Fractional quantum Hall effect and duality

Dam T. Son (University of Chicago)
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Plan
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- General prologue: Fractional Quantum Hall Effect (FQHE)
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- Composite fermions
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- The puzzle of particle-hole symmetry
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• General prologue: Fractional Quantum Hall Effect (FQHE)
• Composite fermions
• The puzzle of particle-hole symmetry
• Dirac composite fermions
General Prologue

- QCD is a prime example of a strongly coupled theory
- The particle excitations of the vacuum are very different from the microscopic degree of freedom
- A very similar situation in FQHE
The setup

\[ H = \sum_a \frac{(p_a + eA_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|x_a - x_b|} \]
Integer quantum Hall effect

- Ignore Coulomb interactions
- When electrons moving in 2D in a magnetic field, energy is quantized: Landau level
- IQHE: electrons filling $n$ Landau levels

\[
\Delta = \frac{B}{m}
\]

\[
\text{degeneracy } \frac{BA}{2\pi}
\]
Plateaux require energy gap
Fractional QHE

Assume we have less particles than states on LLL

\[
\begin{align*}
\text{\_\_\_\_\_\_\_\_}\quad n=3 \\
\text{\_\_\_\_\_\_\_}\quad n=2 \\
\text{\_\_\_\_\_\_\_}\quad n=1 \\
\end{align*}
\]

In the approximation of noninteracting electrons: exponential degeneracy of states
Fractional QHE

Assume we have less particles than states on LLL

\[ \ldots \quad n=3 \]

\[ \ldots \quad n=2 \]

\[ \bullet \circ \circ \circ \circ \quad n=1 \]

In the approximation of noninteracting electrons: exponential degeneracy of states
Why the FQH problem is hard

- degenerate perturbation theory
- Starting point: exponentially large number of degenerate states
- Any small perturbation lifts the degeneracy
- no small parameter
Lowest Landau level limit

\[ H = \sum_a \frac{(p_a + eA_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|x_a - x_b|} \]
Lowest Landau level limit

\[ H = \sum_a \frac{(p_a + eA_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|x_a - x_b|} \]

\[ m \to 0 \]

\[
\begin{array}{ccc}
& & n=1 \\
\downarrow & \downarrow & \downarrow \\
\frac{B}{m} & \to & \infty \\
& n=0 & \\
\end{array}
\]
Lowest Landau level limit

\[ H = \sum_a \frac{(p_a + eA_a)^2}{2m} + \sum_{\langle a,b \rangle} \frac{e^2}{|x_a - x_b|} \]

\[ m \rightarrow 0 \]

\[ H \rightarrow P_{\text{LLL}} \sum_{a,b} \frac{e^2}{|x_a - x_b|} \]

Projection to lowest Landau level

\[ n=0 \]

\[ n=1 \]
Experimental hints
Jain’s sequences of QH plateaux

\[ \nu = \frac{n + 1}{2n + 1} \quad \nu = \frac{n}{2n + 1} \]
Systematics of Jain’s sequences

- Gapped states
- Energy gap goes down $\sim 1/n$ for $n \rightarrow \infty$
- $n=\infty$: gapless, likely Fermi liquid state
A powerful theory with a flaw
Flux attachment

(Wilczek 1982, Jain 1989)

- Flux attachment: statistics does not change by attaching an even number of flux quanta
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\[ (-1) \]
Flux attachment: statistics does not change by attaching an even number of flux quanta

\((-1) \exp(i\pi) = (+1)\)

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- Flux attachment: statistics does not change by attaching an even number of flux quanta

\[ (-1) \exp(2i\pi) = (-1) \]
Flux attachment

(Wilczek 1982, Jain 1989)

- Flux attachment: statistics does not change by attaching an even number of flux quanta

\[-1 \exp(2i\pi) = (-1)\]

\[\text{e} = \text{CF}\]
Composite fermion

$\nu = 1/3$ FQH
Composite fermion

$\nu = 1/3$ FQH
Composite fermion

\( \nu = 1/3 \) FQH

\[ \text{per } e \quad \text{average} \quad \text{per } \text{cf} \]
Composite fermion

\[ \nu = \frac{1}{3} \text{ FQH} \]

