Emergence of hydrodynamic behavior in rapidly expanding fluids

JORGE NORONHA

University of São Paulo

Canterbury Tales of Hot QFT's in the LHC Era, Oxford, 2017

OUTLINE

I) Motivation

II) Exact solutions of the (nonlinear) Boltzmann equation

III) Hydrodynamic attractor behavior of rapidly expanding fluids (Bjorken, Gubser flows)

IV) A different way to handle 2^{nd} order viscous corrections

V) Conclusions

The ubiquitousness of fluid dynamics

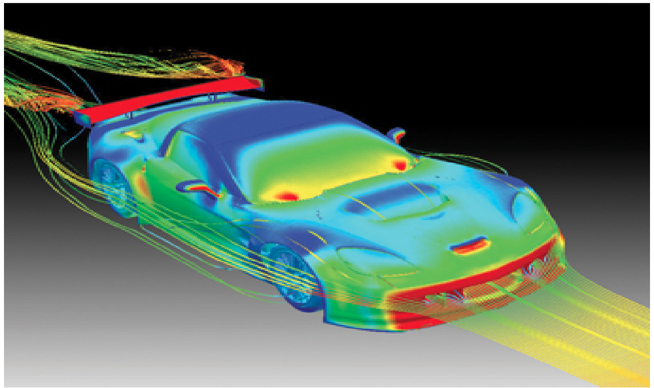
Based on conservations laws + large separation of length scales

Separation of scales $\rightarrow$ macroscopic: L microscopic: ℓ

Knudsen number expansion:

$$K_N \sim \frac{\ell}{L} \ll 1$$

→ **FLUID**



L ~ 1 m $\ell \sim 10^{-7} m$

Macroscopic: Gradient of velocity field

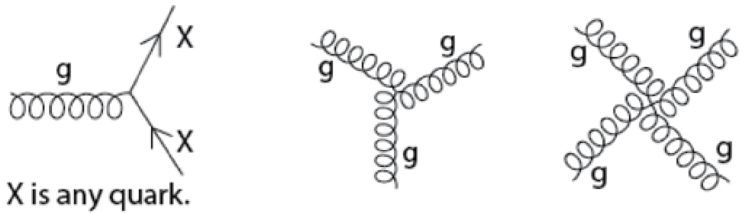
$$\partial v \sim 1/L$$

Example of microscopic scale (gas):

$$\ell \sim 1/(n\sigma)$$ mean free path

Quark-gluon plasma: the primordial liquid

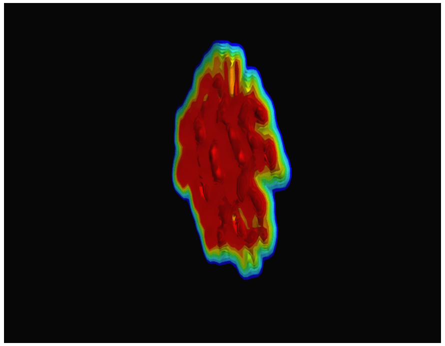
QCD = confinement + asymptotic freedom



gluon self-interactions

?

Quark-Gluon Plasma



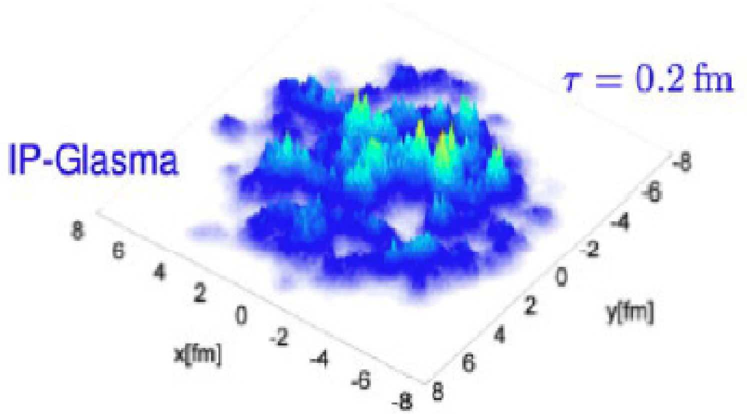
Ex: Schenke, Jeon, Gale, PRL 2011

QGP perfect fluidity: $\frac{\eta}{s} \sim \frac{1}{4\pi} \rightarrow$ emergent property of QCD??

Is this present even in elementary proton+proton collisions ???

Fluid dynamics at length scales of the size of a proton ???

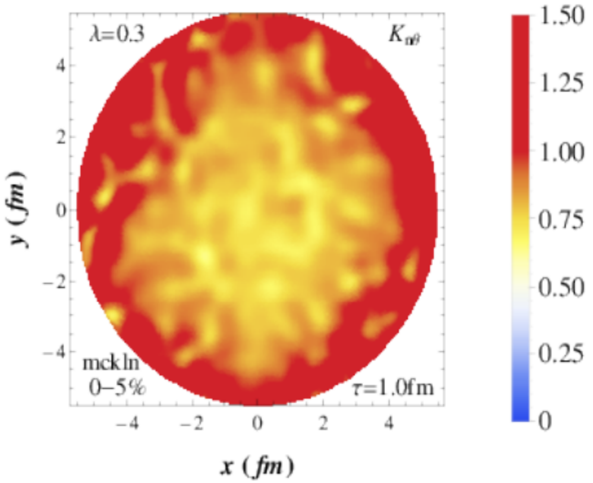
QGP initial condition



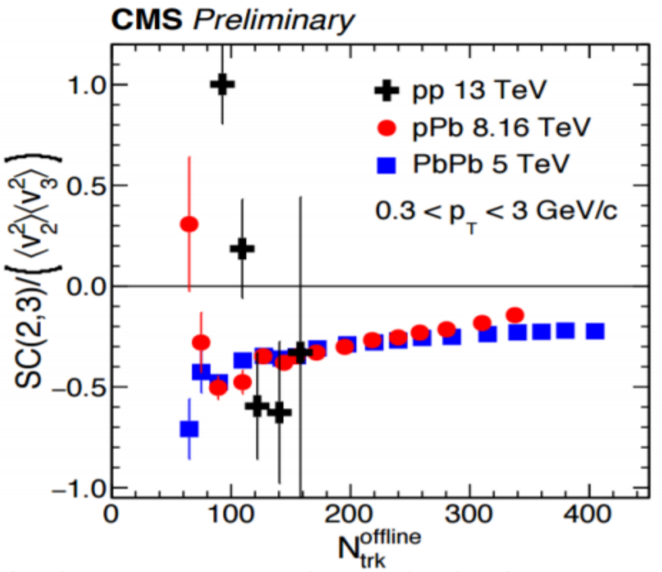
Schenke, Tribedy, Venugopalan, PRL 2012

J. Noronha-Hostler, JN, M. Gyulassy, PRC 2016

Knudsen number event-by-event



Hydrodynamic behavior in small systems????



macro scale $\partial\varepsilon/\varepsilon_0 \sim \Lambda_{QCD}$

microscopic scale ????

There is no reason to believe that Kn has to be small in this case.

Hydrodynamics at its edge ...

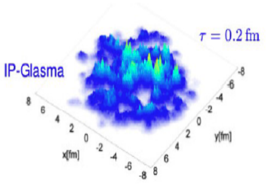
What happens to a many-body system when $K_N \sim 1$???

- This must be figured out to properly interpret collectivity in pp.

- Causality + divergence of gradient expansion → resummation

Heller, Spalinski PRL 2015

- Hydrodynamic attractor, Kn resummation, in complicated flow profiles?



- Interplay between hydro and non-hydro modes when Kn ~ 1?

II – Exact solutions of the Boltzmann equation

(full nonlinear collision kernel)

Let us first focus solely on the dynamics of non-hydro modes in rapidly expanding systems …

Simplest example: kinetic theory in an expanding Universe

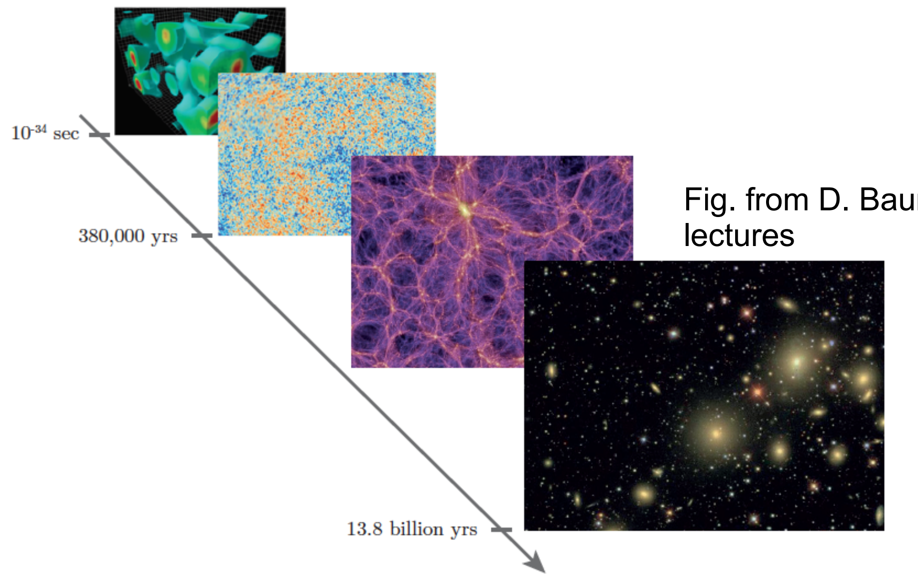


Fig. from D. Baumann's lectures

Symmetries are so powerful that only non-hydro modes have nontrivial dynamics

Friedmann-Robertson-Lemaitre-Walker (FRLW) spacetime

Maximally (spatially) symmetric spacetime

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - K\,r^2} + r^2\,d\Omega^2\,.\right]$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\,T_{\mu\nu}$$

