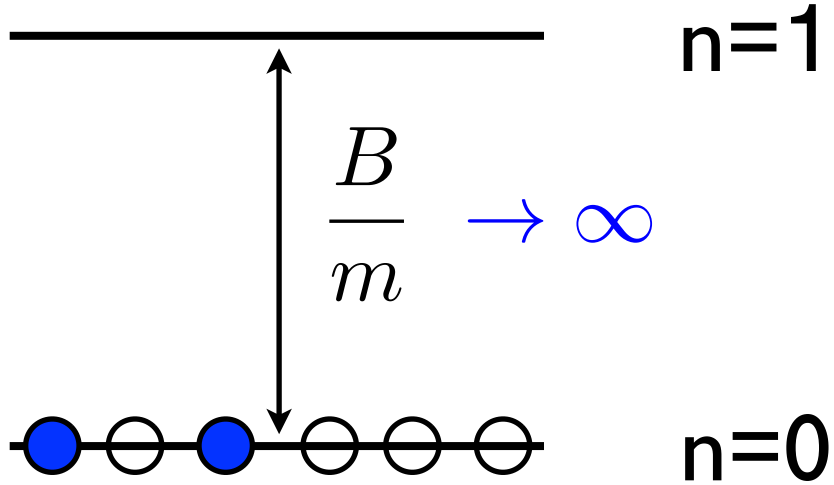
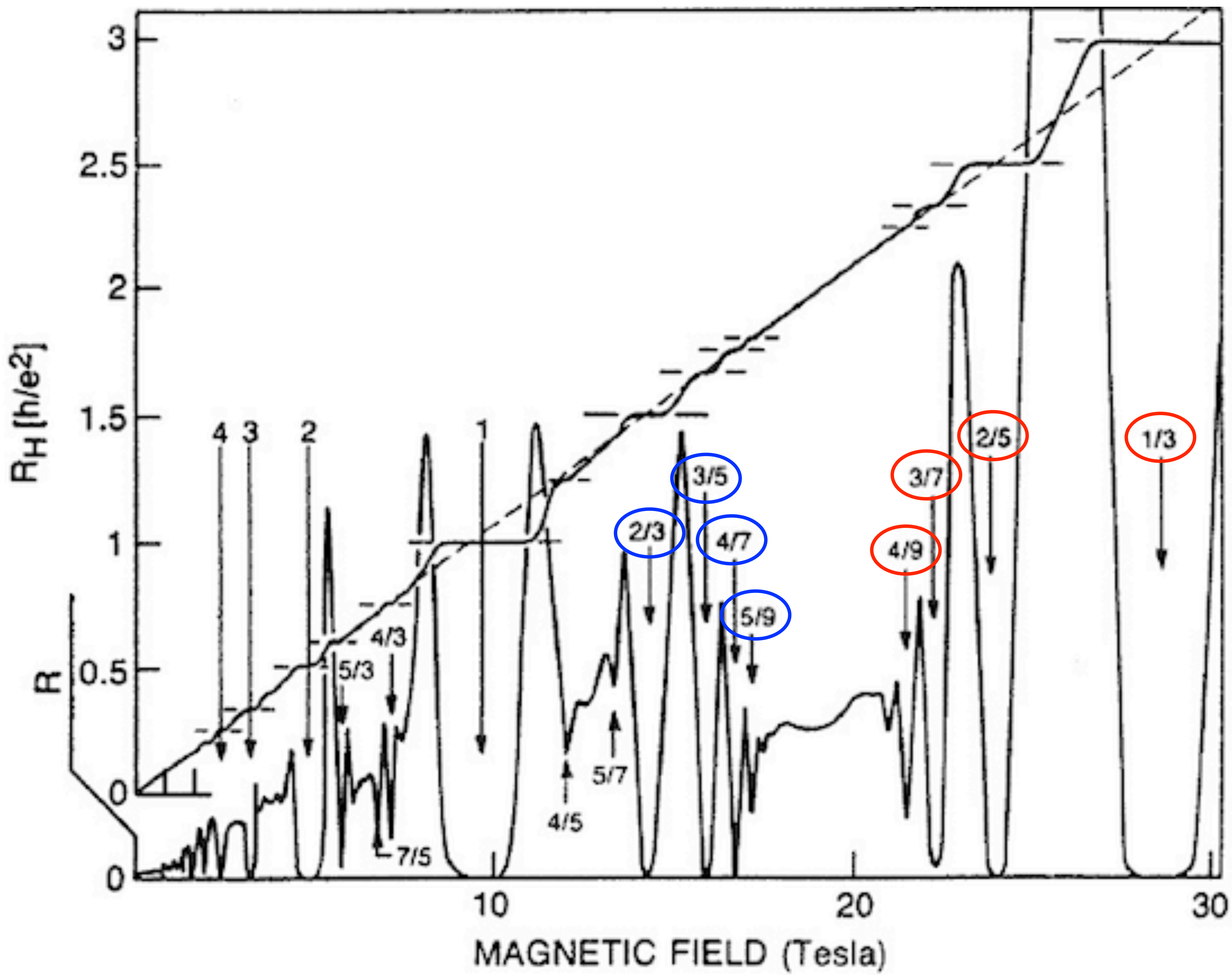



$$H = P_{\mathrm{LLL}} \sum_{a,b} \frac{e^2}{|\mathbf{x}_a - \mathbf{x}_b|}$$

Projection to
lowest Landau level

# Experimental hints

# Jain's sequences of QH plateaux




$$\nu = \frac{n+1}{2n+1} \qquad \nu = \frac{n}{2n+1}$$

# Systematics of Jain's sequences

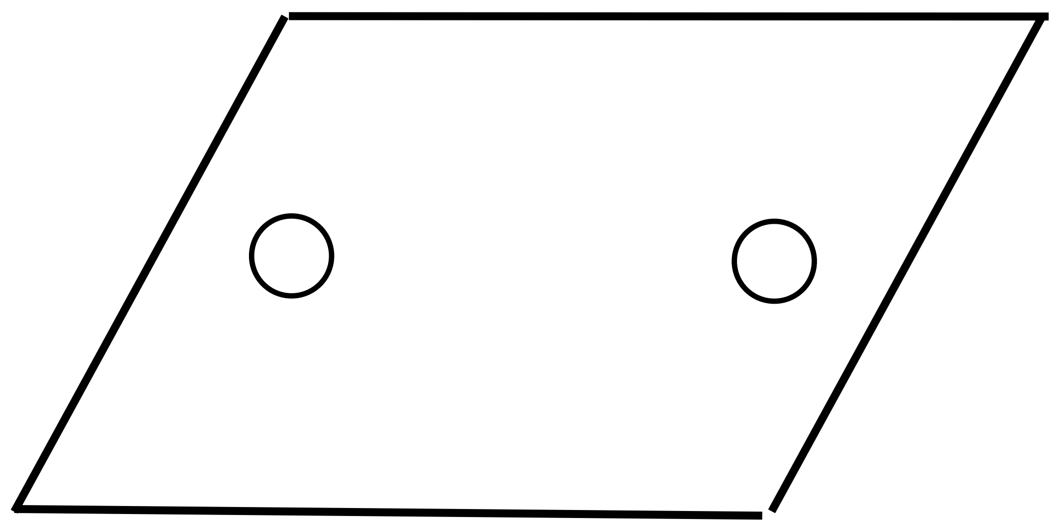
- Gapped states
- Energy gap goes down ~ $1/n$ for $n \to \infty$
- $n = \infty$: gapless, likely Fermi liquid state

# A powerful theory with a flaw

# Flux attachment

(Wilczek 1982, Jain 1989)

- Flux attachment: statistics does not change by attaching an even number of flux quanta



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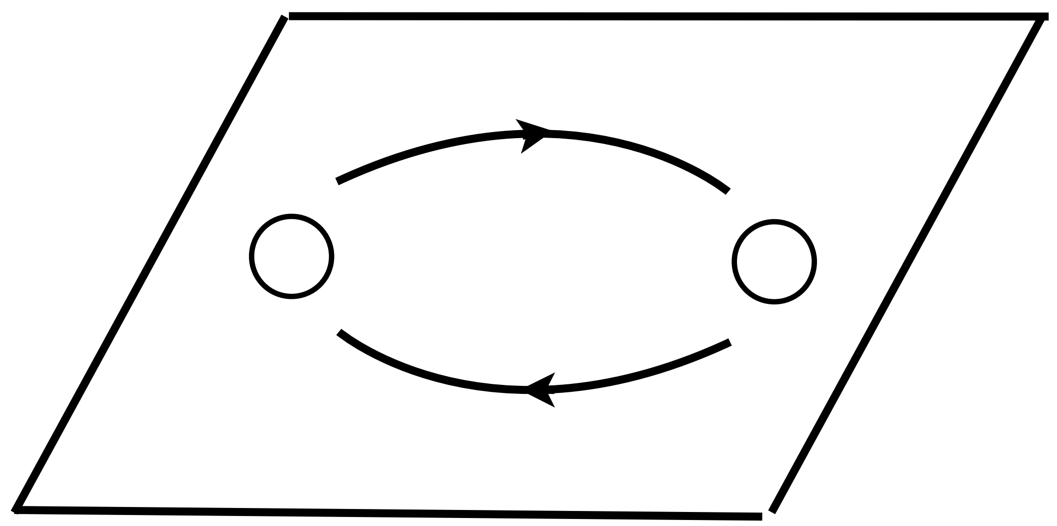

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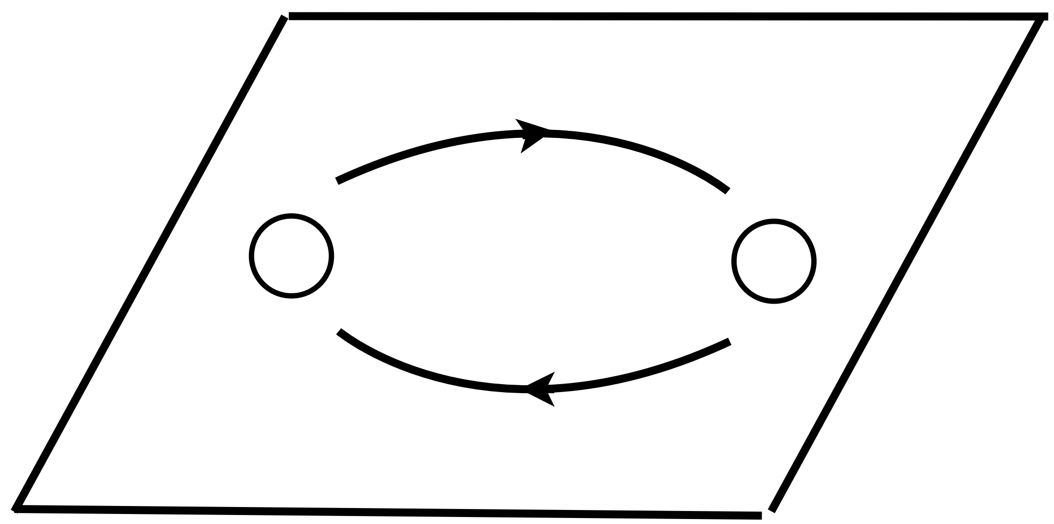

$$(-1)$$

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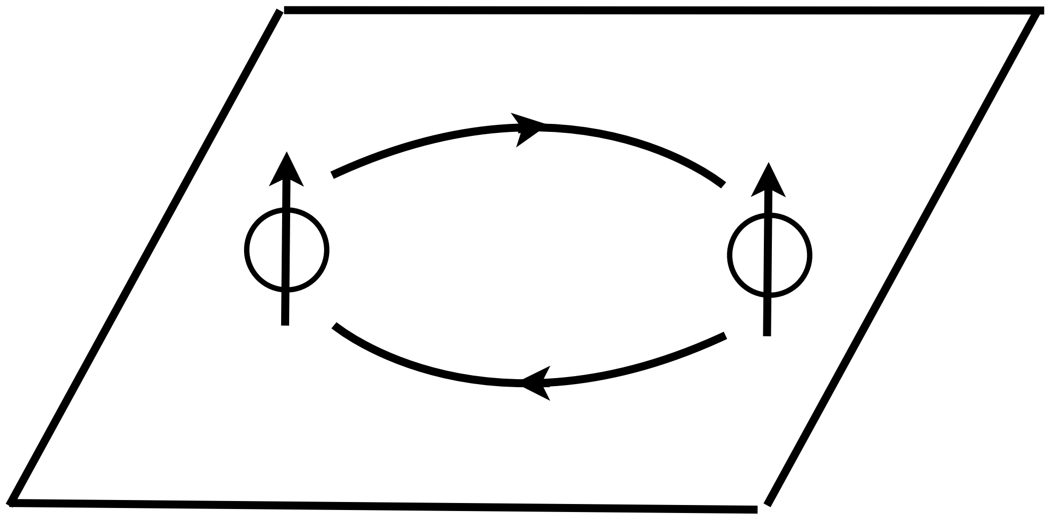