IQHE of CFs with \( \nu = 1 \)
Composite fermion

\[ \nu = \frac{2}{3} \text{FQH} \]

FQHE for original fermions = IQHE for composite fermions (n=2)
HLR field theory

\[ \mathcal{L} = i\psi^\dagger (\partial_0 - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2} \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \]

\[ b = \nabla \times a = 2 \times 2\pi \psi^\dagger \psi \quad \text{“flux attachment”} \]

mean field: \[ B_{\text{eff}} = B - b = B - 4\pi n \]

\[ \nu = \frac{1}{2} \quad B_{\text{eff}} = 0 \]
Jain’s sequence of plateaux

- Using the composite fermion most observed fractions can be explained

<table>
<thead>
<tr>
<th>Electrons</th>
<th>Composite fermions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu = \frac{n}{2n+1}$</td>
<td>$\nu_{CF} = n$</td>
</tr>
<tr>
<td>$\nu = \frac{n+1}{2n+1}$</td>
<td>$\nu_{CF} = n+1$</td>
</tr>
</tbody>
</table>
Prediction for $\nu = 1/2$ state

Halperin Lee Read 1993
Prediction for $\nu = 1/2$ state

Halperin Lee Read 1993
Prediction for \( \nu = 1/2 \) state

Halperin Lee Read 1993

Zero B field for \( \text{cf} \)
Prediction for $\nu = 1/2$ state

Halperin Lee Read 1993

CFs form a Fermi liquid; HLR theory
\[ \nu = \frac{1}{2} \]
Is the composite fermion real?

- Composite fermion can be detected as a quasiparticle near half-filling
- Large semiclassical orbit when magnetic fields do not exactly cancel
agree with the experimental data for we assume that factors and the 2D electron density, as seen by the vertical positions of the FQHSs we observe in the same sample the remainder of the Letter. We emphasize that the field lines in Fig. 1(b) are formed by the minority carriers, i.e., holes, with solid vertical lines labeled $i\hbar = eB/\nu n$ for $\nu = 1$ is the CF cyclotron radius, $\nu$ is the period of the modulation, exhibiting strong CF commensurability oscillations near $B>B_i$ which assumes that CFs are formed by the minority carriers in the LLL (hatched parts of the broadened level).

On the other hand, if we assume that, for $B<B_i$ the above commensurability condition leads to a quadratic equation for the expected positions for the density equals $n$, then the expected positions of the FQHSs, based on the 2D electron density. (b) The CF commensurability oscillations are shown in Fig. 1(b), the expected positions of the resistance minima when the CF density ($\nu = 1/2$), the expected values, as is clearly seen in the LLL are formed by the minority carriers, i.e., holes, flanked by shoulders of higher resistivity for $\nu = 2, 3$, with a periodic potential modulation. Dotted vertical lines mark the expected positions of the FQHSs, based on the 2D electron density. (b) The CF commensurability oscillations are shown in Fig. 1(b) for the case when $\nu = 4/9$, the expected positions where the deepest minimum is seen. The period are given next to each trace, and the expected values for $\nu = 1$, $\nu = 2$, and $\nu = 3$, with a periodic potential modulation. Dotted vertical lines mark the expected positions of the FQHSs, based on the 2D electron density. (Kamburov et al, 2014)
• For a long time it was thought that the HLR theory (zoomed in the near Fermi surface region) gives the correct low-energy effective theory

• There is one crucial problem
The problem of particle-hole symmetry
Particle-hole symmetry

\[ \Theta |\text{empty}\rangle = |\text{full}\rangle \]
\[ \Theta c_k^\dagger \Theta^{-1} = c_k \]
\[ \Theta i \Theta^{-1} = -i \]

\[ \nu \to 1 - \nu \]

exact symmetry the Hamiltonian on the LLL, when mixing of higher LLs negligible
PH symmetry in the CF theory