Einstein's equations

K ~ 0 (spatially flat –> our universe)

K = 1, -1

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon - \frac{K}{a^2}$$

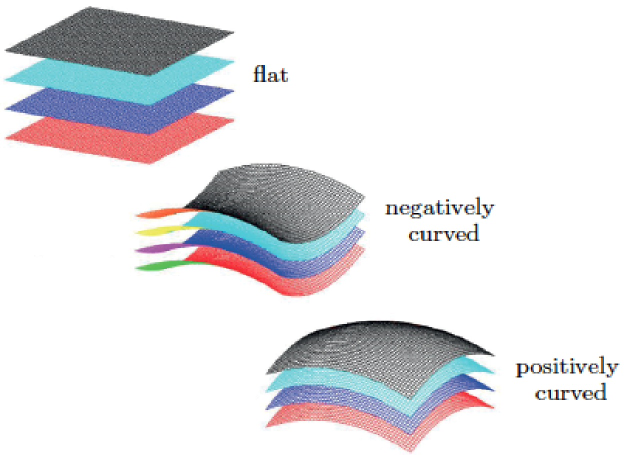
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\varepsilon + 3P)$$

$$\mathcal{E} \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^{0} & \text{vacuum} \end{cases}$$

FLRW spacetime

Spatial isotropy + homogeneity

Universe



Isotropic and homogeneous expanding FLRW spacetime

(zero spatial curvature)

Ex: metric

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

Determined from Einstein's equations

Friedmann-Lemaitre-Robertson-Walker spacetime

We consider an isotropic and homogeneous expanding FRW spacetime

(zero spatial curvature)

metric $$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

Cosmological scale factor (e.g., radiation) $a(t) \sim t^{1/2}$

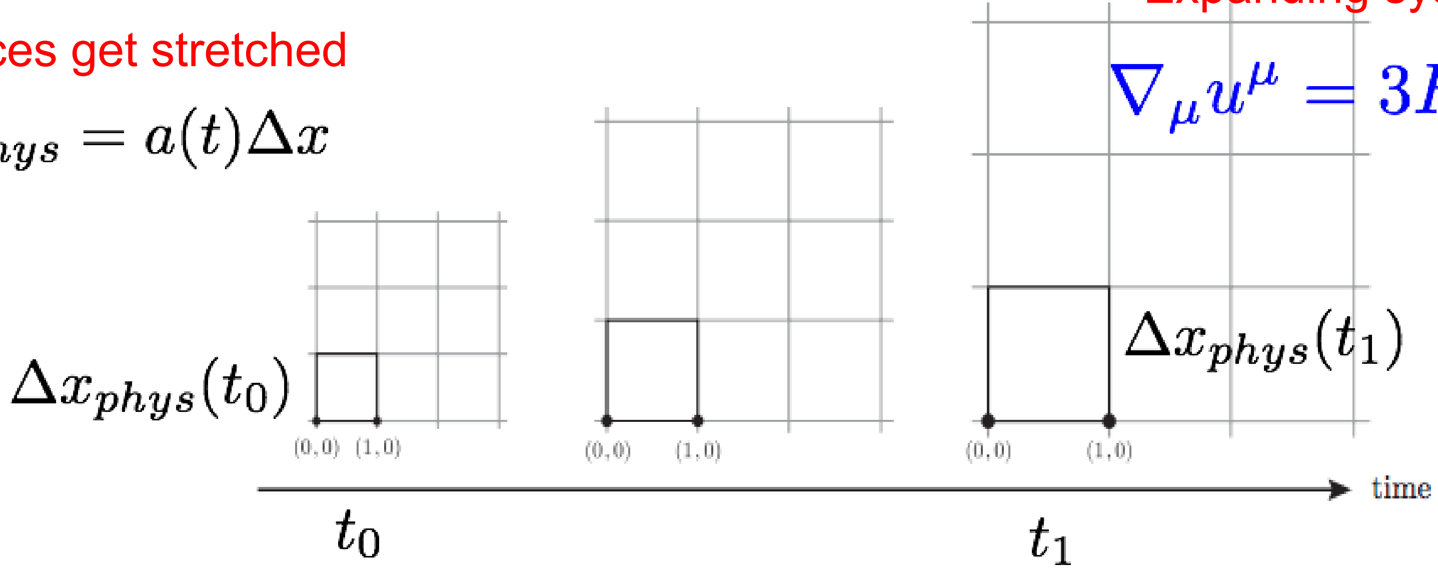
Hubble parameter $H = \dot{a}/a > 0$

Expanding system!

$$\nabla_\mu u^\mu = 3H \neq 0$$

Distances get stretched

$$\Delta x_{phys} = a(t)\Delta x$$



General relativistic Boltzmann equation

- Dilute gases display complex non-equilibrium dynamics.

- The Boltzmann equation has been instrumental in physics and mathematics (e.g., 2010 Fields Medal).

General Relativistic Boltzmann equation

$$k^\mu (u_\mu D + \nabla_\mu) f(x,k) + k_\lambda k^\mu \Gamma^\lambda_{\mu i} \frac{\partial f(x,k)}{\partial k_i} = \mathcal{C}[f],$$

Space-time variation

on-shell

Collision term

- It describes how the particle distribution function $f_k(x,k)$ varies in time and space due to the effects of collisions (and external fields).

Boltzmann Equation in FLRW spacetime

Simplest toy model for an out-of-equilibrium Universe:

- Massless particles, classical statistics, constant cross section: σ

- Weakly coupled QCD at high T is much more complicated than this

- However, I will solve also the case of a massless $\lambda\phi^4$ field ...

- Here $\theta = \nabla_\mu u^\mu \neq 0$ but $\dot{u}^\mu, \sigma_{\mu\nu}, \omega_{\mu\nu} \to 0$

$k^0 = k/a(t)$ with $k = |\boldsymbol{k}|$

$\int_k \equiv \int d^3k / \left[(2\pi)^3 \sqrt{-g}\, k^0\right]$

Our Boltzmann equation:

$$k^0 \partial_t f_k = \frac{(2\pi)^5}{2} \sqrt{-g}\, \sigma \int_{k'pp'} s\, \delta^4(k+k'-p-p')(f_p f_{p'} - f_k f_{k'})$$

$s = (k^\mu + k'^\mu)(k_\mu + k'_\mu)$

This equation includes general relativistic effects + full nonlinear collision dynamics

We want to find solutions for the distribution function

Given an initial condition: $f(t_0, k)$ and $n(t_0), \varepsilon(t_0)$

How does one solve this type of nonlinear integro-differential equation?

The moments method

- Originally introduced by Grad (1949) and used by Israel and Stewart (1979) in the relativistic regime.

- Applications in HIC: see DNMR, Phys. Rev. D 85 (2012) 114047

- Used more recently in Phys. Rev. Lett. 116 (2016) 2, 022301

The idea is simple

Instead of solving for the distribution function itself directly, one uses the Boltzmann eq. to find exact equations of motion for the moments of the distribution function.

Ex: The particle density $n(t) = \int_k (u.k) f_k(t)$ is a scalar moment

with equation $\partial_t n + 3n\, H(t) = 0$

Ex: The energy density $\varepsilon(t) = \int_k (u.k)^2 f_k(t)$ is a scalar moment

with equation $\partial_t \varepsilon + 4\varepsilon\, H(t) = 0$

Clearly, due to the symmetries, here only scalar moments can be nonzero.

Thus, if we can find the time dependence of the scalar moments

$$\rho_m(t) = \int_{\boldsymbol{k}} (u\cdot k)^{m+1} f_k(t) \qquad (m \in \mathbb{N}_0)$$

$\rho_0 = n,\ \rho_1 = \varepsilon$

via solving their exact equations of motion, one should be able to recover $f_k(t)$

Defining the scaled time: $\hat{t} = t/\ell_0$ where $\ell_0 = 1/(\sigma\, n(t_0))$

(constant mean free path)

And the normalized moments $M_m(\hat{t}) = \dfrac{\rho_m(\hat{t})}{\rho_m^{eq}(\hat{t})}$ which obey the **exact** set of eqs:

See Bazow, Denicol, Heinz, Martinez, JN, PRL 2016, arXiv:1507.07834 [hep-ph]

ALL THE NONLINEAR BOLTZMANN DYNAMICS IS ENCODED HERE

$$a^3(\hat{t})\,\frac{\partial}{\partial \hat{t}} M_m(\hat{t}) + M_m(\hat{t}) = \frac{1}{m+1}\sum_{j=0}^{m} M_j(\hat{t}) M_{m-j}(\hat{t})$$

GR effect

Simple recursive nonlinearity

Conservation laws require $M_0 = M_1 = 1$

“Fourier” transforming the Boltzmann equation

G. Denicol and JN, to appear

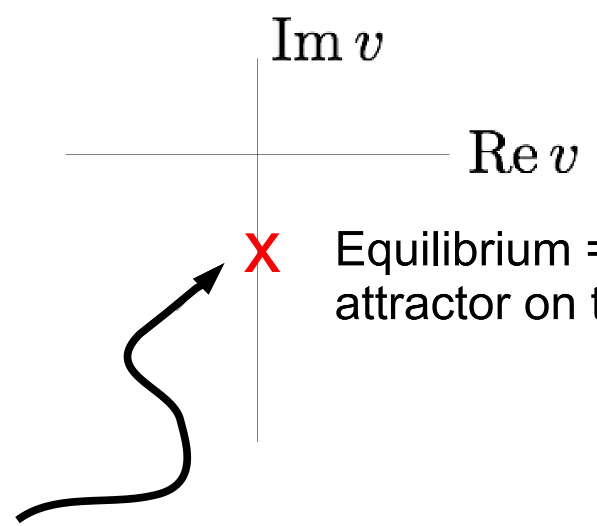
If the moments are what we want, it makes sense to define the generating function

$$\Phi(t,v) \equiv \int_k e^{iv(u\cdot k)}(u\cdot k) f_k(t) \Longrightarrow \rho_n(t) = \left.\frac{\partial^n \Phi(t,v)}{i^n \partial v^n}\right|_{v=0}$$

where v is a <u>complex number</u>

Thermalization → development of a pole at $v = -1/T$

$$\Phi(t,v) \to \Phi^{eq}(v) = \frac{\rho_0}{(1 - iTv)^3}$$



Equilibrium = global attractor on the plane

Thermalization process is mapped onto how the analytical structure of this function changes with time.