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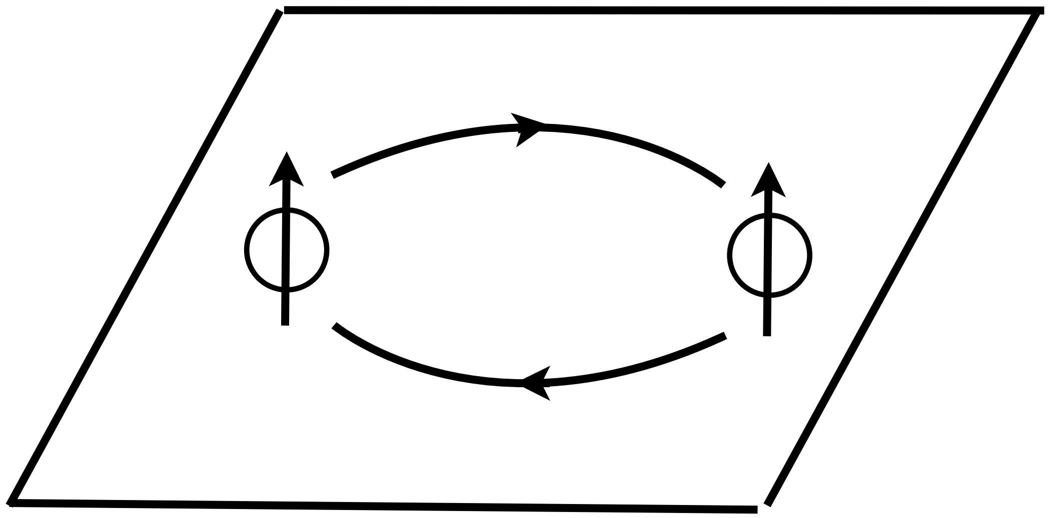

$$(-1)\exp(i\pi) = (+1)$$

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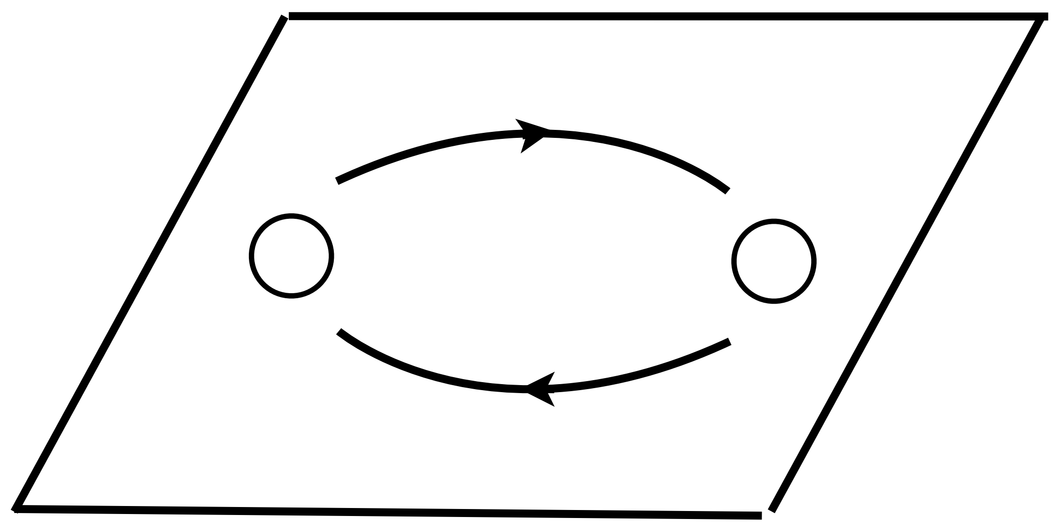

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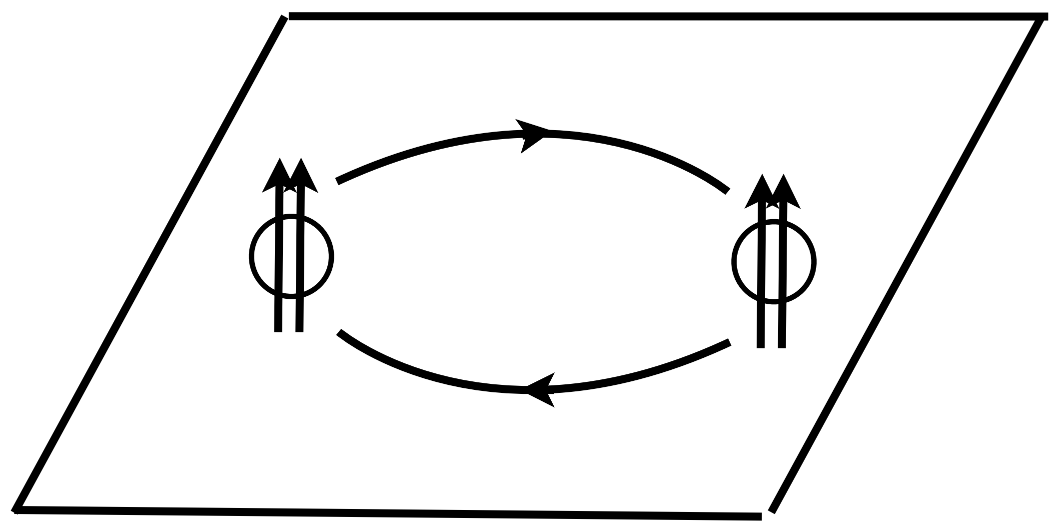

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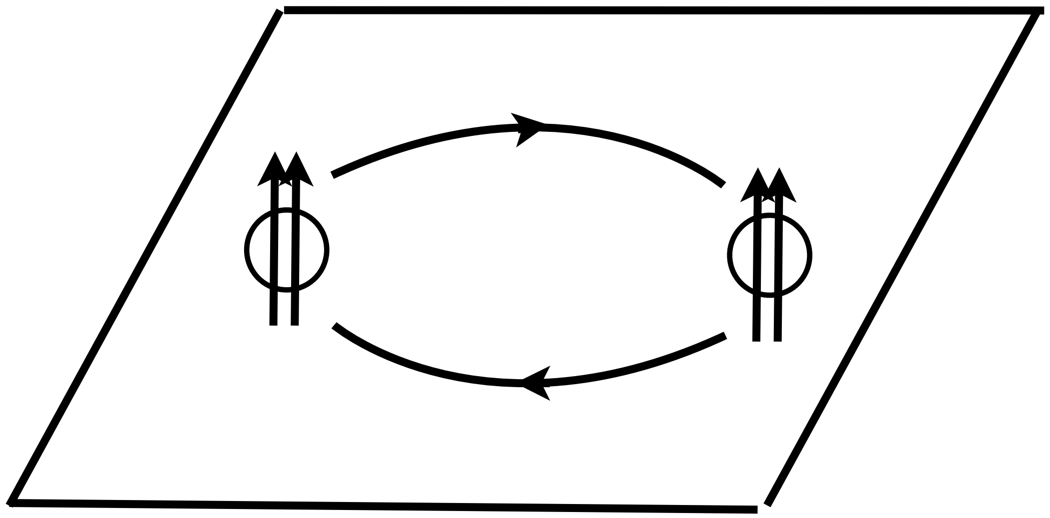

$$(-1)\exp(2i\pi) = (-1)$$

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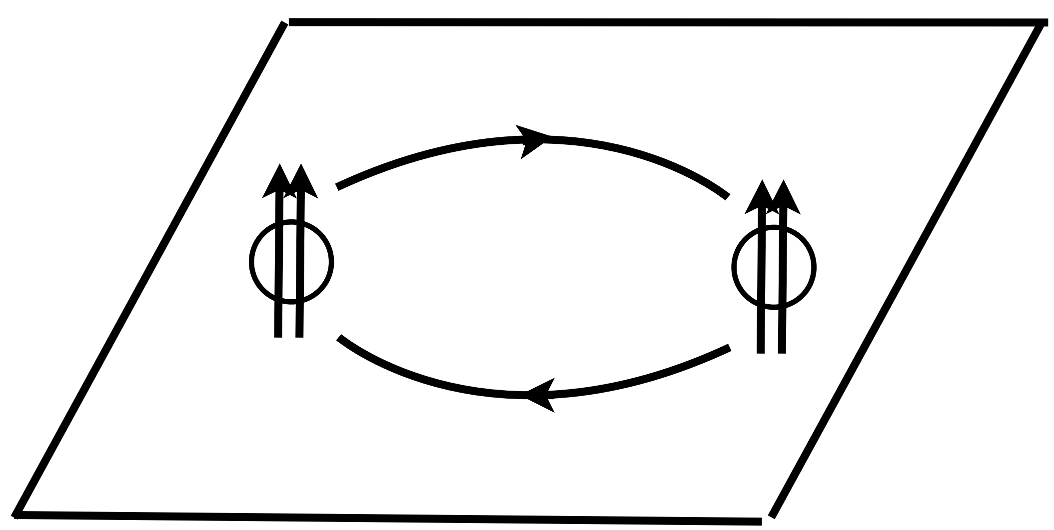

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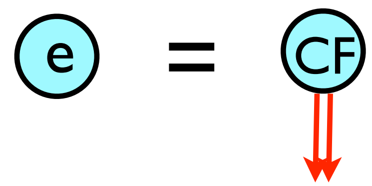
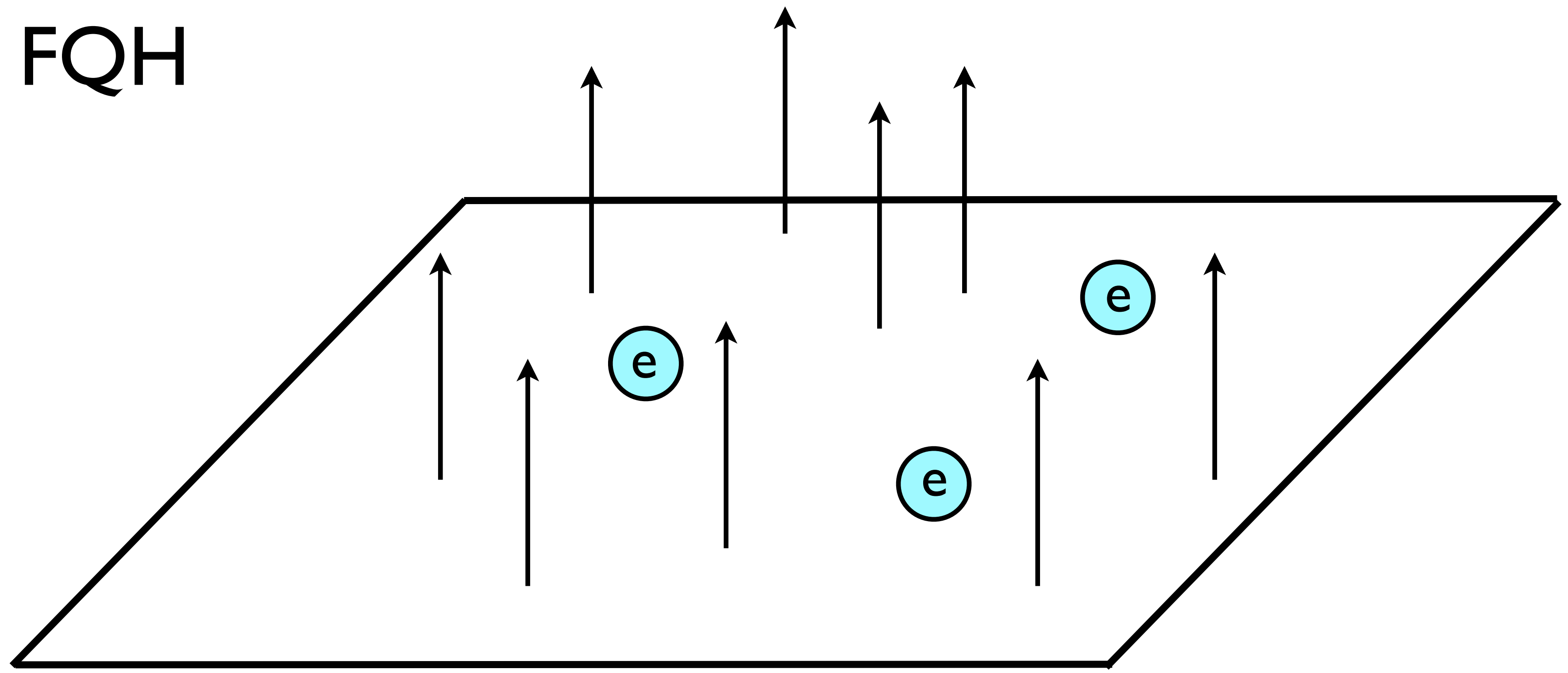
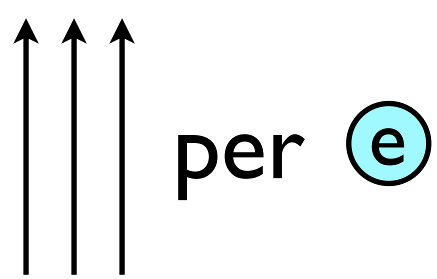


# Composite fermion

$\nu = 1/3$ FQH



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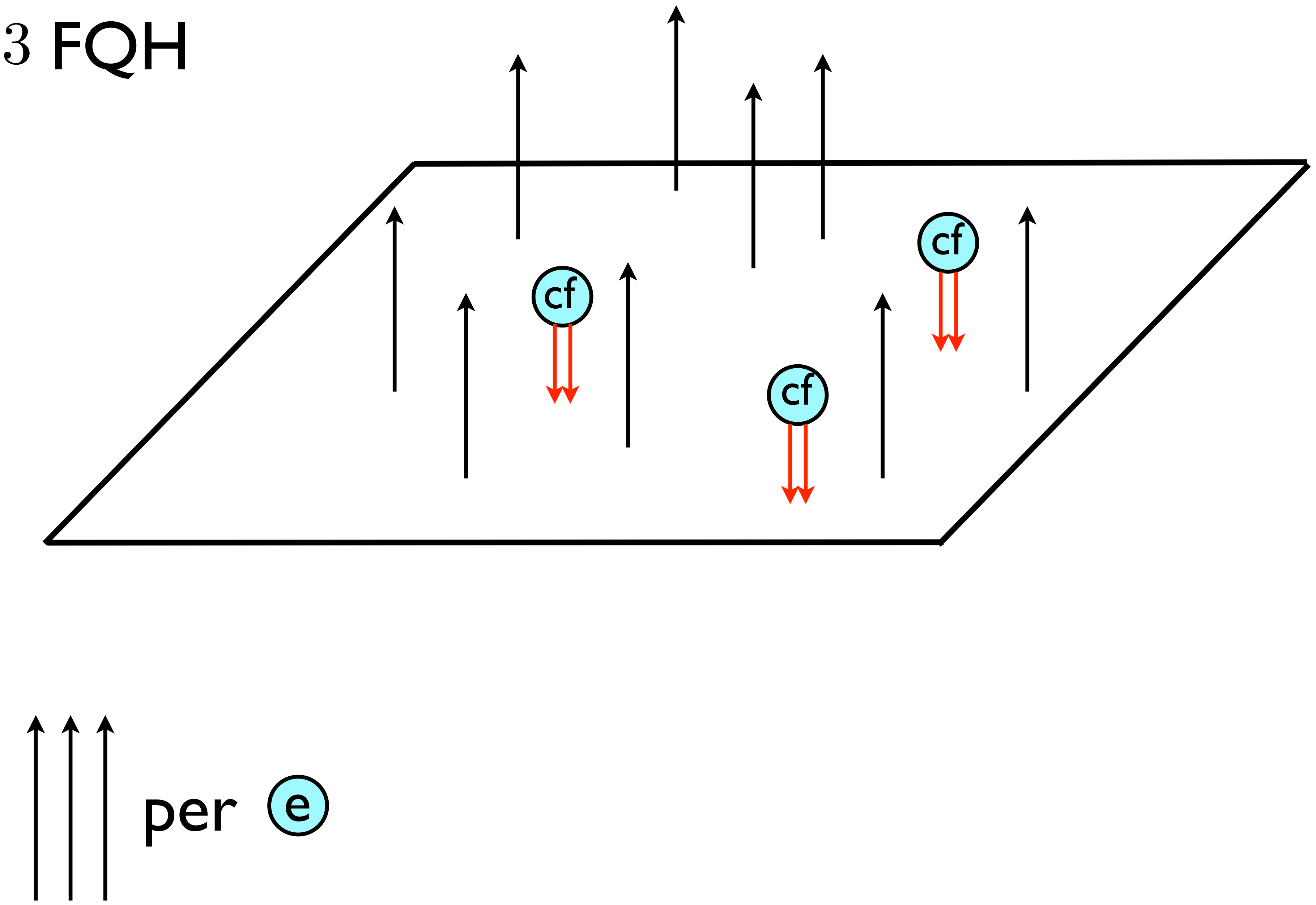