PH conjugate pairs of FQH states

\[ \nu = \frac{n}{2n + 1} \quad \text{and} \quad \nu = \frac{n + 1}{2n + 1} \]

\[ \nu = 1/3 \quad \text{and} \quad \nu = 2/3 \]
PH symmetry in the CF theory

PH conjugate pairs of FQH states

\[ \nu = \frac{n}{2n + 1} \quad \text{and} \quad \nu = \frac{n + 1}{2n + 1} \]

\[ \nu = 2/5 \quad \text{and} \quad \nu = 3/5 \]
PH symmetry in the CF theory

PH conjugate pairs of FQH states

\[ \nu = \frac{n}{2n + 1} \]

\[ \nu = \frac{n + 1}{2n + 1} \]

\( \nu = \frac{3}{7} \)

\( \nu = \frac{4}{7} \)
PH symmetry in the CF theory

PH conjugate pairs of FQH states

\[ \nu = \frac{n}{2n + 1} \quad \text{and} \quad \nu = \frac{n + 1}{2n + 1} \]

\( \nu = 3/7 \) \quad \text{and} \quad \nu = 4/7

CF picture does not respect PH symmetry
PH symmetry of a Fermi liquid?
PH symmetry of a Fermi liquid?
PH symmetry of a Fermi liquid?
PH symmetry of a Fermi liquid?
PH symmetry in HLR

- HLR Lagrangian does not have any symmetry that can be identified with PH symmetry \( \sim 1997 \)
- The problem was considered “hard” as it requires projection to lowest Landau level
- PH conjugation acts nonlocally
Sharpening the problem

- Consider a 2-component massless Dirac fermion
- Can realize fractional quantum Hall effect
- Natural particle-hole symmetry at zero density
The puzzle of QHE for Dirac fermion

- Half filled Landau level arises naturally at zero chemical potential
- Turn on a magnetic field: ground state is a Fermi liquid
- Volume of Fermi sphere \( \sim \) magnetic field
- Which conserved charge in Luttinger’s theorem ???
Solution to the problem of particle-hole symmetry
Prelude to solution: particle-vortex duality

Peskin; Dasgupta, Halperin

\[ \mathcal{L}_1 = -|\partial_\mu \Phi|^2 - m^2 |\Phi|^2 - \lambda |\Phi|^4 \]

\[ \mathcal{L}_2 = -|(\partial_\mu - a_\mu) \phi|^2 - \tilde{m}^2 |\phi|^2 - \tilde{\lambda} |\phi|^4 \]

Goldstone boson
particle

photon
vortex
Coupling to external gauge field

\[ \mathcal{L}_1 = -| (\partial_\mu - A_\mu) \phi |^2 - m^2 |\phi|^2 - \lambda |\phi|^4 \]

\[ \mathcal{L}_2 = -| (\partial_\mu - a_\mu) \phi |^2 - \tilde{m}^2 |\phi|^2 - \tilde{\lambda} |\phi|^4 + \frac{1}{2\pi} \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu a_\lambda \]

\[ j^\mu = \frac{1}{2\pi} \epsilon^{\mu \nu \lambda} \partial_\nu a_\lambda \]
Hypothetical duality

DTS 2015
Metlitski, Vishwanath 2015
Wang, Senthil 2015

“electron theory” \[ \mathcal{L} = i \bar{\psi}_e \gamma^\mu (\partial_\mu - iA_\mu) \psi_e \]

CF theory \[ \mathcal{L} = i \bar{\psi} \gamma^\mu (\partial_\mu - ia_\mu) \psi - \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \]
Particle-vortex duality

\[ S = \int d^3 x \left[ i \bar{\psi} \gamma^\mu (\partial_\mu - i a_\mu) \psi - \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right] \]

\[ \rho = \frac{\delta S}{\delta A_0} = - \frac{b}{4\pi} \quad \frac{\delta S}{\delta a_0} = 0 \rightarrow \langle \psi \tilde{\gamma}^0 \psi \rangle = \frac{B}{4\pi} \]