"Fourier" transforming the Boltzmann equation

G. Denicol and JN, to appear

This way to see the thermalization process is valid for any type of cross section (does not depend on the mass, quantum statistics changes the pole)

It is easy to show that this →

$$\partial_t(a^3\Phi) = \frac{\sigma}{2}\int_{kk'} f_k f_{k'} s(2\pi)^5 \int_{pp'} \delta^{(4)}(k+k'-p-p')e^{iv(u\cdot p)}$$
$$-\frac{\sigma}{2}\int_{kk'} f_k f_{k'} e^{iv(u\cdot k)} s(2\pi)^5 \int_{pp'} \delta^{(4)}(k+k'-p-p')$$

Becomes this:

$$\partial_t(a^3\Phi) + \sigma\rho_0\Phi = \sigma\left(\int_0^1 d\alpha\,\Phi(t,\alpha v)\right)^2 - \sigma\left(\int_0^1 d\alpha\,(2\alpha-1)\Phi(t,\alpha v)\right)^2$$

Taking derivatives w.r.t. v one can easily find the equation for the moments

Full Analytical Solution

Redefining time

Using the moments equations in this form

$$\tau = \int_{\hat{t}_0}^{\hat{t}} dt'/a^3(t')$$

$$\partial_\tau M_m(\tau) + M_m(\tau) = \frac{1}{m+1}\sum_{j=0}^{m} M_j(\tau) M_{m-j}(\tau).$$

One can show that

$$M_m(\tau) = \mathcal{K}(\tau)^{m-1}\left[m - (m-1)\mathcal{K}(\tau)\right] \quad (m \geq 0)$$

is an analytical solution of the moments equations !

$$\mathcal{K}(\tau) = 1 - \frac{e^{-\tau/6}}{4} \sim e^{-1/K_N}$$

Non-perturbative in $K_N = \ell(t) H(t)$

Full Analytical Solution

Analytical solution of the Boltzmann equation for an expanding gas

BDHMN, PRL (2016) arXiv:1507.07834 [hep-ph]

λ = fugacity

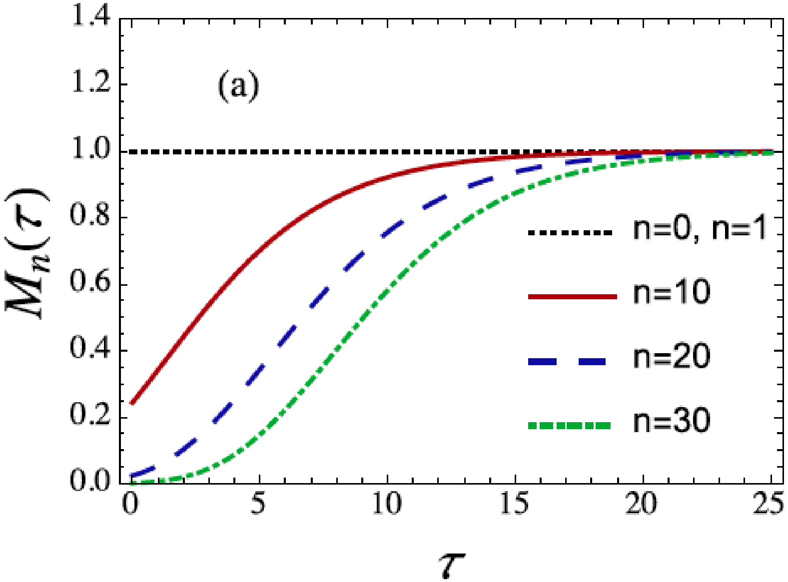
$$f_k(\tau) = \lambda \exp\left(-\frac{u\cdot k}{\mathcal{K}(\tau)T(\tau)}\right) \times \left[\frac{4\mathcal{K}(\tau)-3}{\mathcal{K}^4(\tau)} + \frac{u\cdot k}{T(\tau)}\left(\frac{1-\mathcal{K}(\tau)}{\mathcal{K}^5(\tau)}\right)\right]$$

Initial condition $f_k(0) = \frac{256}{243}\,(k/T_0)\,\lambda\,\exp[-4k/(3T_0)] > 0$

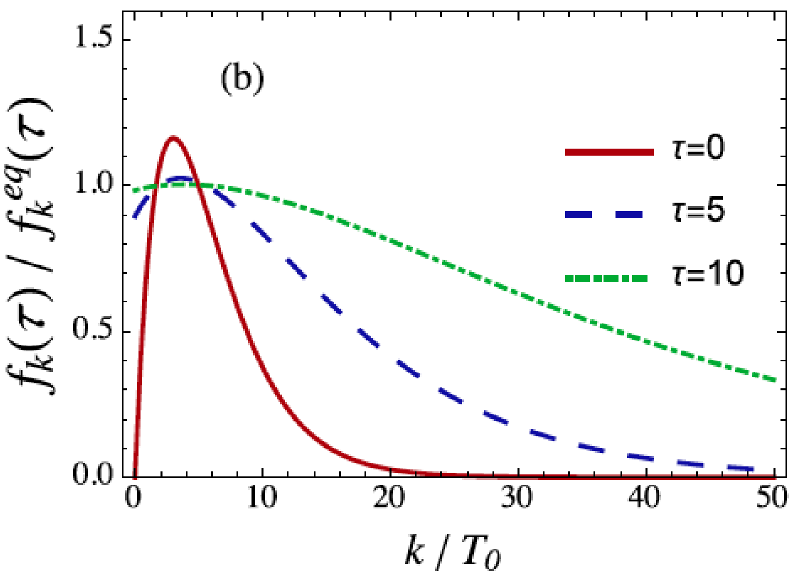
See BDHMN arXiv:1607.05245 for many more details about this and other solutions

Full Analytical Solution

Time evolution



Momentum dependence



For radiation dominated universe higher order moments will certainly not erase the info about initial conditions → system never equilibrates due to expansion.

The approach to equilibrium here depends on the occupancy of each moment.

Full Analytical Solution – Generating function

For the analytical solution

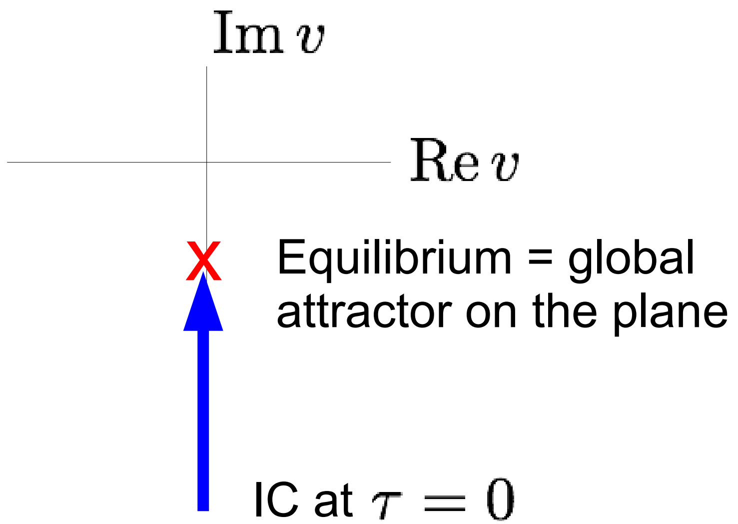
$$M_m(\tau) = \mathcal{K}(\tau)^{m-1}\left[m - (m-1)\mathcal{K}(\tau)\right] \quad (m \geq 0)$$

one finds

$$\Phi(t,v) = \rho_0 \frac{\left[1 + iTv(3 - 4\mathcal{K}(\tau))\right]}{\left[1 - iTv\mathcal{K}(\tau)\right]^4}$$

Time dependent pole at

$$v = -i/(T\mathcal{K}(\tau))$$



- Thermalization process of different initial conditions correspond to other trajectories on the plane.

- Thermalization vs. non-hydro modes???? Universality?