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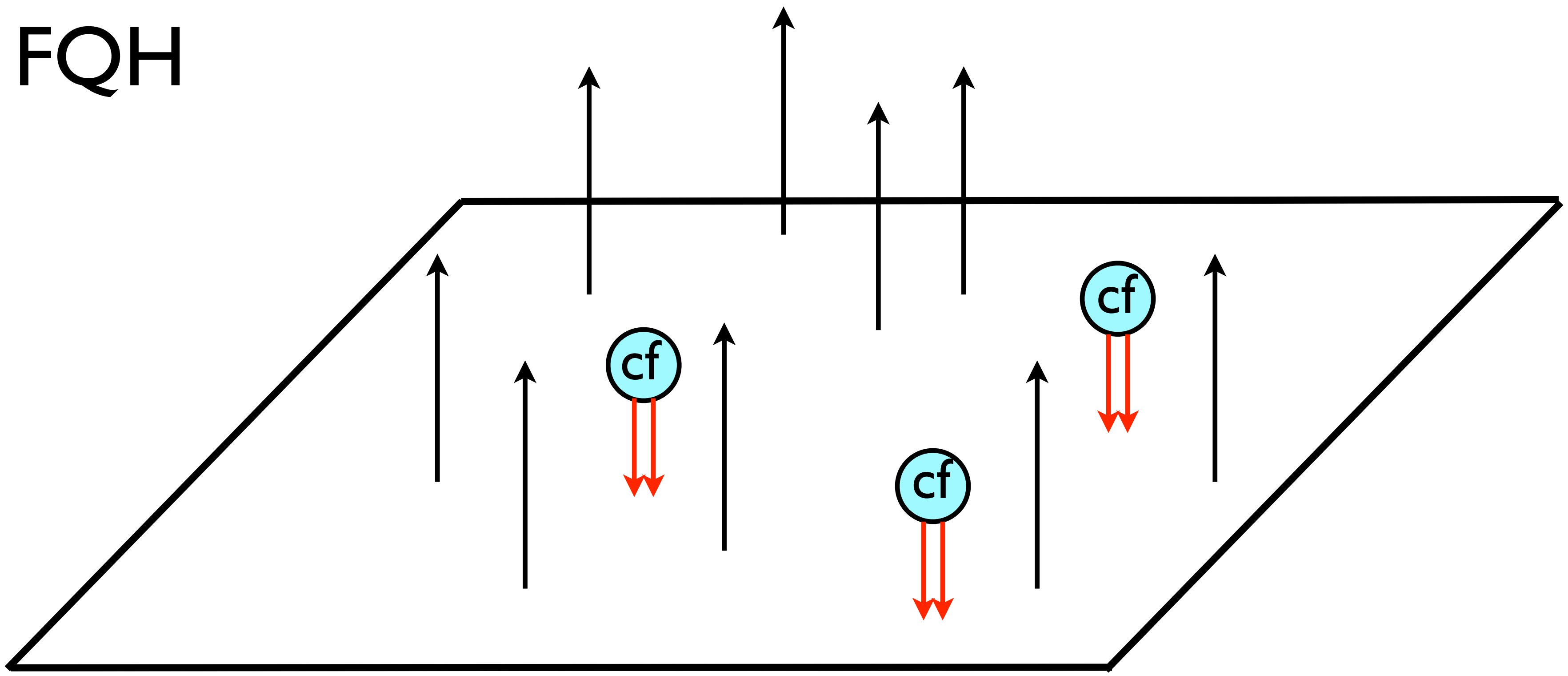


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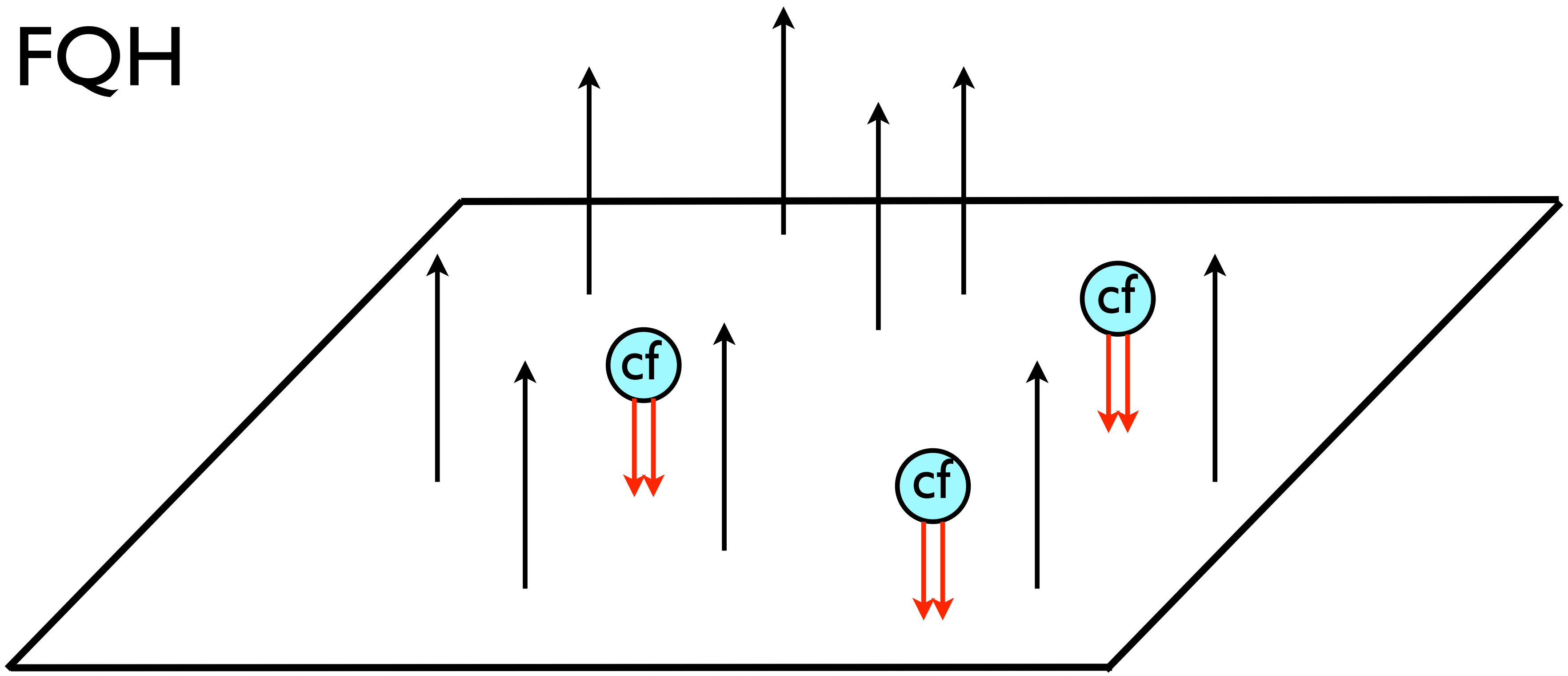
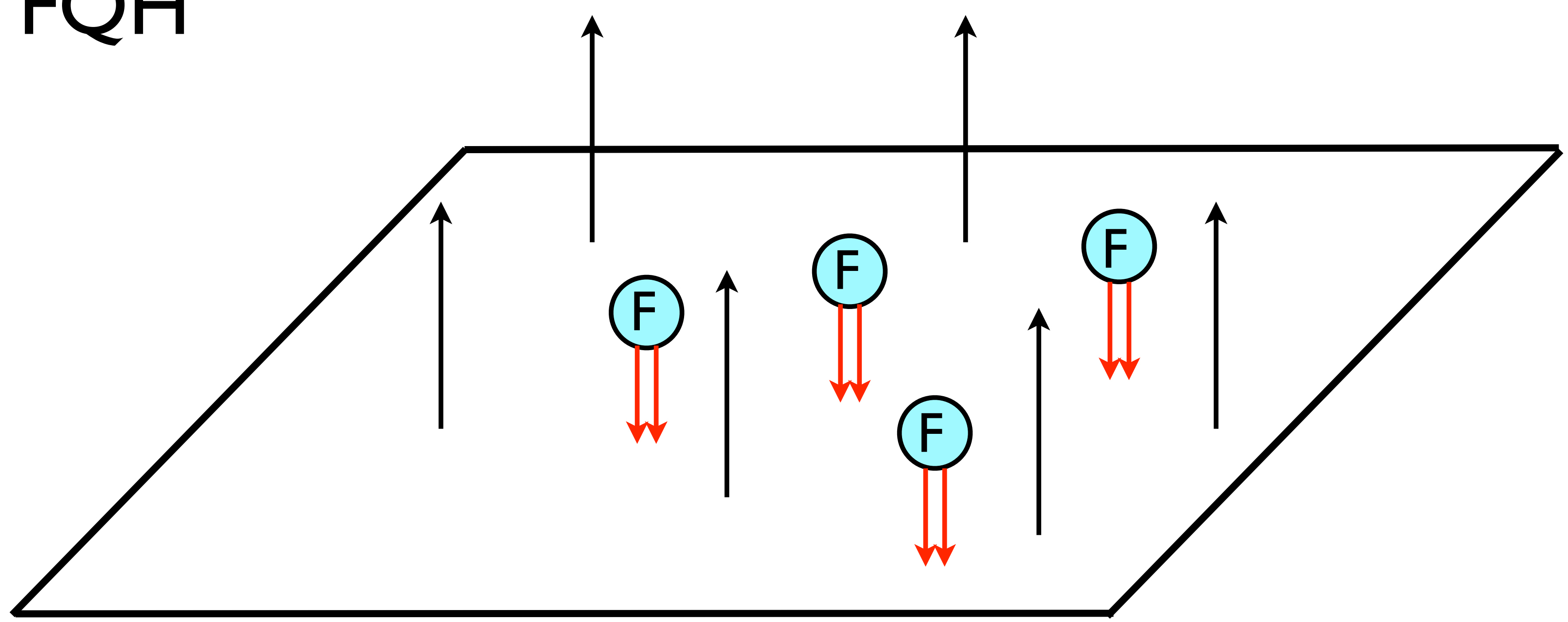


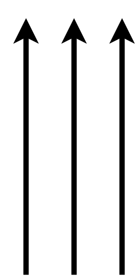
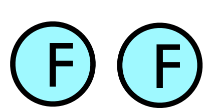


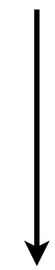
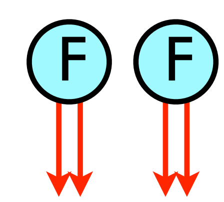
IQHE of CFs with $\nu=1$

# Composite fermion

$\nu = 2/3$ FQH




 per 

average  per 

FQHE for
original fermions

=

IQHE for
composite fermions (n=2)

# HLR field theory

$$\mathcal{L} = i\psi^\dagger(\partial_0 - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

$$b = \nabla \times a = 2 \times 2\pi\psi^\dagger\psi$$

"flux attachment"

mean field: $$B_{\mathrm{eff}} = B - b = B - 4\pi n$$

$$\nu = \frac{1}{2} \qquad B_{\mathrm{eff}} = 0$$

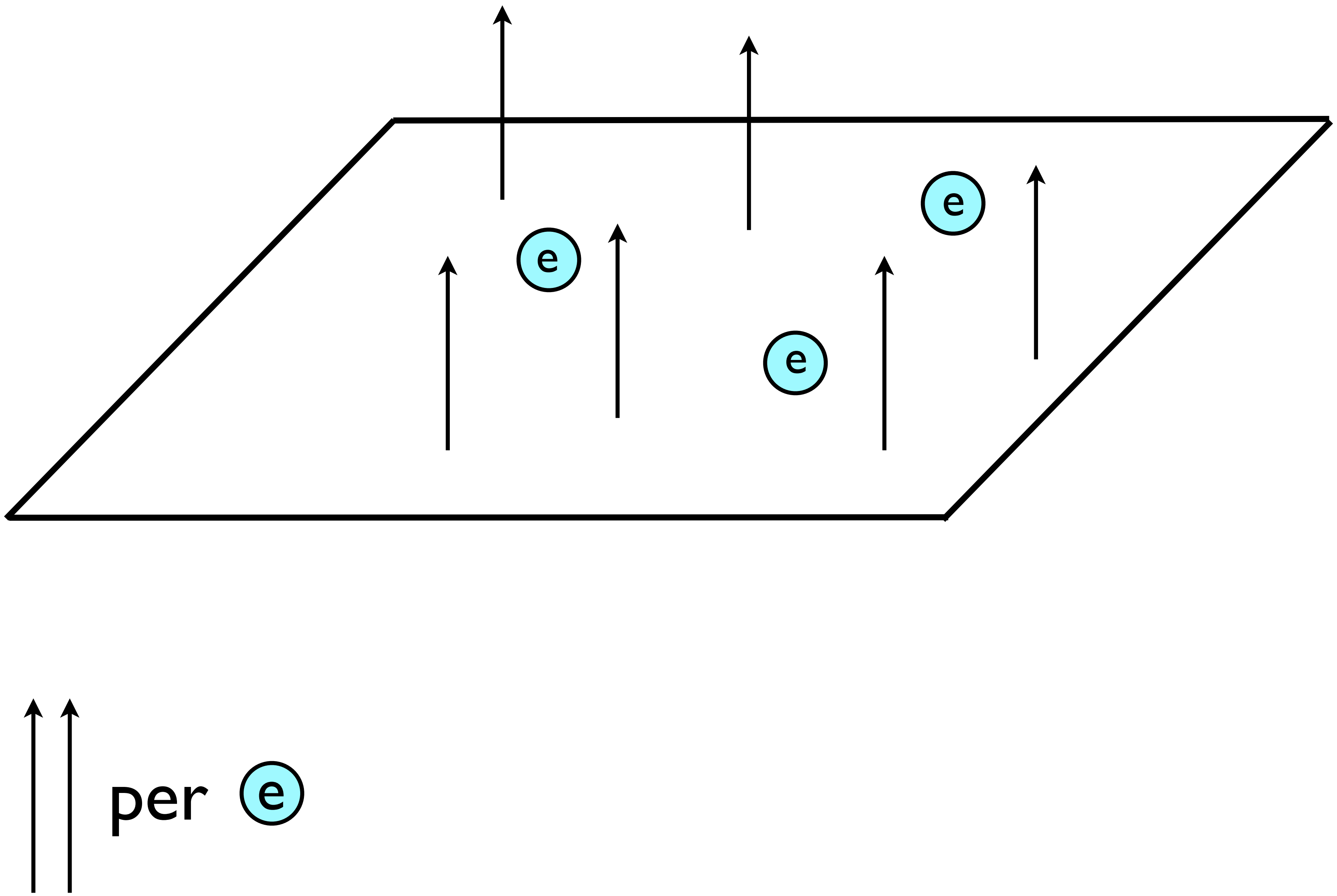
# Jain's sequence of plateaux

- Using the composite fermion most observed fractions can be explained

| Electrons | Composite fermions |
| --- | --- |
| $\nu = \frac{n}{2n+1}$ | $\nu_{\mathrm{CF}} = n$ |
| $\nu = \frac{n+1}{2n+1}$ | $\nu_{\mathrm{CF}} = n+1$ |