Turn on magnetic field lead to a finite density
Landau’s reasoning: Fermi surface

original fermion \( \psi \)\n
magnetic field\n
density\n
composite fermion \( \psi_e \)\n
density\nmagnetic field
Dirac composite fermion

- Low energy dynamics of a half-filled Landau level is described by a low-energy effective theory of a new fermion ("composite fermion") coupled to a dynamical gauge field
- The composite fermion is electrically neutral
- Density of composite fermion = physical magnetic field
Particle-hole symmetry as CT symmetry

- Magnetic field breaks C, P, T
- preserves PT, CT, CP
- Particle-hole symmetry of the n=0 Landau level can be identified with CT
- Effective theory of the composite fermion has CT symmetry
Action of CT

\[ A_0(t, x) \rightarrow -A_0(-t, x) \quad a_0(t, x) \rightarrow a_0(-t, x) \]
\[ A_i(t, x) \rightarrow A_i(-t, x) \quad a_i(t, x) \rightarrow -a_i(-t, x) \]
\[ \psi(t, x) \rightarrow -i\sigma_2\psi(-t, x) \]
CT on composite fermion
CT on composite fermion
CT on composite fermion

\[
\begin{array}{c}
\text{PH} \\
\end{array}
\]
CT on composite fermion

\[ \text{PH} \]
CT on composite fermion

Particle-hole symmetry maps particle to particle

\[ k \rightarrow -k \]

\[ \psi \rightarrow i\sigma_2\psi \]
CT on composite fermion

Particle-hole symmetry maps particle to particle

\[ k \rightarrow -k \]

\[ \psi \rightarrow i\sigma_2\psi \]

DTS 2015
Away from half filling

\[ S = \int d^3x \left[ i\bar{\psi}\gamma^\mu (\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \right] \]

\[ \rho = \frac{\delta S}{\delta A_0} = -\frac{b}{4\pi} \]

\[ \frac{\delta S}{\delta a_0} = 0 \rightarrow \langle \psi \tilde{\gamma}^0 \psi \rangle = \frac{B}{4\pi} \neq 0 \]

Now the CFs move in large circular orbits
agree with the experimental data for the commensurability condition: 

The field leads to commensurability oscillations, seen in Fig. 1(b). The CF commensurability oscillations are shown in the LLL are formed by the minority carriers, i.e., holes, which assumes that CFs are formed by the minority carriers in the LLL (hatched parts of the broadened level). 

On the other hand, if we assume that, for \( \nu = \frac{4}{9} \), the density of the minority carriers in the LLL \( \nu \) equals the minority density, then the expected positions for the density equals \( \nu \) are shown in Fig. 1(b). The inset schematically shows the commensurability oscillations, which are the same as before because the minority carrier density equals the minority density, then the expected positions for the density equals \( \nu \) are shown in Fig. 1(b).

Having established a possible explanation for the asymmetry of the CF commensurability minima is not unique to 2DESs. It persists in 2DHSs whose data are well captured by the minority density model. In the remainder of the Letter. We emphasize that the field leads to commensurability oscillations, seen in Fig. 1(b). The CF commensurability oscillations are shown in the LLL are formed by the minority carriers, i.e., holes, which assumes that CFs are formed by the minority carriers in the LLL (hatched parts of the broadened level).
Mapping Jain’s sequences

\[ \nu = \frac{n}{2n + 1} \quad \rightarrow \quad \nu_{\text{CF}} = n \]

\[ \nu = \frac{n + 1}{2n + 1} \quad \rightarrow \quad \nu_{\text{CF}} = n + 1 \]
Mapping Jain’s sequences

\[ \nu = \frac{n}{2n + 1} \]

\[ \nu_{CF} = n + \frac{1}{2} \]
Mapping Jain’s sequences

\[ \nu = \frac{n}{2n + 1} \]

\[ \nu_{\text{CF}} = n + \frac{1}{2} \]