Solving conformal kinetic theory

G. Denicol and JN, to appear

- Assume tree level ~ $\lambda\phi^4$, massless, classical statistics

FLRW conformal gauge

$$ds^2 = a^2(\tau)\left(d\tau^2 - dx^2 - dy^2 - dz^2\right)$$

Weyl transformation

$$g_{\mu\nu}(x) \rightarrow e^{-2\Omega(x)} g_{\mu\nu}(x)$$

Weyl symmetry emerges when m=0 (BRSSS, 2007)

$$p^\mu \partial_\mu f + \Gamma^\lambda_{\mu i} p_\lambda p^\mu \frac{\partial f}{\partial p_i} - \mathcal{C}[f] = 0 \Longrightarrow e^{2\Omega}\left(p^\mu \partial_\mu f + \Gamma^\lambda_{\mu i} p_\lambda p^\mu \frac{\partial f}{\partial p_i} - \mathcal{C}[f]\right) = 0$$

One can solve the simple "static" dynamics and can Weyl back to the expanding state we desire at the end

Solving conformal kinetic theory

G. Denicol and JN, to appear

Boltzmann equation

$$k\partial_\tau f_k = \frac{\lambda^2}{2}\int_{k'pp'}(2\pi)^5\delta^{(4)}\left(k_\mu + k'_\mu - p_\mu - p'_\mu\right)\left(f_p f_{p'} - f_k f_{k'}\right)$$

General solution: $f_k(\tau) = e^\alpha e^{-k}\sum_{n=0}^{\infty} c_n(\tau) L_n^{(2)}(k)$ See arXiv:1607.05245

Laguerre polynomials

Moments

Mode-by-mode coupling: $f_k(\tau) \to \{c_n(\tau)\}$ $c_n \sim \int_k k\, L_n^{(2)}(k)$

Equilibrium: $c_0 = 1\,, c_{n>1} = 0$

Solving conformal kinetic theory

G. Denicol and JN, to appear

While for the constant cross section case dynamics was simple

$$\frac{dc_n}{d\tau} + c_n = \frac{1}{n+1}\sum_{m=0}^{n} c_{n-m}c_m$$

BDHMN arXiv:1607.05245

For the case of scalar field the exact equation for the moments is

$$\frac{dc_n}{d\tau} = \frac{1}{8}\frac{n!}{(n+2)!}\sum_{m=0}^{n-2}\frac{(n-m)!}{(n-m-2)!}\left[\sum_{q=0}^{m} c_{m-q}c_q + (m+1)(c_{m+1}-c_m)\left(\sum_{p=0}^{\infty} c_p\right)\right]$$

"Debye" mass squared

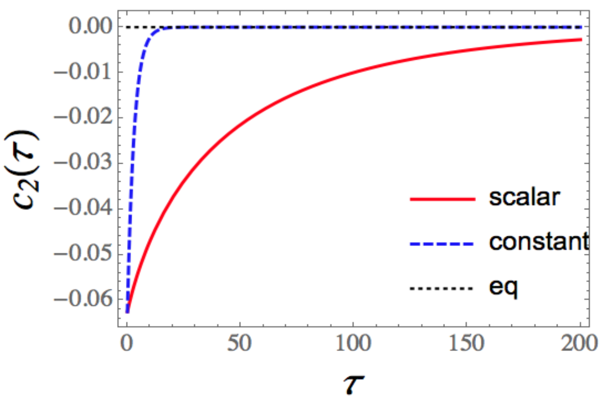
$$\int_k f_k(\tau) = \frac{\rho_0}{2}\sum_{n=0}^{\infty} c_n(\tau)$$

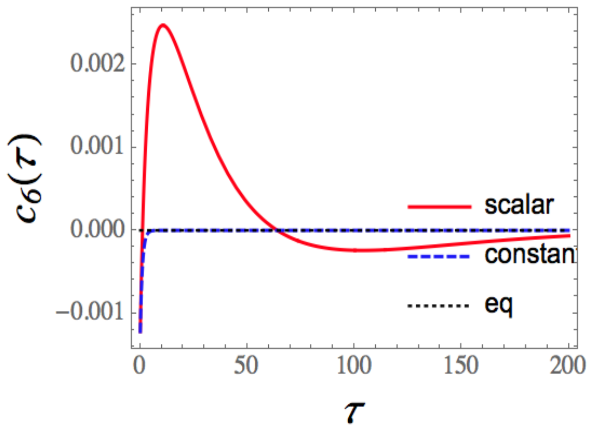
Resummation!!!!

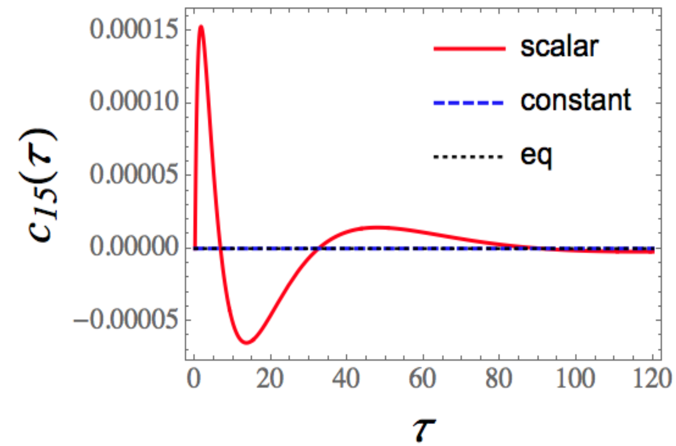
Solving conformal kinetic theory

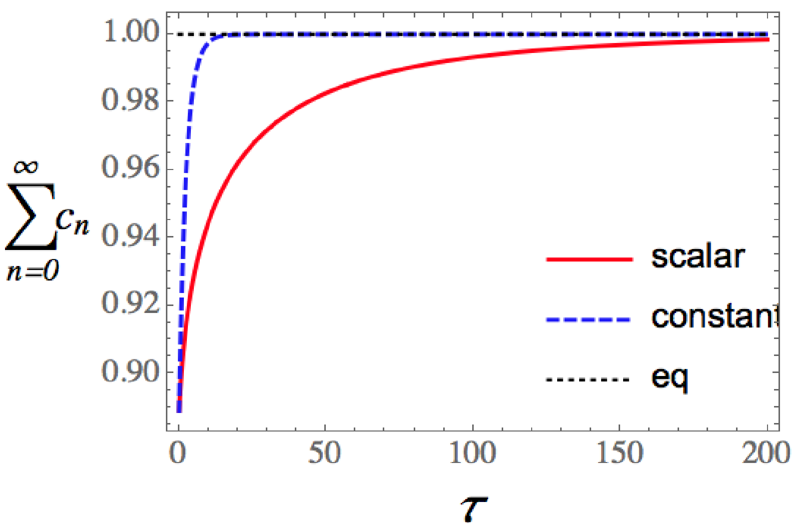
IC: $c_n(0) = \dfrac{(1-n)}{4^n}$

G. Denicol and JN, to appear









"Thermalization time" increases significantly for the scalar field case

- It has been a challenge to generalize our approach to anisotropic flows (e.g. Bjorken)

III – Hydrodynamic behavior of rapidly expanding fluids

Hydrodynamics as a series expansion

D. Hilbert, 1912

Scaled Boltzmann equation

$$p^\mu \partial_\mu f = \frac{\mathcal{C}[f,f]}{K_N}$$

Knudsen number

$$K_N \sim \frac{\ell}{L}$$

Formal solution via a series

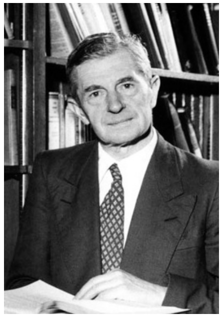
Hilbert Series

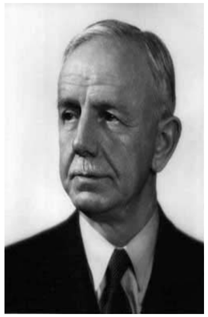
$$f(x^\mu, p^\mu, K_N) = \sum_{n=0}^{\infty} (K_N)^n f_n(x^\mu, p^\mu)$$

$f_0 \sim e^{-p\cdot u/T}$ $\rightarrow$ derives (does not assume) ideal fluid dynamics

No statement about existence of series is made

Hydrodynamics as a gradient expansion


S. Chapman


D. Enskog

Assume deviations from local equilibrium

Chapman-Enskog series

$$f = f_{eq}\left(1 + \sum_{n=1}^{\infty}(K_N)^n f_n(x^\mu, p^\mu)\right)$$

Convergence???

- At 0th order in Kn → Euler equations, at 1st order → Navier-Stokes

- Very hard to carry out procedure to higher orders

- Solution of Boltzmann completely defined in terms of local hydrodynamic fields:

$$T,\ \mu,\ u^\mu$$

Hydrodynamics as a gradient expansion

- Chapman-Enskog solution, though systematic, is highly contrived

- Kn itself depends on the flow properties

- Problems with instabilities and acausality in the relativistic domain

- Procedure cannot describe all possible solutions of Boltzmann …

Ex: Homogeneous relaxation

$$\partial_t f = \mathcal{C}[f, f]$$ → Dynamics contains only non-hydro modes

Non-hydro modes → defined by nonzero eigenvalues of collision operator

$$\lim_{k \to 0} \omega_{nh}(k) \neq 0$$

Chapman-Enskog expansion: Non-relativistic regime

Santos, Brey, Dufty, 1571, vol 51 PRL (1986)

Newtonian fluid: $P_{xy} = -\eta_0 \partial u_x / \partial y$

Uniform shear flow

$$\partial u_i / \partial x_j = \dot{\gamma} \delta_{ix} \delta_{jy}$$

$$\nabla n = \nabla T = \nabla \dot{\gamma} = \mathbf{0}$$

Pressure tensor

$$P_{xy} = -\sum_{k=0}^{\infty} \eta_k \left(\partial u_x / \partial y \right)^{2k+1}$$

BGK Boltzmann

$$(\partial_t + \mathbf{v} \cdot \nabla) f = -\nu (f - f_0)$$

Series converges if

$$\nu \sim const$$

(Maxwell molecules)

DIVERGES $\nu \sim p^{\alpha}, \quad \alpha = (n-4)/2n \ (r^{-n} \text{ potential})$

(e.g., hard spheres, n=2)

Hydrodynamics from the method of moments

Harold Grad, 1948

$$p^\mu \partial_\mu f = \mathcal{C}[f,f]$$

Define infinite set of moments such as

$$\varepsilon = \int_p (u\cdot p)^2 f \qquad\qquad T^{\mu\nu} = \int_p p^\mu p^\nu f$$

energy density — Energy-momentum tensor

- Use Boltzmann equation to find exact equations for the moments
- Reconstruct solution of Boltzmann using a complete set of moments
- In the relativistic domain, 14 moments truncation → Israel-Stewart eqs.