# Prediction for nu=1/2 state

Halperin Lee Read 1993



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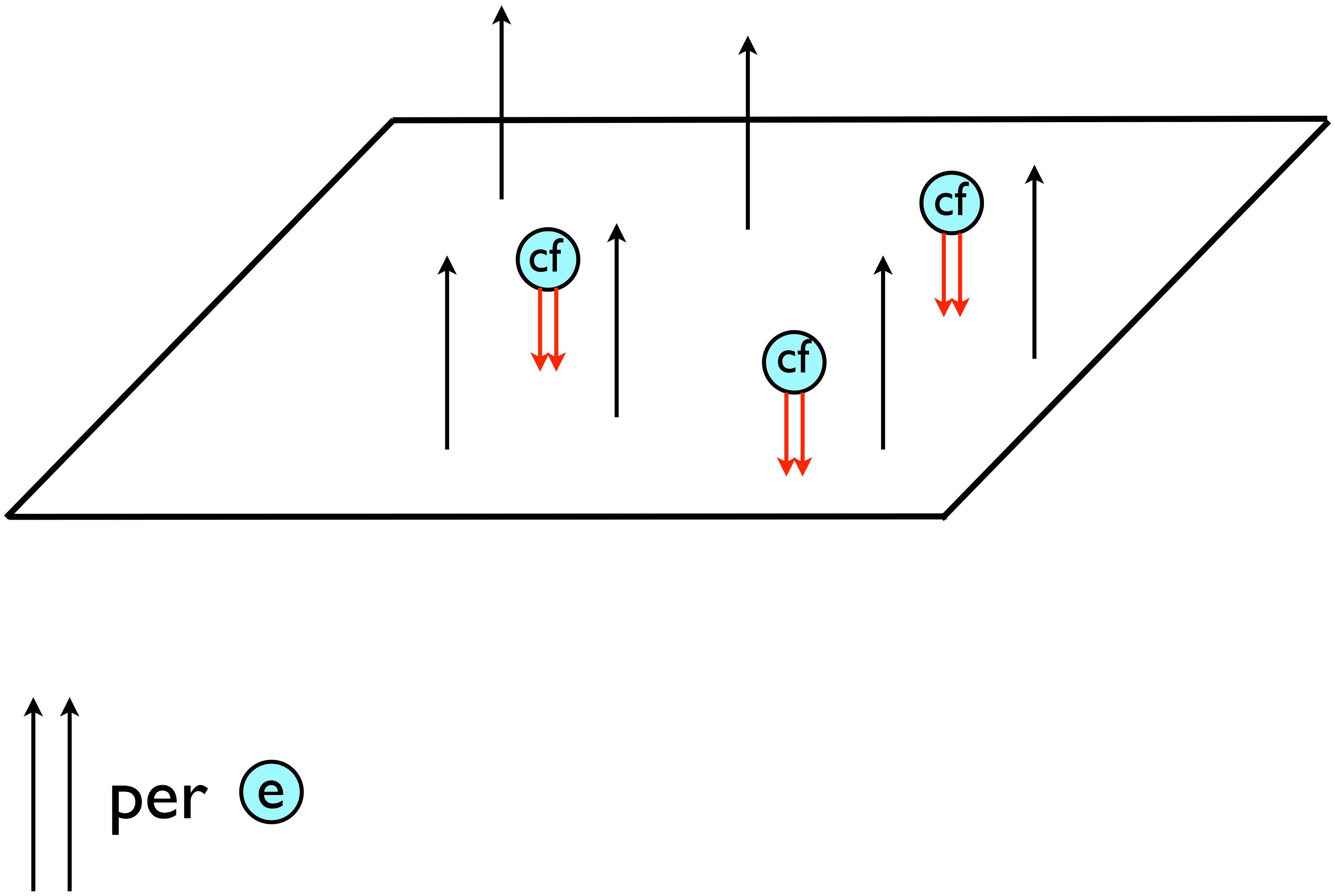

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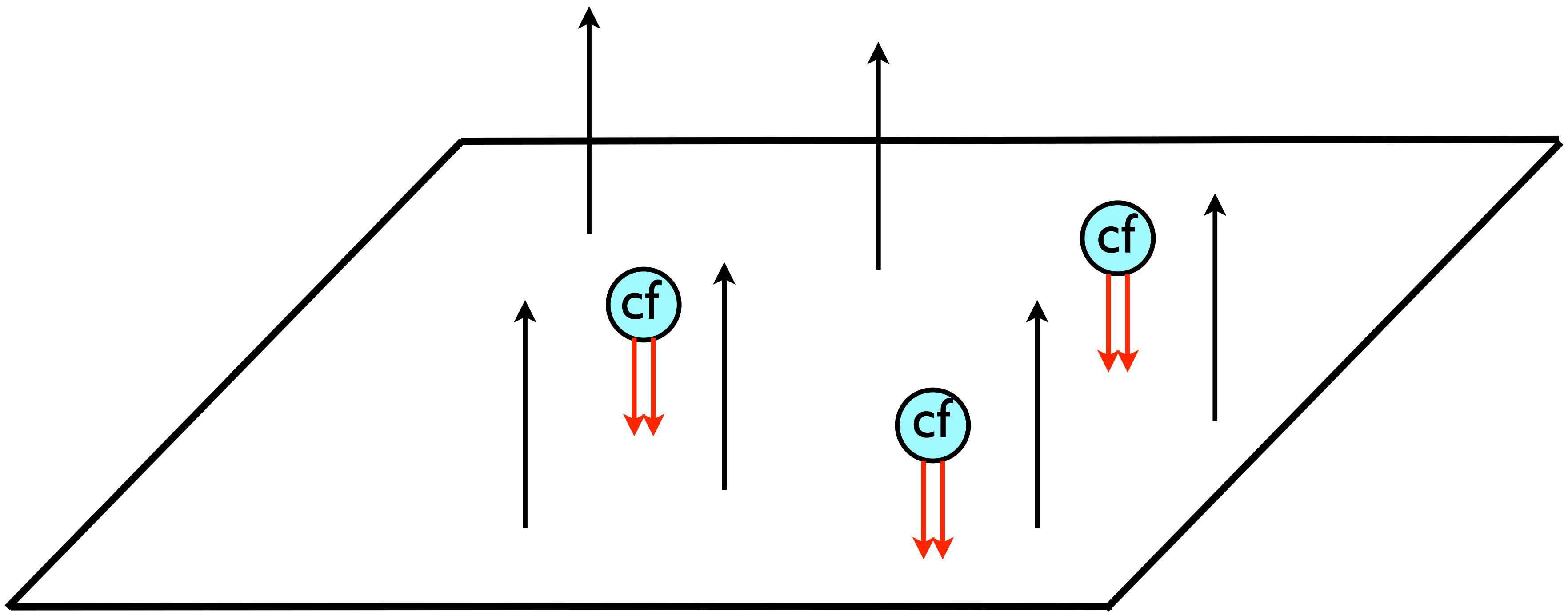


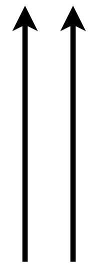

per e

Zero B field for cf

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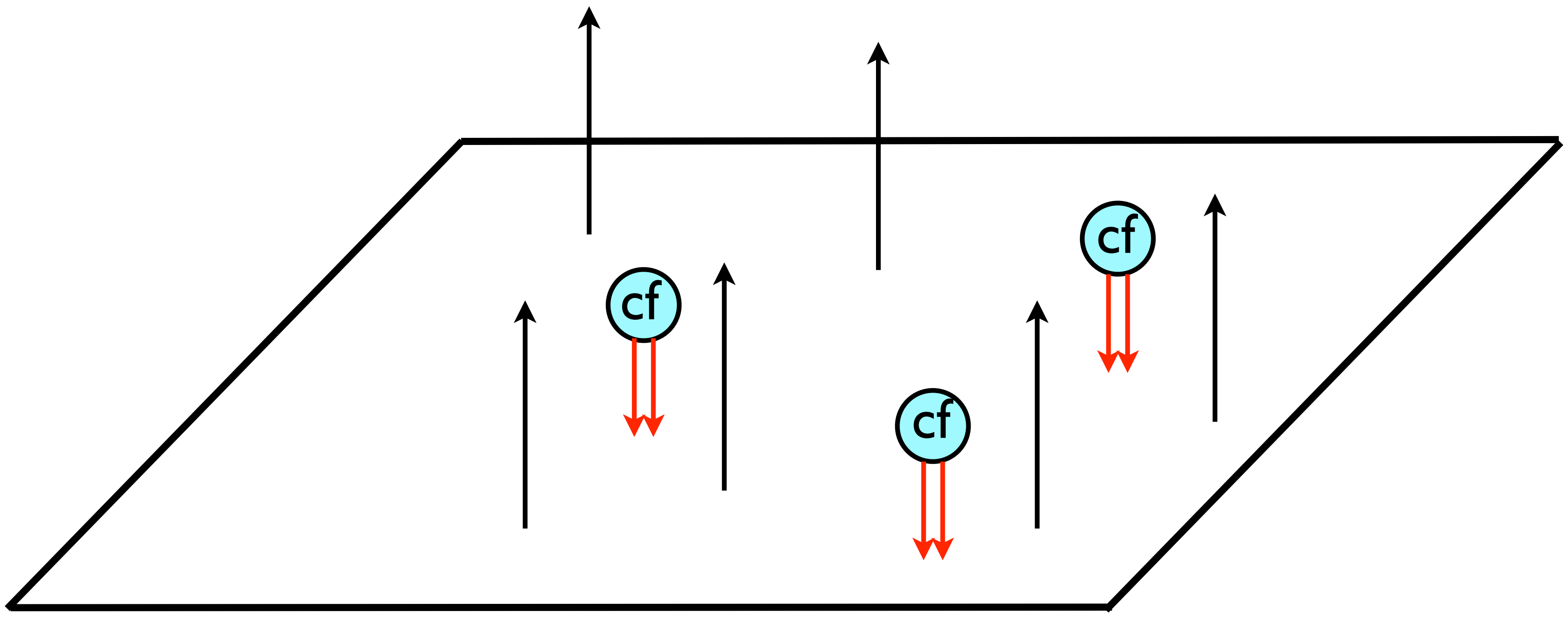
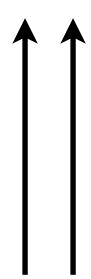
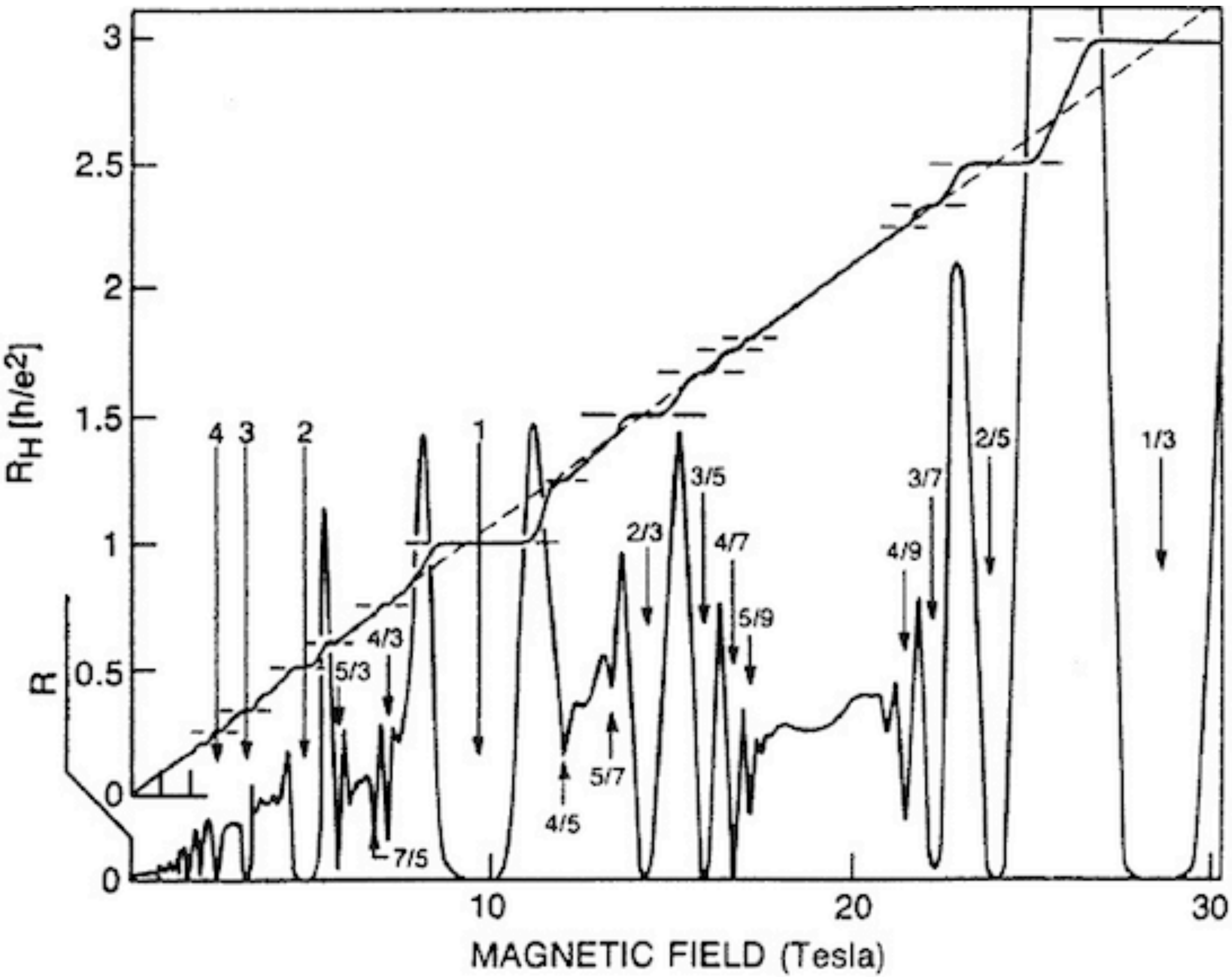