CFs form an IQH state at half-integer filling factor: must be a Dirac fermion
IQHE in graphene

\[ \sigma_{xy} = \left( n + \frac{1}{2} \right) \frac{e^2}{2\pi\hbar} \]

**Figure 4 | QHE for massless Dirac fermions.** Hall conductivity \( \sigma_{xy} \) and longitudinal resistivity \( \rho_{xx} \) of graphene as a function of their concentration at \( B = 14 \) T and \( T = 4 \) K. \( \sigma_{xy} = (4e^2/h) \nu \) is calculated from the measured...
(Particle-hole)²

\[\Theta \rightarrow \Theta^2 = \pm 1\]
On a single Landau level

1 2 3 ... M
On a single Landau level

\[ \Theta^2 = (-1)^{M(M-1)/2} \]
On a single Landau level

\[ \Theta^2 = (-1)^{M(M-1)/2} \]

\[ M = 2N_{CF} \]

\[ \Theta^2 = (-1)^{N_{CF}} \]
On a single Landau level

\[ \Theta^2 = (-1)^{M(M-1)/2} \]

\[ M = 2N_{\text{CF}} \]

Dirac CF: \[ \psi \rightarrow (-i\sigma_2)^2 \psi = -\psi \]

Geraedts, Zaletel, Mong, Metlitski, Vishwanath, Motrunich; Levin, Son
Consequences of PH symmetry

\[ j = \sigma_{xx} E + \sigma_{xy} E \times \hat{z} + \alpha_{xx} \nabla T + \alpha_{xy} \nabla T \times \hat{z} \]

- Conductivities
- Thermoelectric coefficients

- At exact half filling, in the presence of particle-hole symmetric disorders

\[ \sigma_{xy} = \frac{e^2}{2h} \quad \alpha_{xx} = 0 \]

HLR

\[ \rho_{xy} = \frac{2h}{e^2} \]

Potter, Serbyn, Vishwanath 2015
A new gapped state

• The composite fermions can form Cooper pairs
• Simplest pairing does not break particle-hole symmetry
  \[ \langle \epsilon^{\alpha\beta} \psi_\alpha \psi_\beta \rangle \neq 0 \]
• A new gapped state: PH-Pfaffian state
Consequences of Dirac CF

Suppression of Friedel oscillations in correlations of particle-hole symmetric observables

$$\hat{O} = (\rho - \rho_0) \nabla^2 \rho$$

Geraedts, Zaletel, Mong, Metlitsky, Vishwanath, Montrunich, 2015

Direct proof of Berry phase $\pi$ of the composite fermion
A window to duality

- Fermionic particle-vortex duality can be derived of a more “elementary” fermion-boson duality Karch, Tong; Seiberg, Senthil, Wang, Witten
  - small N version of duality between CS theories, tested at large N
- New dualities can be obtained
  - Example: Nf=2 QED3 is self-dual Cenke Xu
The elementary duality

\[ \mathcal{L} = L[\psi, A] - \frac{1}{2} \frac{1}{4\pi} AdA \]

\[ \mathcal{L} = L[\phi, a] + \frac{1}{4\pi} ada + \frac{1}{2\pi} Ada \]
Comments on duality

• One side of duality is a theory of a single 2-component Dirac fermion coupled to U(1) gauge field

• It is usually thought QED3 with Nf fermions is unstable with respect to SSB for Nf<Nf*. Nf* ~ 6 in Schwinger-Dyson

• One lattice simulation (Karthik, Narayanan 2015) indicates Nf* < 2.

• The stability of theory with Nf=1 is not required for fractional quantum Hall effect (finite density)
Conclusion

• The low-energy quasiparticle of half-filled Landau level is completely different from the electron

• Symmetries allowed to guess the form of the low-energy effective theory

• Hints on new field-theoretic dualities in 2+1 D
References

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