PROS:

- Moments method played a major role in the derivation of the hydrodynamic equations for the QGP – linear stability and causality!!!

- Used in many approaches: Israel-Stewart, DNMR, (v)AHYDRO ...

- Describe interactions between hydro and non-hydro modes

- Can provide consistent (and convergent) solution of Boltzmann

CONS:

- Absence of a small expansion parameter

- Very hard to derive general equations in practice (full collision term), unless flow too simple (such as FLRW)

Divergence of the Chapman-Enskog (CE) expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Heavy ion collisions → perfect arena to study CE expansion

Bjorken expanding (conformal, transversely homogeneous) fluid:

$$x^\mu = (\tau, x, y, \eta)$$

Spacetime rapidity

$$\eta = \tanh^{-1}(z/t)$$

Milne propertime

$$\tau = \sqrt{t^2 - z^2}$$

Flow velocity

$$u^\mu = (1, 0, 0, 0)$$

Boltzmann equation

$$\partial_\tau f = \mathcal{C}[f, f]$$

By symmetry:

- $f \to f(\tau, k_0, k_\eta)$

- Any gradient $\sim \dfrac{1}{\tau}$

Divergence of the Chapman-Enskog (CE) expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Relaxation time τ_R approximation

$$\frac{\partial_\tau T}{T} + \frac{1}{3\tau} - \frac{\pi}{12\tau} = 0,$$
$$\partial_\tau f_{\mathbf{k}} = -\frac{f_{\mathbf{k}} - f_{\text{eq}}}{\tau_R},$$

Knudsen number

$$K_N = \frac{\tau_R}{\tau}$$

constant τ_R

Shear stress tensor

$$\pi^\eta_\eta = \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau} k_0 \left[\frac{1}{3} - \left(\frac{k_\eta}{k_0\tau}\right)^2\right] f_{\mathbf{k}}$$

Landau matching condition

$$\varepsilon = \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau}\, k_0 f_{\mathbf{k}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau}\, k_0 f_{\text{eq}}$$

$$f_{\text{eq}} = \exp\left(-k_0/T\right)$$

Massless particles, constant relaxation time

Divergence of the Chapman-Enskog (CE) expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Method of moments: $$\rho_{n,\ell} = \int \frac{d^3\mathbf{k}}{(2\pi)^3\tau} \left(k^0\right)^n \left(\frac{k_\eta}{k^0\tau}\right)^{2\ell} f_{\mathbf{k}}$$

$$\partial_\tau M_{n,\ell} + \frac{1}{\tau_R} M_{n,\ell} + \frac{6\ell - n}{3\tau} M_{n,\ell} - \frac{n+3}{12\tau} \underline{M_{1,1}\left(1 + M_{n,\ell}\right)} + \frac{1}{\tau}\frac{(n-2\ell)(1+2\ell)}{2\ell+3} M_{n,\ell+1} = -\frac{1}{\tau}\frac{4\ell(n+3)}{3(2\ell+3)}$$

Dimensionless moments $$M_{1,1} = -\pi/P$$

$$M_{n,\ell} \equiv \frac{\rho_{n,\ell} - \rho_{n,\ell}^{\mathrm{eq}}}{\rho_{n,\ell}^{\mathrm{eq}}}$$

Nonlinearity

Divergence of the Chapman-Enskog (CE) expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Solution of Boltzmann → reconstructed via $M_{n,\ell}(\tau)$

Chapman-Enskog series:

$$M_{n,\ell} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(n,\ell)}}{\hat{\tau}^p}$$

Knudsen number

$$K_N = \frac{\tau_R}{\tau} = \frac{1}{\hat{\tau}}$$

Exact recursive relation

$$\alpha_0^{(n,\ell)} = 0$$

$$\alpha_1^{(n,\ell)} = -\frac{4\ell\,(n+3)}{3\,(2\ell+3)}$$

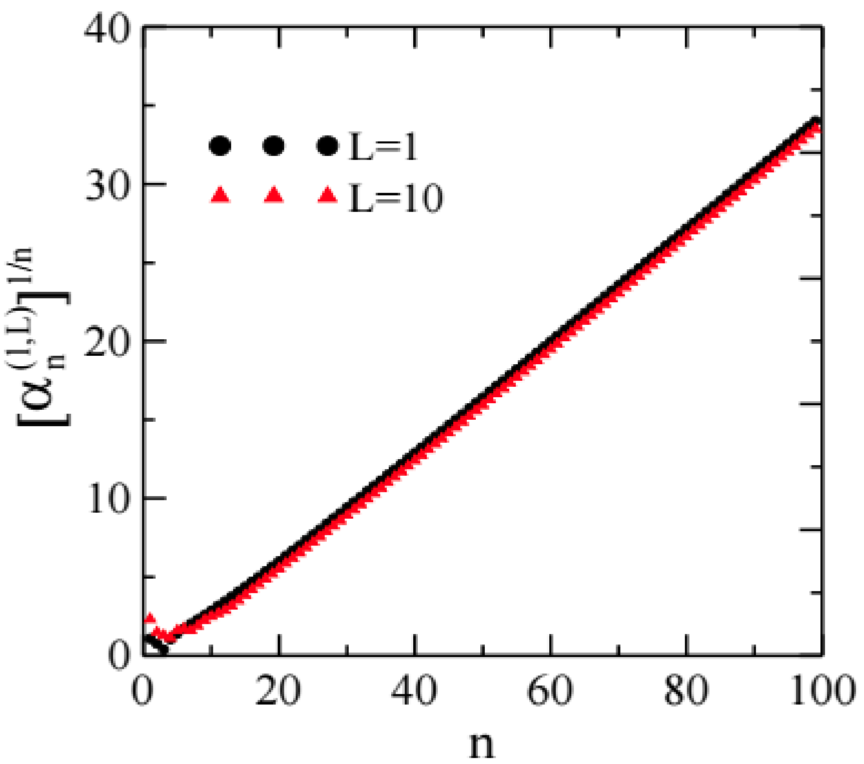
$$\alpha_{m+1}^{(n,\ell)} = -\frac{6\ell - n - 3m}{3}\alpha_m^{(n,\ell)} + \frac{n+3}{12}\alpha_m^{(1,1)} - \frac{(n-2\ell)\,(1+2\ell)}{2\ell+3}\alpha_m^{(n,\ell+1)} + \frac{n+3}{12}\sum_{p=0}^{m}\alpha_p^{(1,1)}\alpha_{m-p}^{(n,\ell)}.$$

Divergence of the Chapman-Enskog (CE) expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Chapman-Enskog series is clearly divergent !!!

$$\lim_{m \gg 1} \alpha_m^{(n,\ell)} \sim m!$$



See also Heller, Kurkela, Spalinski, arXiv:1609.04803

Divergence of the Chapman-Enskog (CE) expansion

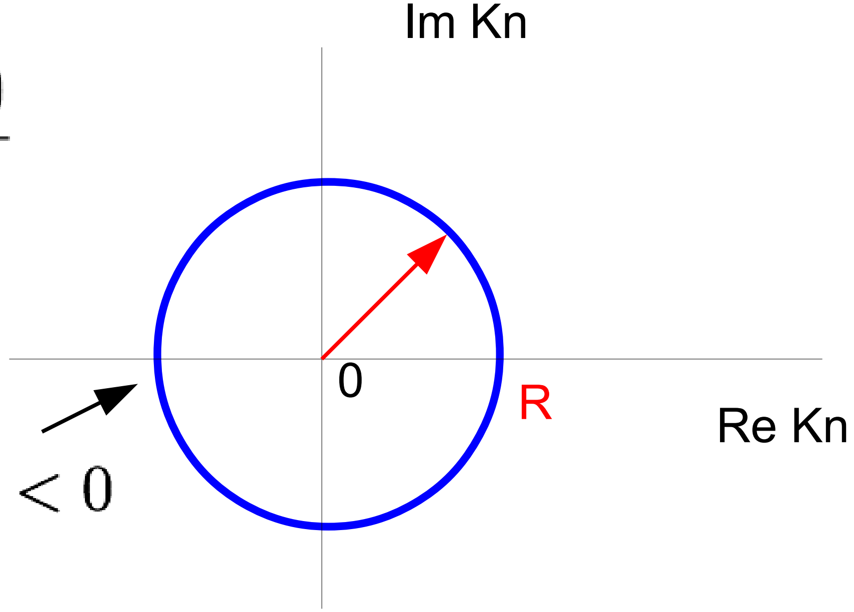
Simple argument (a la Dyson) to show that the series must diverge

If series converged around $K_N = \frac{\tau_R}{\tau} \rightarrow 0$, there would be a nonzero radius of convergence R

But for RTA $\partial_\tau f = -\frac{(f - f_{eq})}{\tau_R}$

Since $\tau > 0$

$K_N < 0 \rightarrow \tau_R < 0$



INSTABILITY !!!!!!