per e

Zero B field for cf

CFs form a Fermi liquid; HLR theory



3
2.5
2
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R
0
$R_H$ [h/e$^2$]
4
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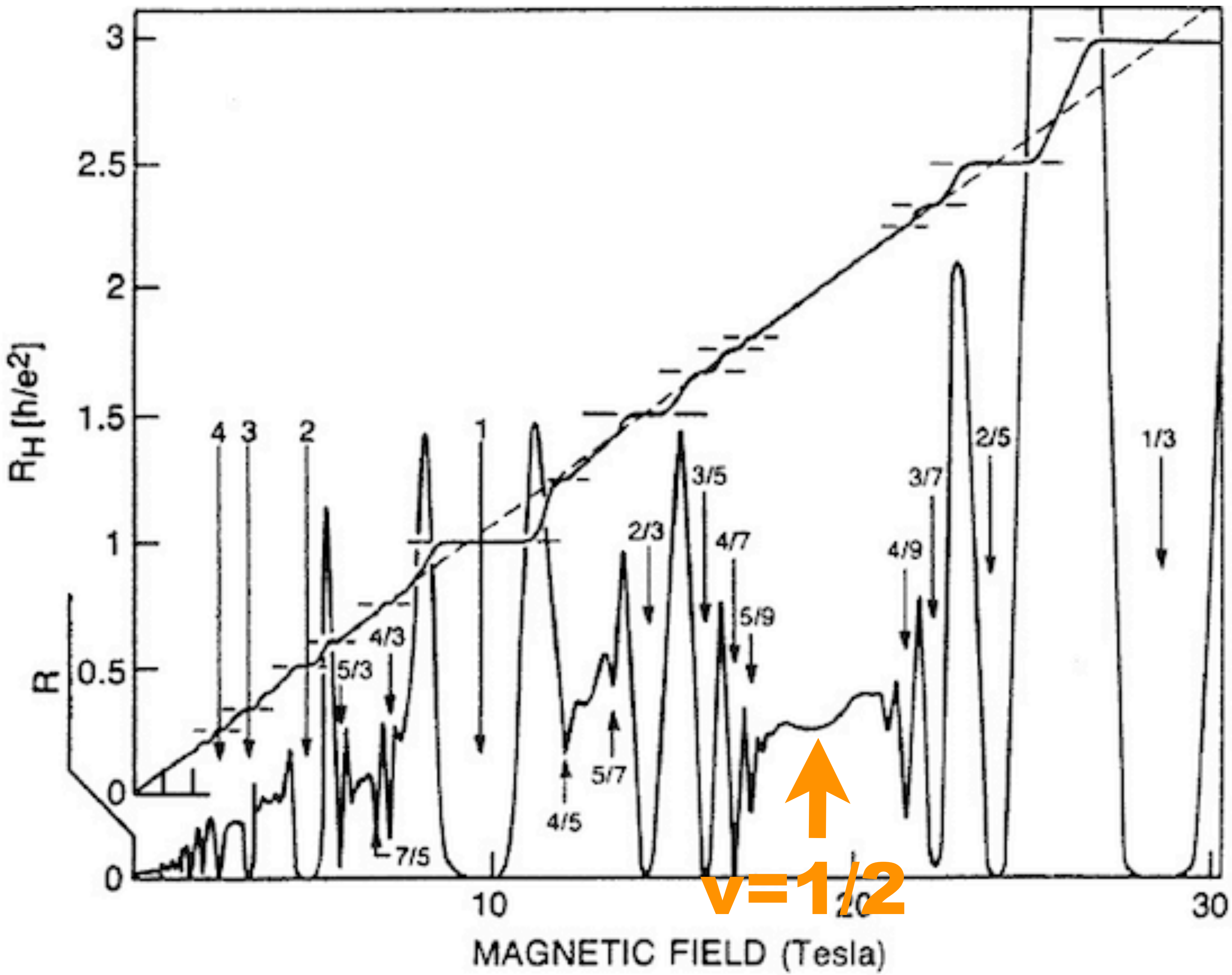
10
20
30
MAGNETIC FIELD (Tesla)


R_H [h/e²]
R
3
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0.5
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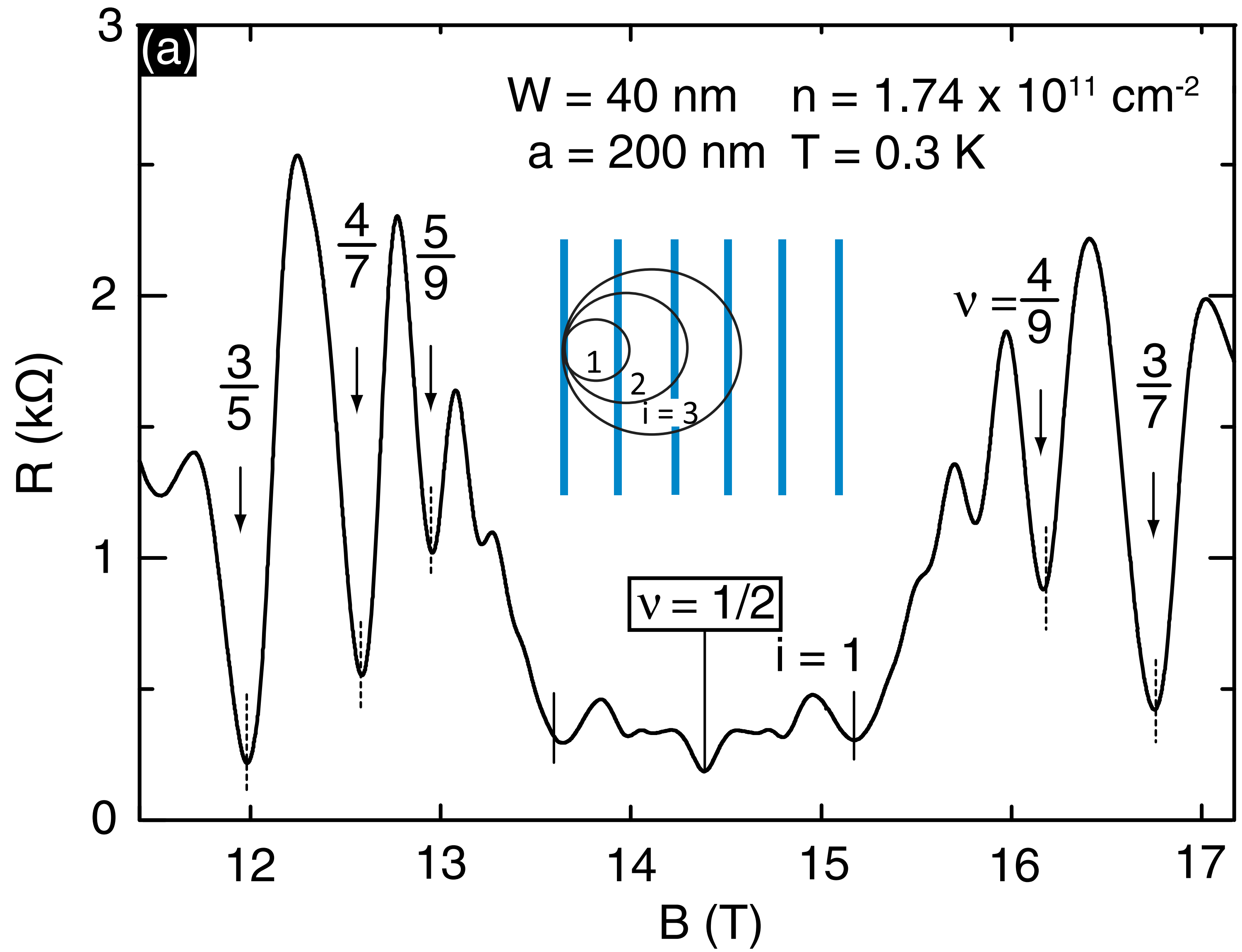
7/5
4/5
5/7
2/3
3/5
4/7
5/9
4/9
3/7
2/5
1/3
v=1/2
10
20
30
MAGNETIC FIELD (Tesla)

# Is the composite fermion real?

- Composite fermion can be detected as a quasiparticle near half-filling
  - large semiclassical orbit when magnetic fields do not exactly cancel

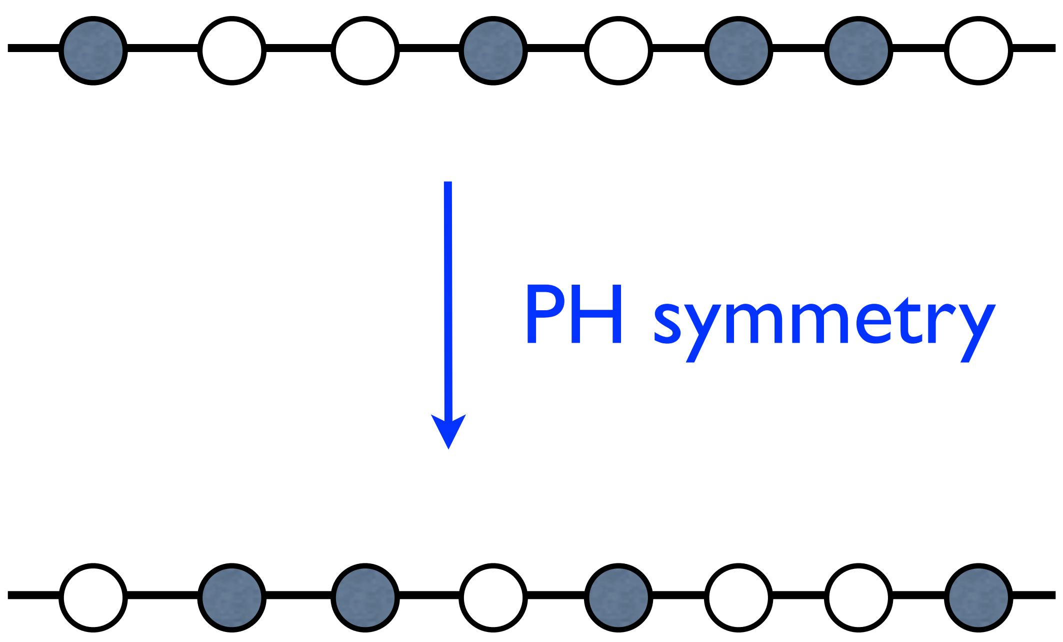



(Kamburov et al, 2014)

- For a long time it was thought that the HLR theory (zoomed in the near Fermi surface region) gives the correct low-energy effective theory
- There is one crucial problem

# The problem of particle-hole symmetry

# Particle-hole symmetry




$$\Theta|\text{empty}\rangle = |\text{full}\rangle$$

$$\Theta c_k^\dagger \Theta^{-1} = c_k$$

$$\Theta\, i\, \Theta^{-1} = -i$$

$$\nu \to 1 - \nu$$

exact symmetry the Hamiltonian on the LLL, when mixing of higher LLs negligible

# PH symmetry in the CF theory

PH conjugate pairs of FQH states

$$\nu = \frac{n}{2n+1} \qquad \qquad \nu = \frac{n+1}{2n+1}$$

ν=1/3 ν=2/3

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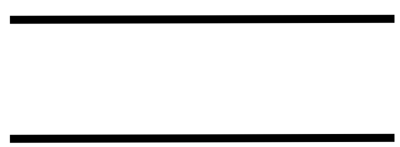
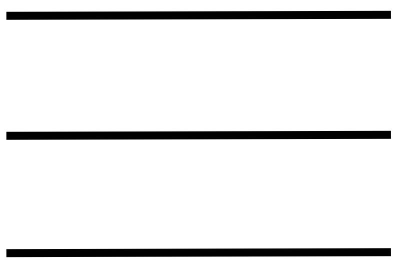


ν=2/5



ν=3/5

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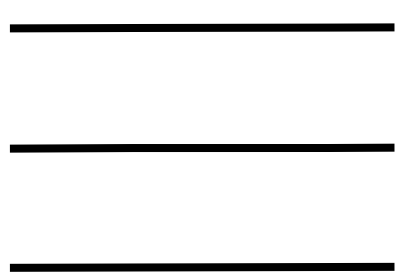
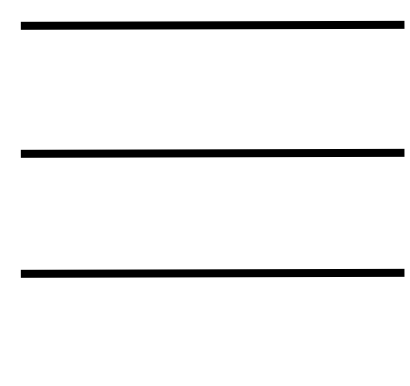


ν=3/7



ν=4/7

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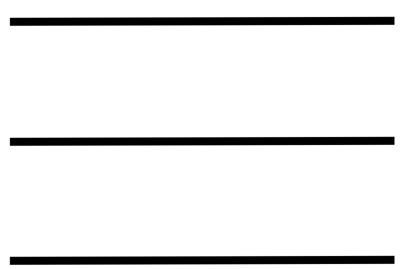

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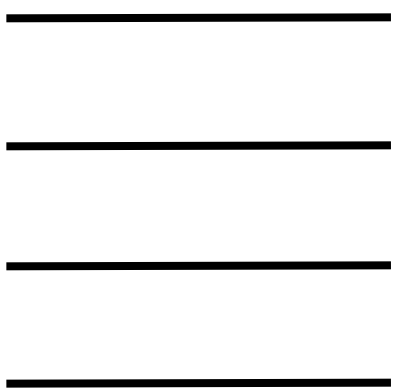
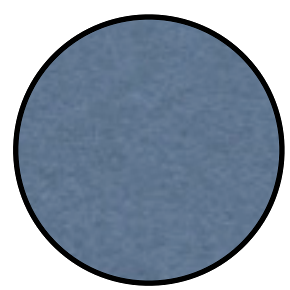


ν=4/7

CF picture does not respect PH symmetry

# PH symmetry of a Fermi liquid?



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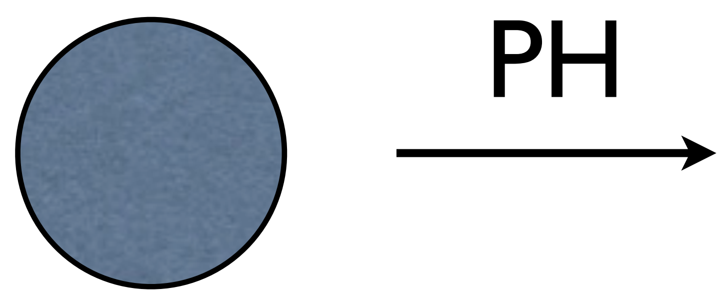

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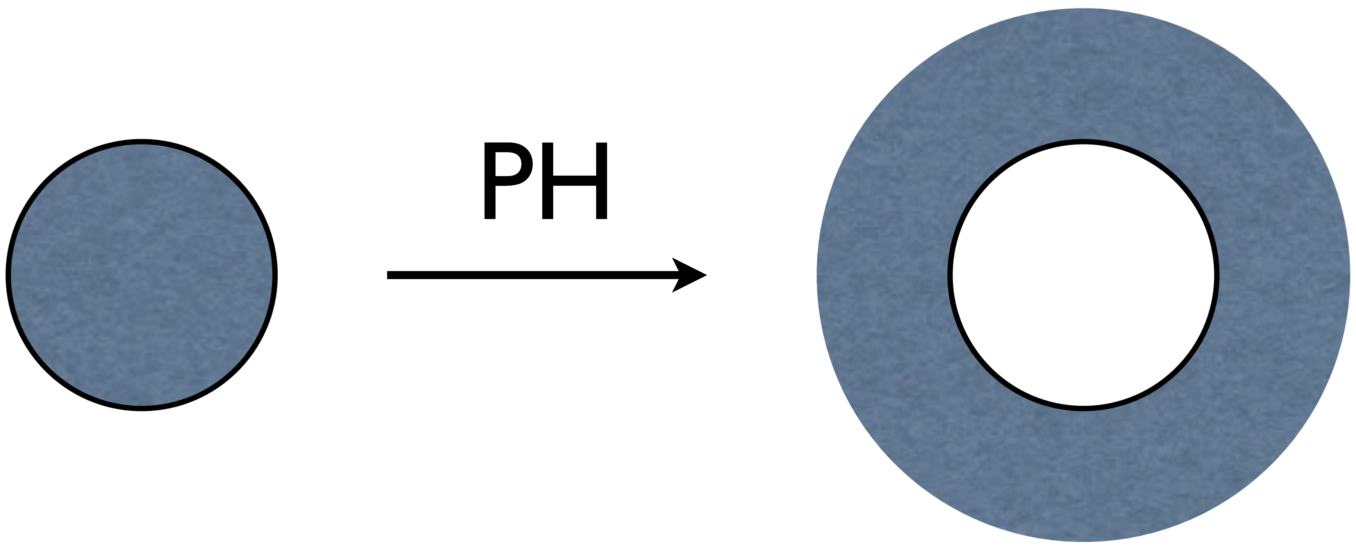

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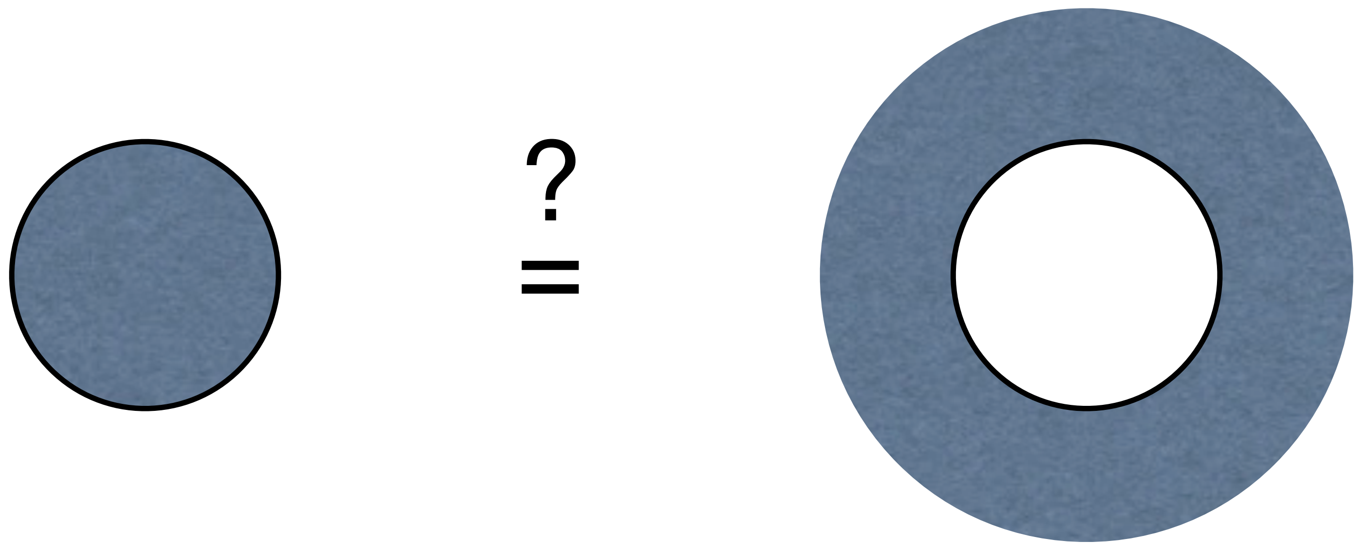
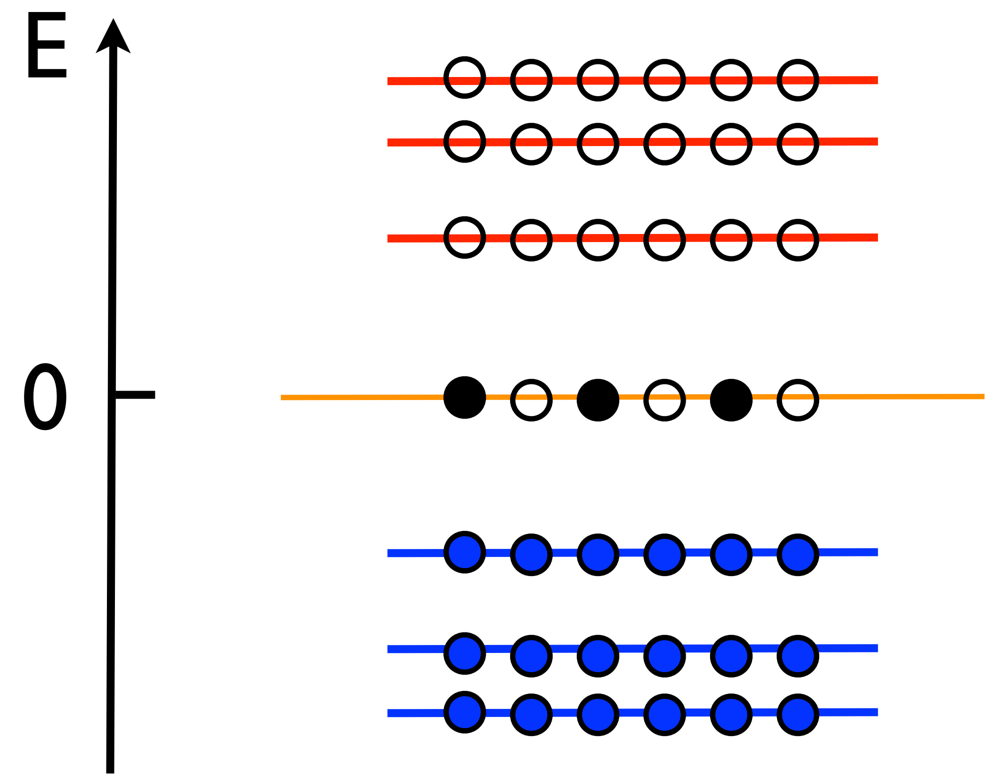


# PH symmetry in HLR

- HLR Lagrangian does not have any symmetry that can be identified with PH symmetry ~1997
- The problem was considered "hard" as it requires projection to lowest Landau level
  - PH conjugation acts nonlocally

# Sharpening the problem

- Consider a 2-component massless Dirac fermion
- Can realize fractional quantum Hall effect
- Natural particle-hole symmetry at zero density



# The puzzle of QHE for Dirac fermion

- Half filled Landau level arises naturally at zero chemical potential
- Turn on a magnetic field: ground state is a Fermi liquid
- Volume of Fermi sphere ~ magnetic field
- Which conserved charge in Luttinger's theorem ???

# Solution to the problem of particle-hole symmetry

# Prelude to solution: particle-vortex duality

Peskin; Dasgupta, Halperin

$$\mathcal{L}_1 = -|\partial_\mu \Phi|^2 - m^2|\Phi|^2 - \lambda|\Phi|^4$$

$$\mathcal{L}_2 = -|(\partial_\mu - a_\mu)\phi|^2 - \tilde{m}^2|\phi|^2 - \tilde{\lambda}|\phi|^4$$

| | |
|---|---|
| Goldstone boson | photon |
| particle | vortex |

# Coupling to external gauge field

$$\mathcal{L}_1 = -|(\partial_\mu - A_\mu)\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4$$

$$\mathcal{L}_2 = -|(\partial_\mu - a_\mu)\phi|^2 - \tilde{m}^2|\phi|^2 - \tilde{\lambda}|\phi|^4 + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda$$

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

# Hypothetical duality

DTS 2015
Metlitski, Vishwanath 2015
Wang, Senthil 2015

"electron theory" $\mathcal{L} = i\bar{\psi}_e \gamma^\mu (\partial_\mu - iA_\mu)\psi_e$

CF theory $\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda$

# Particle-vortex duality

$$S=\int d^3x\left[i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda\right]$$

$$\rho=\frac{\delta S}{\delta A_0}=-\frac{b}{4\pi} \qquad \frac{\delta S}{\delta a_0}=0 \longrightarrow \langle\psi\bar{\gamma}^0\psi\rangle=\frac{B}{4\pi}$$

Turn on magnetic field lead to a finite density
Landau's reasoning: Fermi surface

| original fermion $\psi$ | composite fermion $\psi_e$ |
|---|---|
| magnetic field | density |
| density | magnetic field |

# Dirac composite fermion

- Low energy dynamics of a half-filled Landau level is described by a low-energy effective theory of a new fermion ("composite fermion") coupled to a dynamical gauge field
- The composite fermion is electrically neutral
- Density of composite fermion = physical magnetic field

# Particle-hole symmetry as CT symmetry

- Magnetic field breaks C, P, T
- preserves PT, CT, CP
- Particle-hole symmetry of the n=0 Landau level can be identified with CT
- Effective theory of the composite fermion has CT symmetry

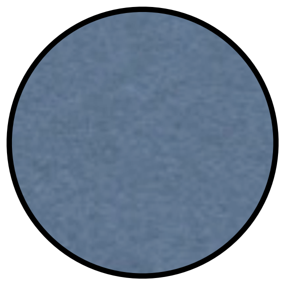
# Action of CT

$$A_0(t, \mathbf{x}) \to -A_0(-t, \mathbf{x}) \qquad a_0(t, \mathbf{x}) \to a_0(-t, \mathbf{x})$$

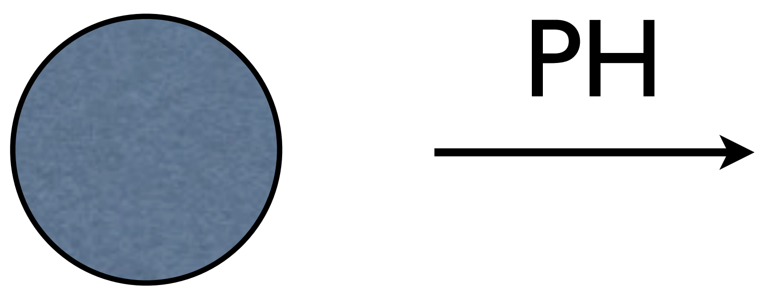
$$A_i(t, \mathbf{x}) \to A_i(-t, \mathbf{x}) \qquad a_i(t, \mathbf{x}) \to -a_i(-t, \mathbf{x})$$