Generalized Chapman-Enskog expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Regularity of the system near initial condition: $\tau \to \tau_0$

+

And the 1st order nature of the ODE's for the moments

Show that at early times $M_{n,\ell}(\hat{\tau}) \sim e^{-(\hat{\tau}-\hat{\tau}_0)} \sim e^{-1/K_N}$

Essential singularity

Dynamics contains highly non-perturbative terms !!!!

Valid also for a nonlinear kernel

Generalized Chapman-Enskog expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

- These new terms cannot be captured by the usual Knudsen series

- They show that initial condition data is not easily “erased”

- They carry information about how non-hydro modes

We define a Generalized Chapman-Enskog (GCE) series

$$M_{n,\ell}\left(\hat{\tau}\right) = \sum_{p=0}^{\infty} \frac{\beta_p^{(n,\ell)}\left(\hat{\tau}\right)}{\hat{\tau}^p}$$

This introduces an expansion parameter in the moments method

Generalized Chapman-Enskog expansion

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

CE series: $\alpha_m^{(n,\ell)} \rightarrow$ obey algebraic relations

GCE series: $\beta_m^{(n,\ell)}(\hat{\tau}) \rightarrow$ obey differential equations

$$\partial_{\hat{\tau}} \beta_{m+1}^{(n,\ell)} + \beta_{m+1}^{(n,\ell)} = -\frac{4\ell\,(n+3)}{3\,(2\ell+3)}\delta_{m,0} - \frac{(n-2\ell)\,(1+2\ell)}{2\ell+3}\beta_m^{(n,\ell+1)} - \frac{6\ell-n-3m}{3}\beta_m^{(n,\ell)} + \frac{n+3}{12}\beta_m^{(1,1)} + \frac{n+3}{12}\sum_{p=0}^{m}\beta_{m-p}^{(1,1)}\beta_p^{(n,\ell)}.$$

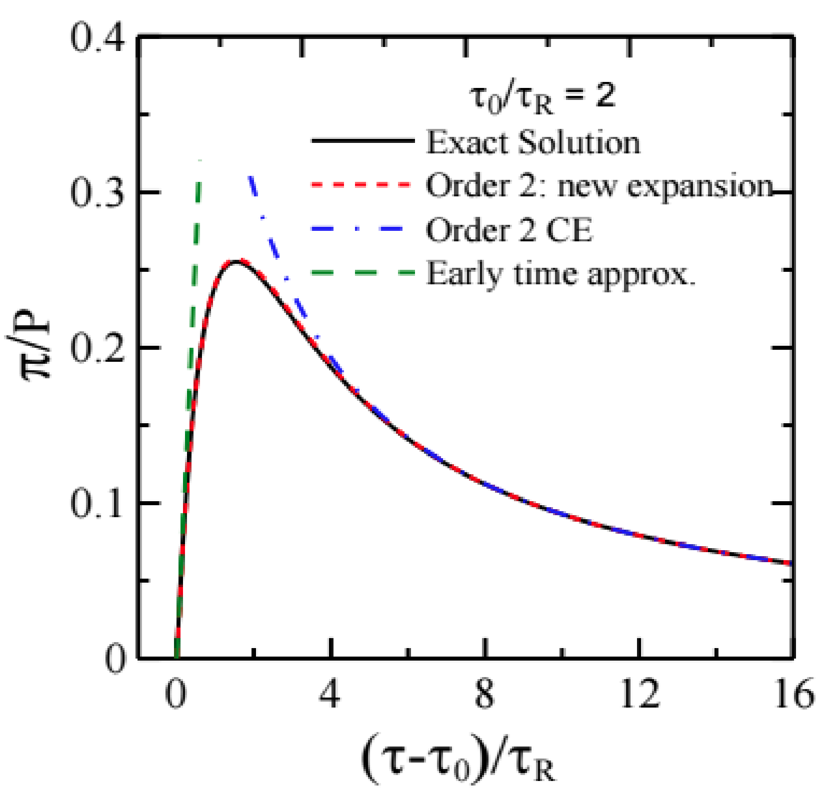
$$\partial_{\hat{\tau}} \beta_0^{(n,\ell)} + \beta_0^{(n,\ell)} = 0$$

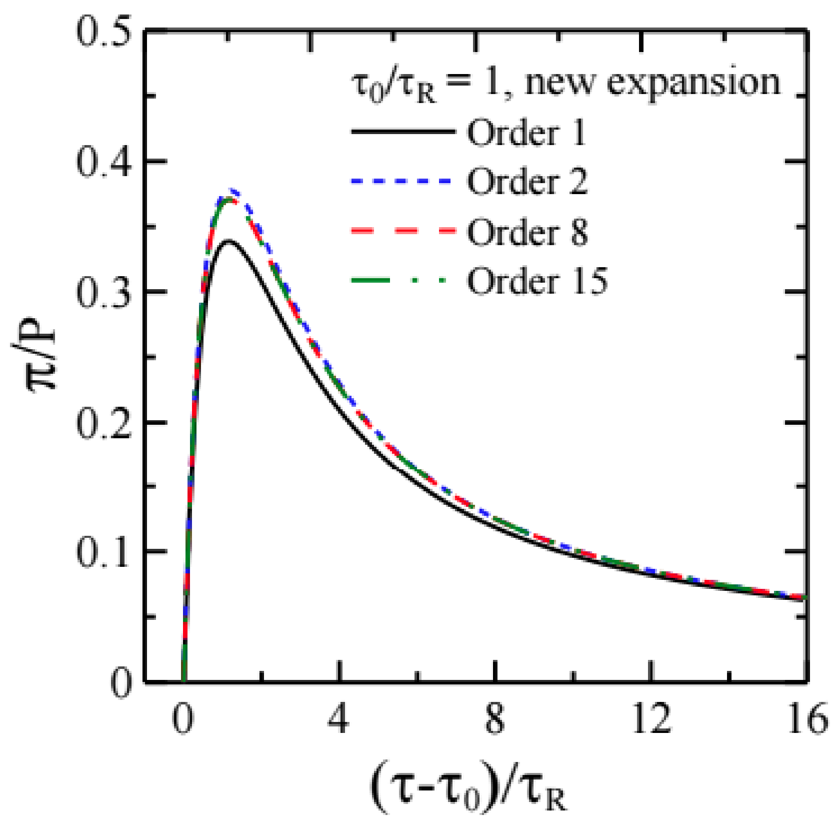
New series describes the whole time evolution since initial condition !!!

Generalized Chapman-Enskog expansion

New series describes the whole time evolution since initial condition !!!

Excellent agreement with full exact solution of Boltzmann already when truncated at 2nd order !!!





One expects that:

- First the higher order moments relax, effectively changing the equation for shear stress tensor

- RTA Boltzmann → Israel-Stewart-like equation → hydrodynamic attractor

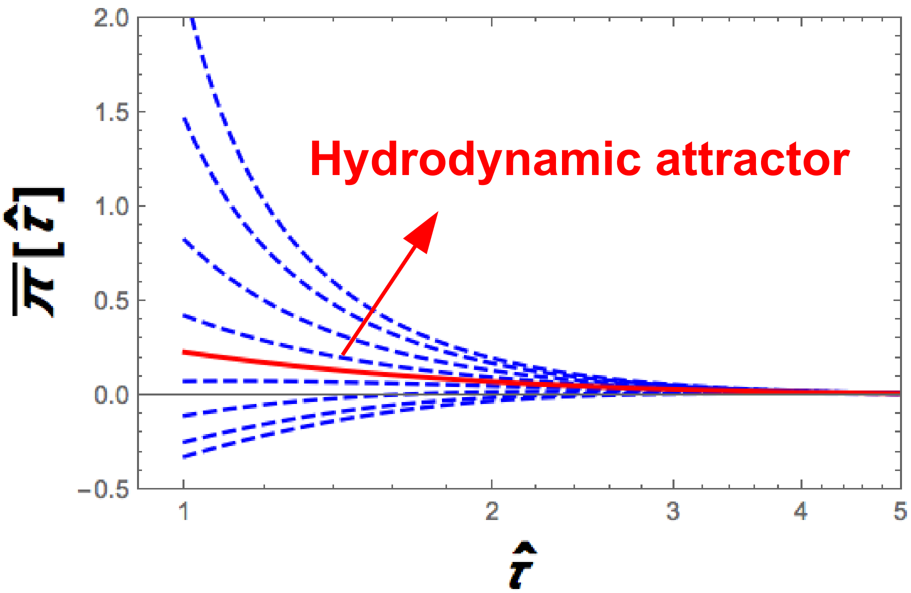
Israel-Stewart-like eqs

$$\begin{aligned} \frac{d\varepsilon}{d\tau} + (\varepsilon + P)\frac{1}{\tau} + \frac{\pi}{\tau} &= 0 \\ \tau_R \frac{d\pi}{d\tau} + \pi + \frac{4\tau_R}{3}\frac{\pi}{\tau} &= \frac{4}{3}\frac{\eta}{\tau}. \end{aligned}$$

$$\tau_R = b\frac{\eta}{\varepsilon + P}$$

$$\pi = \pi^\eta_\eta$$

$$\bar{\pi} \equiv \frac{\pi}{\varepsilon + P}$$



What happens in the case of more complex flows?