$$\psi(t, \mathbf{x}) \to -i\sigma_2\psi(-t, \mathbf{x})$$

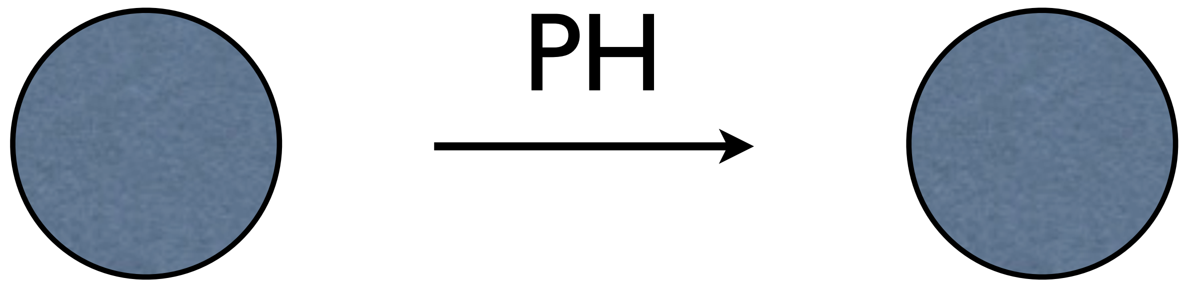
# CT on composite fermion



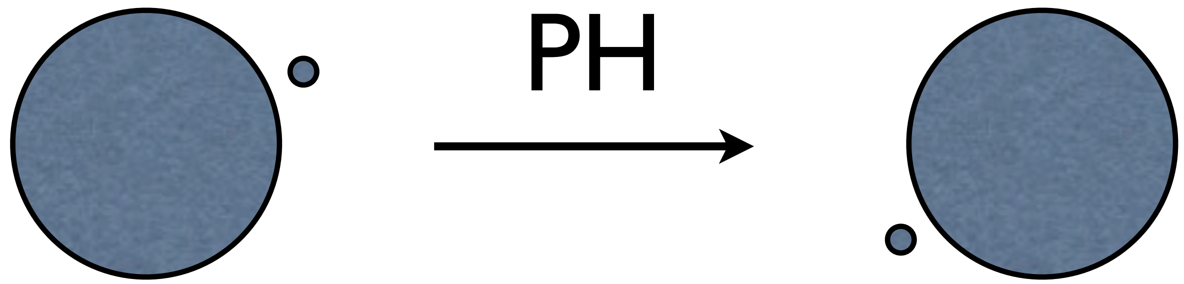
# CT on composite fermion



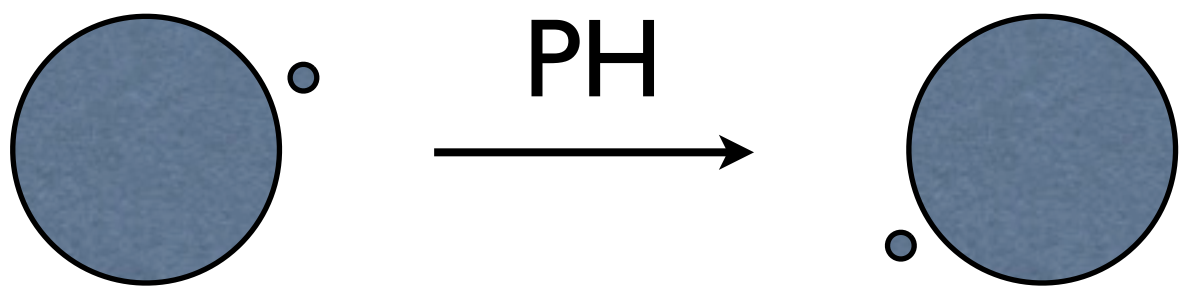
# CT on composite fermion



# CT on composite fermion



# CT on composite fermion

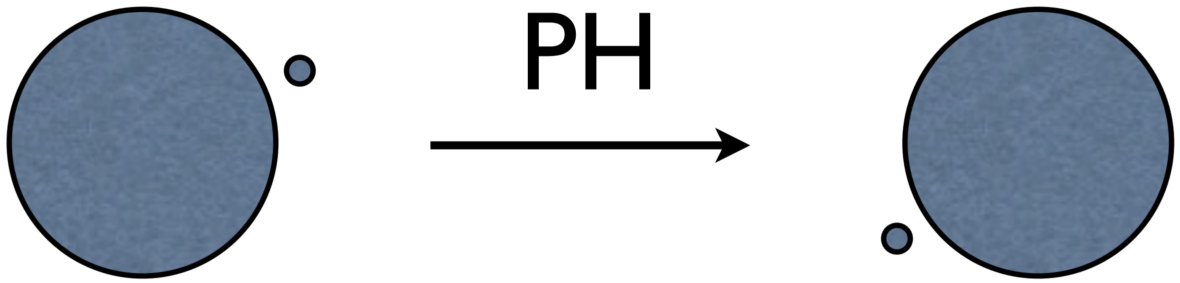



Particle-hole symmetry maps particle to particle

$$\mathbf{k} \to -\mathbf{k}$$

$$\psi \to i\sigma_2 \psi$$

# CT on composite fermion




Particle-hole symmetry maps particle to particle

$$\mathbf{k} \to -\mathbf{k}$$

$$\psi \to i\sigma_2\psi$$

DTS 2015

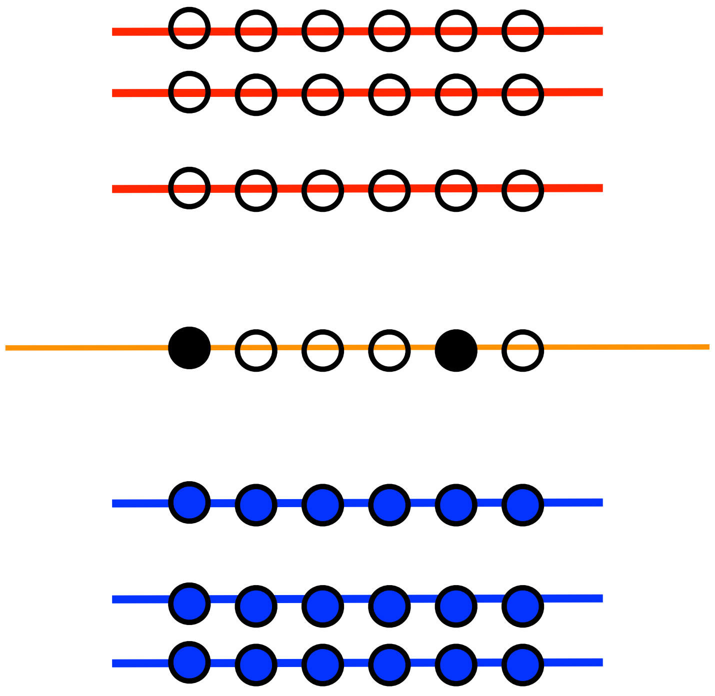
# Away from half filling

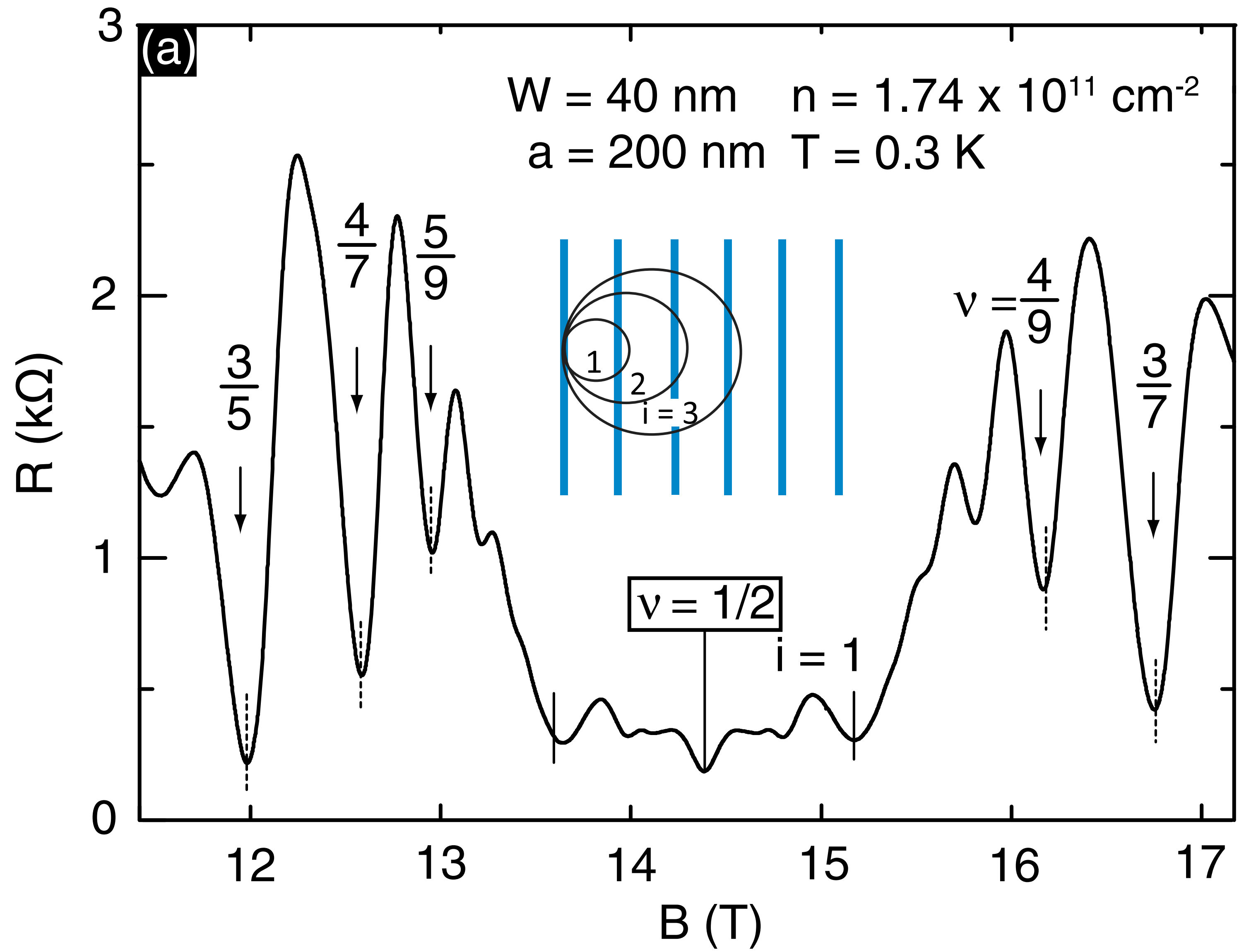
$$S = \int d^3x \left[ i\bar{\psi}\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda \right]$$

$$\rho = \frac{\delta S}{\delta A_0} = -\frac{b}{4\pi} \qquad \frac{\delta S}{\delta a_0} = 0 \longrightarrow \langle \psi\bar{\gamma}^0\psi \rangle = \frac{B}{4\pi}$$

$$\neq 0$$

Now the CFs move in large circular orbits






(Kamburov et al, 2014)

# Mapping Jain's sequences

$$\nu = \frac{n}{2n+1} \longrightarrow \nu_{\mathrm{CF}} = n$$

$$\nu = \frac{n+1}{2n+1} \longrightarrow \nu_{\mathrm{CF}} = n+1$$

# Mapping Jain's sequences

$$\nu = \frac{n}{2n+1}$$

$$\nu = \frac{n+1}{2n+1}$$

$$\nu_{\mathrm{CF}} = n + \frac{1}{2}$$

# Mapping Jain's sequences

$$\nu = \frac{n}{2n+1}$$

$$\nu_{\mathrm{CF}} = n + \frac{1}{2}$$

$$\nu = \frac{n+1}{2n+1}$$

CFs form an IQH state at half-integer filling factor: must be a Dirac fermion

# IQHE in graphene

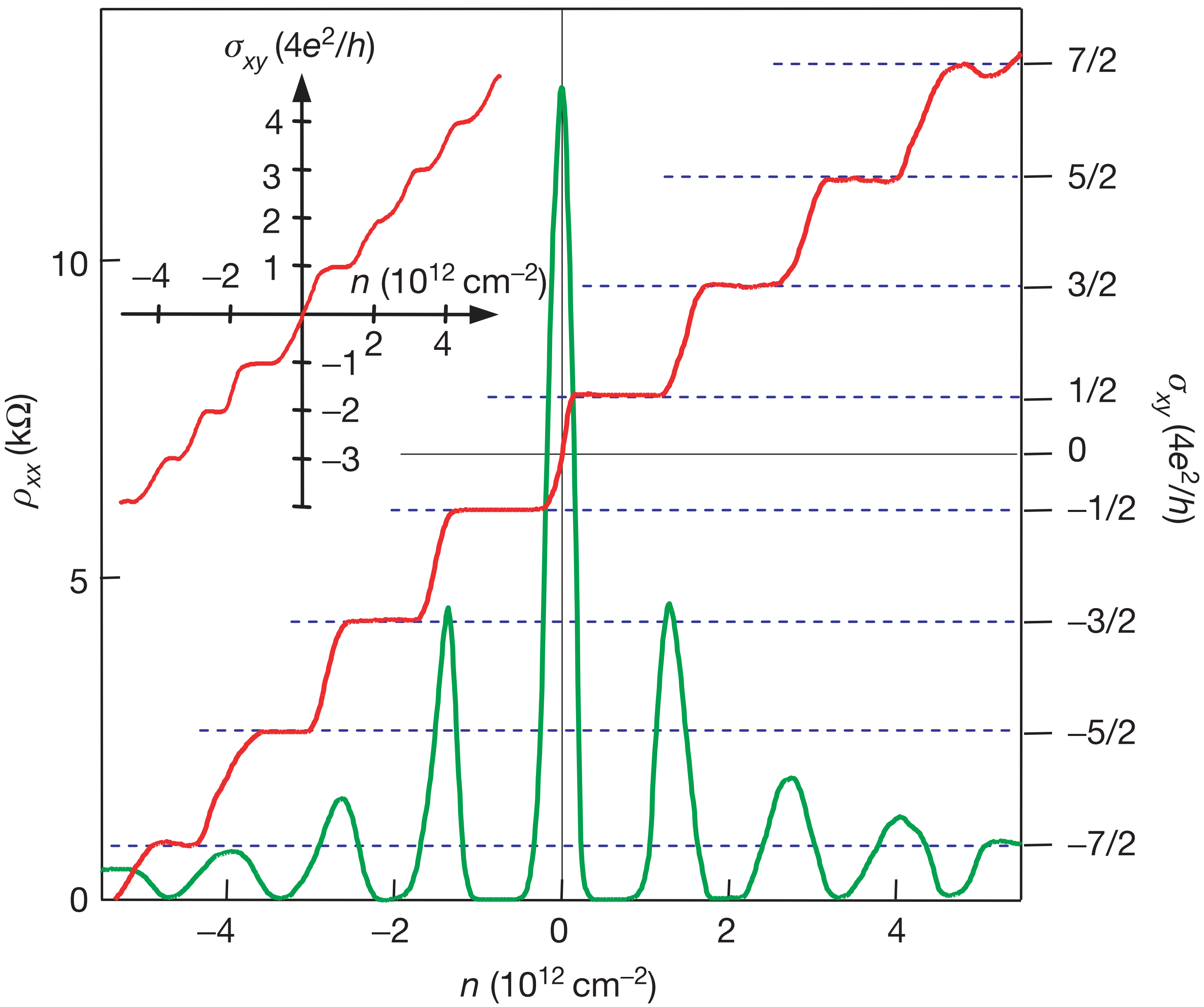
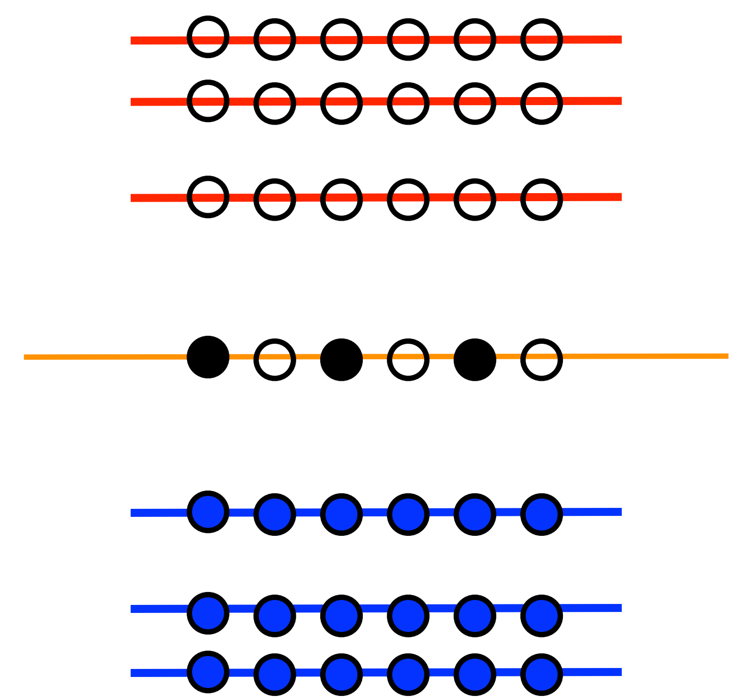