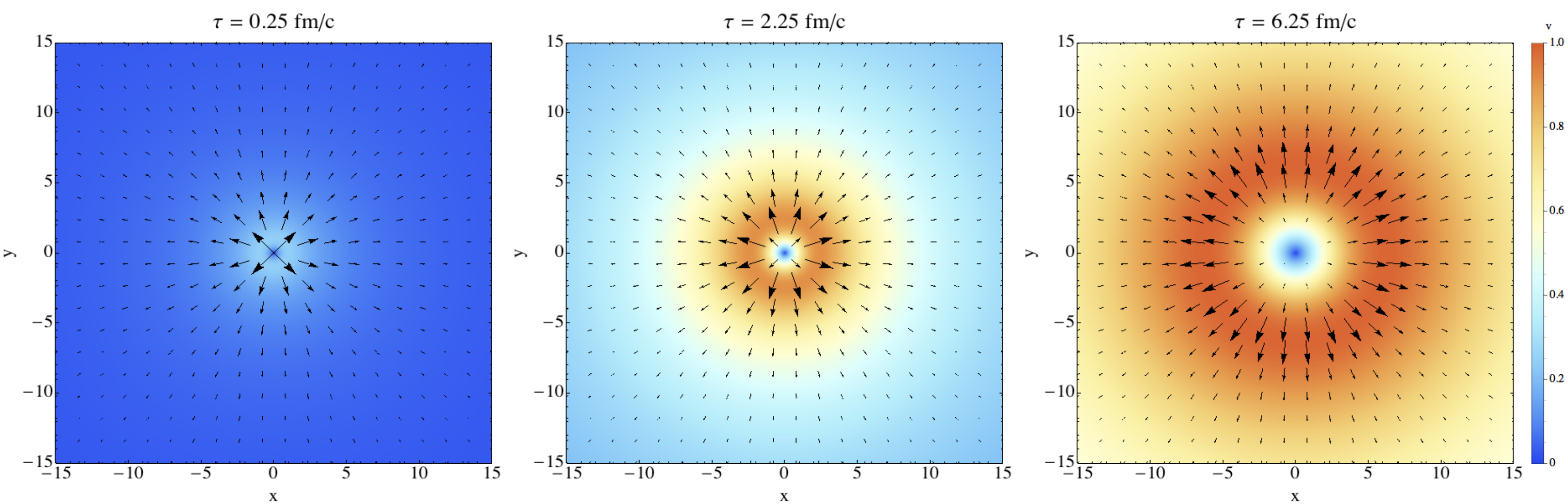
Transverse flow velocity in flat spacetime – Gubser flow

Gubser, PRD82 (2010) 085027.
Gubser and Yarom, NPB846 (2011) 469-511.

Symmetry under $SO(3)_q \otimes SO(1,1) \otimes Z_2$

$$u_r = \sinh\left[\tanh^{-1}\left(\frac{2q\tau r}{1+q^2\tau^2+r^2r^2}\right)\right]$$

Flow velocity



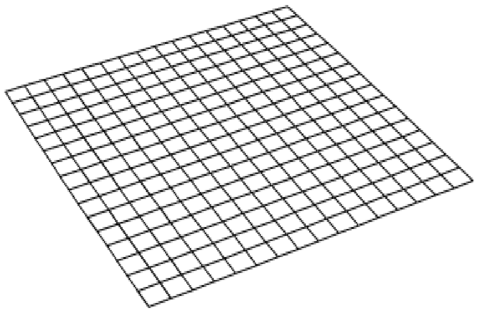
Conformal hydrodynamics undergoing Gubser flow

Gubser and Yarom, NPB846 (2011) 469-511

Weyl transformation

$$g_{\mu\nu}(x) \to e^{-2\Omega(x)} g_{\mu\nu}(x)$$

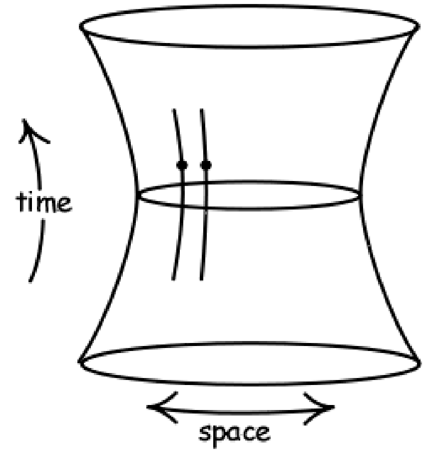
Flat Minkowski space



$$ds^2 = -d\tau^2 + dr^2 + r^2 d\phi^2 + \tau^2 d\xi^2$$

Complicated dynamics

dS_3 x R



3d de Sitter space

$$d\hat{s}^2 = -d\rho^2 + \cosh^2\rho\, d\theta^2 + \cosh^2\rho\, \sin^2\theta\, d\phi^2 + d\xi^2$$

Trivial (locally static) flow

For other flows see:
Hatta, JN, Xiao
PRD89 (2014)

$$\sinh\rho = -\frac{1-\tilde{\tau}^2+\tilde{r}^2}{2\tilde{\tau}}, \qquad \tan\theta = \frac{2\tilde{r}}{1+\tilde{\tau}^2-\tilde{r}^2}.$$

Gubser flow and the Israel-Stewart equations

Marrochio, JN, Denicol, Luzum, Jeon, Gale, PRC 91, 014903 (2015)

Equations of motion

$$\frac{D_\tau T}{T} + \frac{\theta}{3} + \frac{\pi_{\mu\nu}\sigma^{\mu\nu}}{3sT} = 0,$$
$$\frac{\Delta^\mu_\alpha \nabla^\alpha T}{T} + D_\tau u^\mu + \frac{\Delta^\mu_\nu \nabla_\alpha \pi^{\alpha\nu}}{sT} = 0,$$
$$\frac{\tau_R}{sT}\left(\Delta^\mu_\alpha \Delta^\nu_\beta D_\tau \pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\theta\right) + \frac{\pi^{\mu\nu}}{sT} = -\frac{2\eta}{s}\frac{\sigma^{\mu\nu}}{T}$$

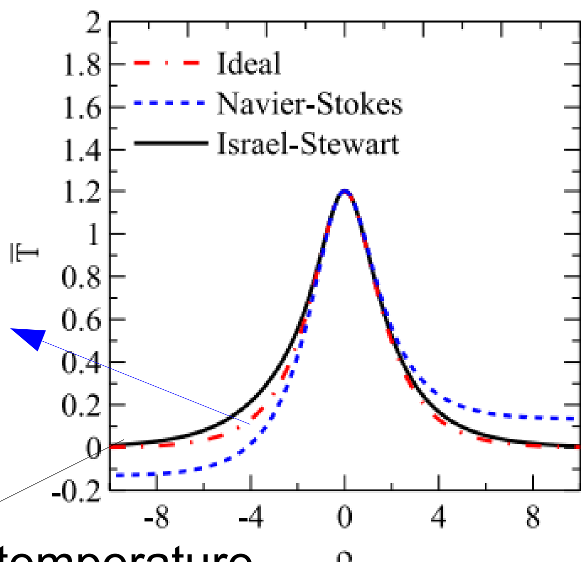
Equations of motion: dS3 x R

$$\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^\xi_\xi(\rho)\tanh\rho,$$
$$\frac{c}{\hat{T}}\frac{\eta}{s}\left[\frac{d\bar{\pi}^\xi_\xi}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\xi_\xi\right)^2\tanh\rho\right] + \bar{\pi}^\xi_\xi = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho,$$

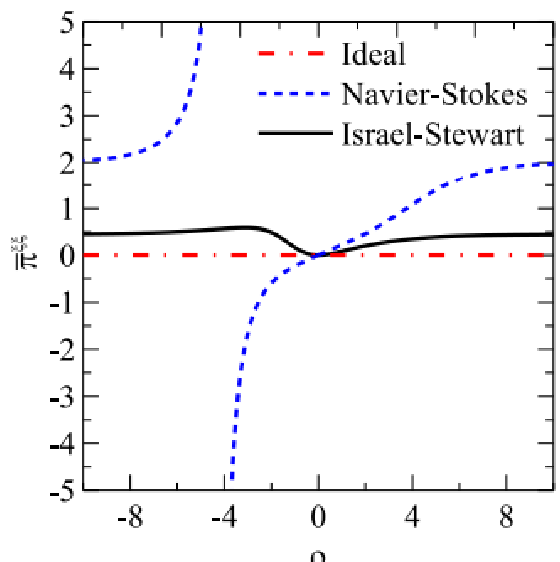
Relaxation time: $\tau_R = c\eta/(Ts)$

$\hat{\theta} \sim \hat{\sigma}^\xi_\xi \sim \tanh\rho$ $\qquad \bar{\pi}^\xi_\xi \equiv \hat{\pi}^\xi_\xi/(\hat{T}\hat{s})$

NS gives negative temperature



IS gives well defined temperature



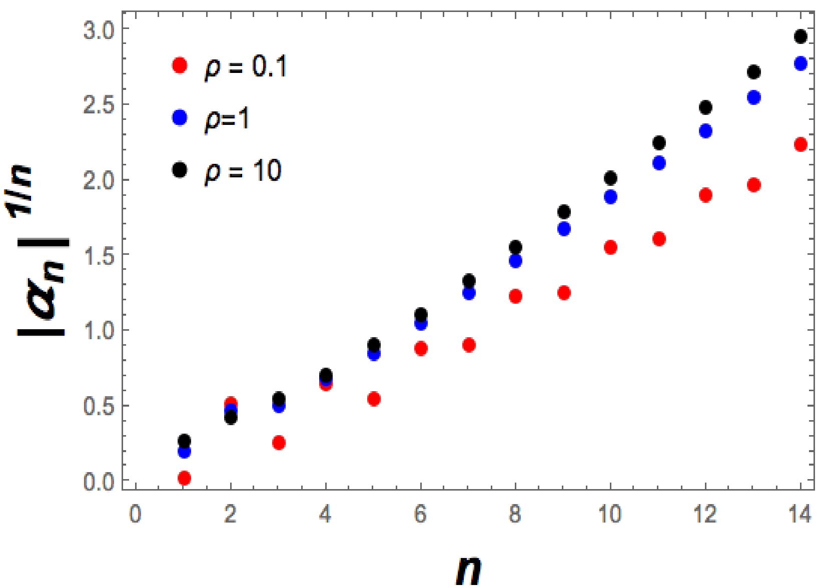
Divergence of the gradient expansion: Gubser flow

G. Denicol, JN, to appear

Israel-Stewart equations can be written as

$$\frac{\hat{\tau}_R}{\tanh^2\rho}\frac{T''}{T} + \frac{3\hat{\tau}_R}{\tanh^2\rho}\left(\frac{T'}{T}\right)^2 + \frac{T'}{\tanh\rho}\left(\frac{19\hat{\tau}_R}{3T} - \frac{\hat{\tau}_R}{T\tanh^2\rho} + \frac{1}{\tanh\rho}\right) + \frac{16}{9}\hat{\tau}_R + \frac{2T}{3\tanh\rho} - \frac{4}{9}\frac{\eta}{s} = 0$$

Expansion in powers of ~ $\hat{\tau}_R$



Gradient series diverges also for more complex flow patterns !!!