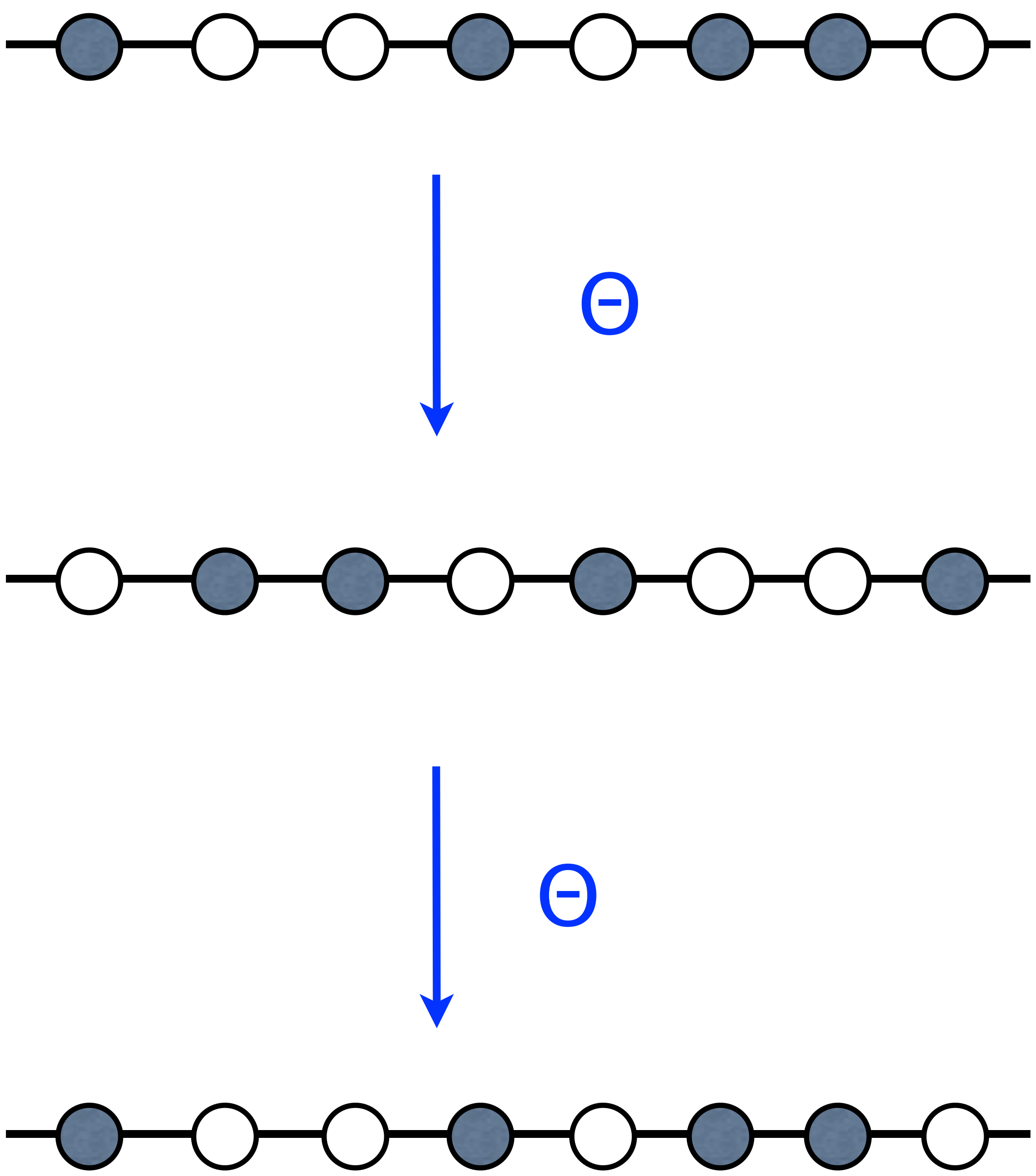

**Figure 4 | QHE for massless Dirac fermions.** Hall conductivity $\sigma_{xy}$ and longitudinal resistivity $\rho_{xx}$ of graphene as a function of their concentration at $B = 14$ T and $T = 4$ K. $\sigma_{xy} \equiv (4e^2/h)\nu$ is calculated from the measured

$$\sigma_{xy} = \left(n + \frac{1}{2}\right)\frac{e^2}{2\pi\hbar}$$



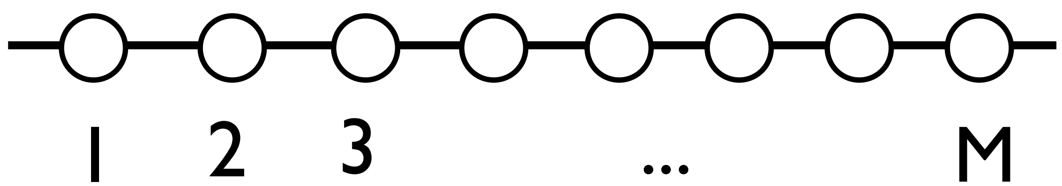
Novoselov et al 2005

# (Particle-hole)$^2$

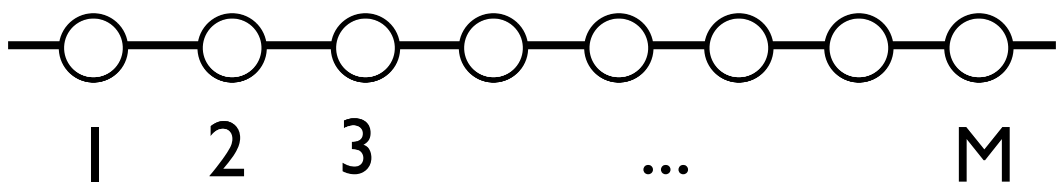



$$\Theta^2 = \pm 1$$

On a single Landau level

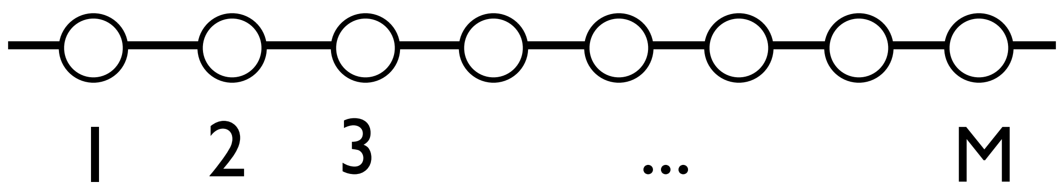


On a single Landau level




$$\Theta^2 = (-1)^{M(M-1)/2}$$

## On a single Landau level

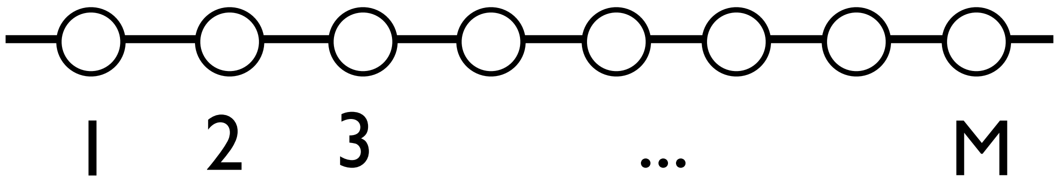



$$\Theta^2 = (-1)^{M(M-1)/2}$$

$$M = 2N_{\mathrm{CF}}$$

$$\Theta^2 = (-1)^{N_{\mathrm{CF}}}$$

On a single Landau level




$$\Theta^2 = (-1)^{M(M-1)/2}$$

$$M = 2N_{\rm CF}$$

$$\Theta^2 = (-1)^{N_{\rm CF}}$$

Dirac CF: $\psi \to (-i\sigma_2)^2 \psi = -\psi$

Geraedts, Zaletel, Mong, Metlitski, Vishwanath, Motrunich; Levin, Son

# Consequences of PH symmetry

$$\mathbf{j} = \sigma_{xx}\mathbf{E} + \sigma_{xy}\mathbf{E}\times\hat{\mathbf{z}} + \alpha_{xx}\boldsymbol{\nabla}T + \alpha_{xy}\boldsymbol{\nabla}T\times\hat{\mathbf{z}}$$

conductivities

thermoelectric coefficients

- At exact half filling, in the presence of particle-hole symmetric disorders

$$\sigma_{xy} = \frac{e^2}{2h} \qquad \alpha_{xx} = 0$$

HLR $\rho_{xy} = \frac{2h}{e^2}$

Potter, Serbyn, Vishwanath 2015

# A new gapped state

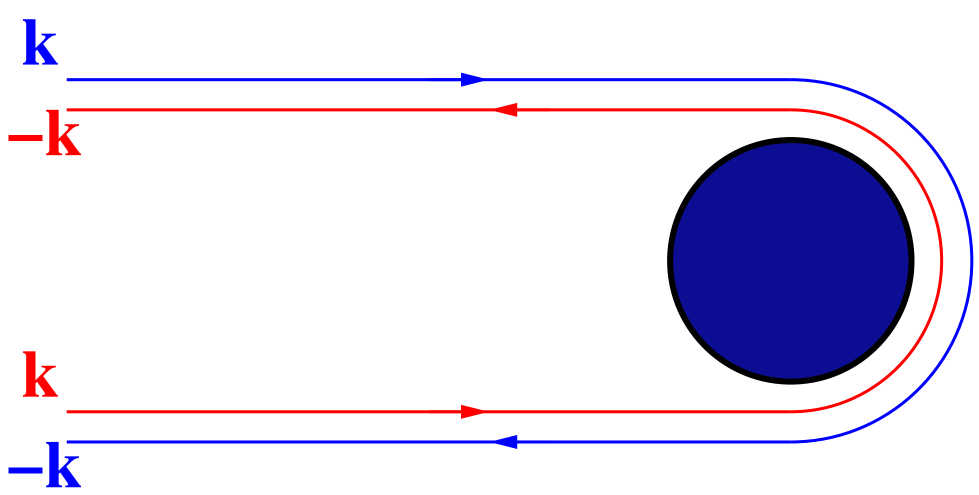
- The composite fermions can form Cooper pairs
- Simplest pairing does not break particle-hole symmetry

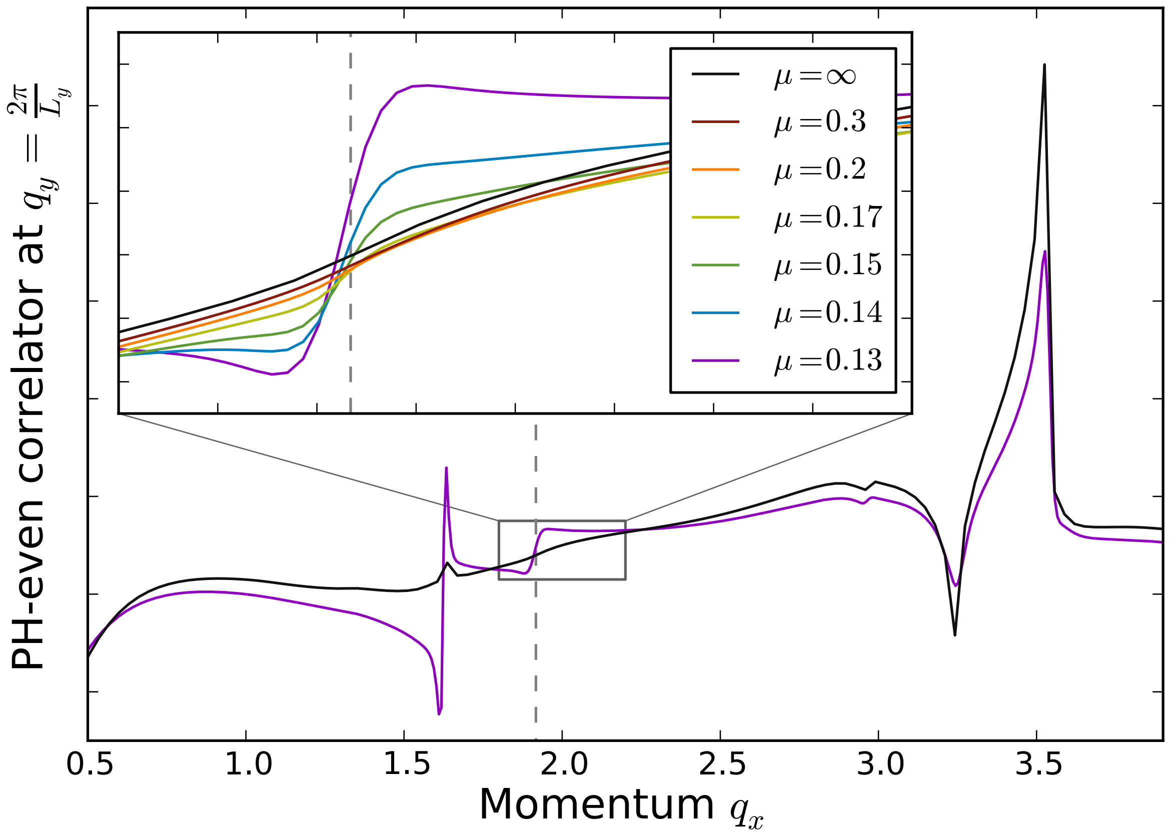
$$\langle \epsilon^{\alpha\beta} \psi_\alpha \psi_\beta \rangle \neq 0$$

- A new gapped state: PH-Pfaffian state

# Consequences of Dirac CF

Suppression of Friedel oscillations in correlations of particle-hole symmetric observables $\hat{O} = (\rho - \rho_0)\nabla^2 \rho$




Geraedts, Zaletel, Mong, Metlitsky, Vishwanath, Montrunich, 2015

Direct proof of Berry phase π of the composite fermion

# A window to duality

- Fermionic particle-vortex duality can be derived of a more "elementary" fermion-boson duality Karch, Tong; Seiberg, Senthil, Wang, Witten
  - small N version of duality between CS theories, tested at large N
- New dualities can be obtained
  - Example: Nf=2 QED3 is self-dual Cenke Xu

# The elementary duality

$$\mathcal{L} = L[\psi, A] - \frac{1}{2}\frac{1}{4\pi}AdA$$

$$\mathcal{L} = L[\phi, a] + \frac{1}{4\pi}ada + \frac{1}{2\pi}Ada$$

# Comments on duality

- One side of duality is a theory of a single 2-component Dirac fermion coupled to U(1) gauge field
- It is usually thought QED3 with Nf fermions is unstable with respect to SSB for Nf<Nf*. Nf* ~ 6 in Schwinger-Dyson
- One lattice simulation (Karthik, Narayanan 2015) indicates Nf* < 2.
- The stability of theory with Nf=1 is not required for fractional quantum Hall effect (finite density)

# Conclusion

- The low-energy quasiparticle of half-filled Landau level is completely different from the electron
- Symmetries allowed to guess the form of the low-energy effective theory
- Hints on new field-theoretic dualities in 2+1 D

# References

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