Attractor dynamics for Gubser flow

G. Denicol, JN, to appear

We follow Heller, Spalinski PRL 2015

$$\frac{c}{\hat{T}}\frac{\eta}{s}\left[\frac{d\bar{\pi}_{\xi}^{\xi}}{d\rho}+\frac{4}{3}\left(\bar{\pi}_{\xi}^{\xi}\right)^{2}\tanh\rho\right]+\bar{\pi}_{\xi}^{\xi}=\frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho$$

Define new variable w $\sim \dfrac{1}{K_N}$

$$w=\frac{T}{\hat{\tau}_R\tanh\rho}$$

Look for attractor behavior $\dfrac{d\bar{\pi}_{\xi}^{\xi}}{dw}\to 0$

Attractor

$$\bar{\pi}_{\xi}^{\xi}\sim-\frac{3w}{8\hat{\tau}_R}+\frac{\sqrt{9w^2+64\hat{\tau}_R(\eta/s)}}{\hat{\tau}_R}$$

Goes to equilibrium

$$\lim_{w\to\infty}\bar{\pi}_{\xi}^{\xi}\to\frac{4}{3w}\frac{\eta}{s}+\mathcal{O}(1/w^3)$$

$$\lim_{w\to 0}\bar{\pi}_{\xi}^{\xi}\to\sqrt{\frac{\eta}{s\hat{\tau}_R}}+\mathcal{O}(w)$$

non-equilibrium state

“Cold plasma limit”

- Are there hydrodynamic attractors for flows without any particular symmetry, such as in event-by-event simulations of heavy ion collisions?

- How does one generalize the resummation procedure when Kn is not “simple” (i.e., series depends on more than one variable)?

- Israel-Stewart leads to causal dynamics (in contrast to Navier-Stokes) and has a hydrodynamic attractor (at least for simple flows such as Bjorken and Gubser).

- Note that causality implies resummation of spatial derivatives:

Linearized modes

$$\lim_{k\to 0} w(k) \sim c_s k - i\#\frac{\eta}{sT}k^2 + \dots \longrightarrow \lim_{k\gg 1} w(k) \sim k + \mathcal{O}(1/k)$$

$w(k)$ cannot be a polynomial in k !!!!

IV – A different way to handle 2nd order corrections

- Asymptotic series are good in physics (QED).

- Why is there is so much interest about the divergence of the gradient expansion in fluid dynamics?

- The problem is that we cannot seem to be able to easily truncate. If we write the viscous correction as

$$\Pi^{\mu\nu} = T^{\mu\nu} - T_0^{\mu\nu} \to \Pi^{\mu\nu}(\partial u, \partial \varepsilon)$$

Landau's choice

$$u_\mu \Pi^{\mu\nu} = 0$$

To 2nd order in the expansion BRSSS (2007) proposed for a conformal fluid

$$\begin{aligned}
\Pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} \\
&\quad + \eta\tau_\Pi \left[{}^{\langle}D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}(\nabla\cdot u) \right] + \kappa \left[R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta \right] \\
&\quad + \lambda_1 \sigma^{\langle\mu}{}_\lambda \sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} .
\end{aligned}$$

If we stop at 1st order → relativistic Navier-Stokes: $\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$

- Initial value problem requires $(\varepsilon, u_\mu, \partial_t u_\mu)$ in the initial spacelike hypersurface

- This theory does not have a well-defined Cauchy problem (Pichon, 1965).

- This theory is linearly unstable around equilibrium (Hiscock, Lindblom, 1984).

On the other hand, the BRSSS 2nd order theory

- Main starting point of current hydrodynamic simulations (after IS-like resummation).

- Initial value problem requires $(\varepsilon, u_\mu, \partial_t u_\mu, \partial_t^2 u_\mu)$ → EOM are of 3rd order

- Causality is a tricky business in theories with higher order derivatives (even if metric is flat) → I don't know how to deal with this.

- Naive linearized analysis shows that this theory is acausal and unstable (at large k)

This problem is very complicated mathematically …

But let us be bold (and somewhat reckless)*

*It is easier to ask for forgiveness than to ask for permission ...

To make it easier to write down a causal theory based on gradients:

- Remove for now the terms with 3rd in derivatives $\sim \dot{\sigma}^{\mu\nu}$

- Remove terms in the energy-momentum tensor $\sim \partial u \partial u, \partial u \partial T$

- Include 2nd order-like terms (easy to handle in a proof of causality)

$$\frac{\dot{T}}{T} + \frac{\theta}{3} = \mathcal{O}(\partial^2) \qquad \dot{u}^\mu + \frac{\nabla_\perp^\mu T}{T} = \mathcal{O}(\partial^2)$$

$$T_{\mu\nu} = \varepsilon \left(u_\mu u_\nu + \frac{\Delta_{\mu\nu}}{3} \right) - \eta \sigma_{\mu\nu} + \Delta T_{\mu\nu}$$

$$T_{\mu\nu} = \varepsilon \left(u_\mu u_\nu + \frac{\Delta_{\mu\nu}}{3} \right) - \eta \sigma_{\mu\nu} + \Delta T_{\mu\nu}$$

Bemfica, Disconzi, JN, work in progress

With

$$\Delta T^{\mu\nu} = 3\lambda \frac{\mathcal{D}T}{T} \left(u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3} \right) + \lambda u^\mu \frac{\mathcal{D}_\perp^\nu T}{T} + \lambda u^\nu \frac{\mathcal{D}_\perp^\mu T}{T}$$

This theory is:

where

$$\frac{\mathcal{D}T}{T} = \frac{\dot{T}}{T} + \frac{\theta}{3}$$

$$\frac{\mathcal{D}_\perp^\mu T}{T} = \dot{u}^\mu + \frac{\Delta^{\mu\nu} \nabla_\nu T}{T}$$

- Causal (full nonlinear level)

- Linearly stable around equilibrium

- Has a well defined Cauchy problem (solution exists)

- Can be dynamically coupled to Einstein's equations (Cauchy + “gravity”)

Existence, uniqueness, and causality

$$T_{\mu\nu} = \varepsilon \left(u_\mu u_\nu + \frac{\Delta_{\mu\nu}}{3} \right) - \eta \sigma_{\mu\nu} + \Delta T_{\mu\nu}$$

Transport coeff.

$$\lambda \geq 4\eta$$

Bemfica, Disconzi, JN

Dynamical variables: $\varepsilon, u_\mu, g_{\mu\nu}$

Theorem 4.4. *Let $\mathcal{I} = (\Sigma, g_0, \kappa, \epsilon_0, \epsilon_1, v_0, v_1)$ be an initial data set for the VECF system. Assume that Σ is compact with no boundary, and that $\epsilon_0 > 0$. Suppose that χ and λ are given by (3.3), where $\eta : (0, \infty) \to (0, \infty)$ is analytic, and assume that $a_1 = 4$ and $a_2 \geq 4$, or that $a_1 \geq 4$ and $a_2 = \frac{a_1}{a_1 - 1}$. Finally, assume that the initial data is in $G^{(s)}(\Sigma)$ for some $1 \leq s < \frac{17}{16}$. Then:*

1) There exists a globally hyperbolic development M of $\mathcal{I}$.

2) M is causal, in the following sense. Let (g, ϵ, u) be a solution to the VECF system provided by the globally hyperbolic development M. For any point $q \in M$, $(g(q), u(q), \epsilon(q))$ depends only on $\mathcal{I}|_{i(\Sigma) \cap J^-(q)}$, where $J^-(q)$ is the causal past of q and $i : \Sigma \to M$ is the embedding associated the globally hyperbolic development M.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

Coupled

$$\nabla_\mu T^{\mu\nu} = 0$$

Hydrodynamic attractor (Bjorken flow)

$u^\mu = (1,0,0,0) \qquad \theta = \nabla_\mu u^\mu = 1/\tau$

$$\lambda \frac{\ddot{T}}{T} + 2\frac{\lambda}{\eta}\left(\frac{\dot{T}}{T}\right) + \frac{7}{3}\frac{\lambda}{\eta}\theta\frac{\dot{T}}{T} + 4\frac{\lambda}{\eta}\frac{\theta^2}{9} + \frac{\lambda}{3\eta}\dot{\theta} + \dot{T} + \frac{\theta T}{3} = \frac{\eta}{6s}\sigma_{\mu\nu}\sigma^{\mu\nu}$$

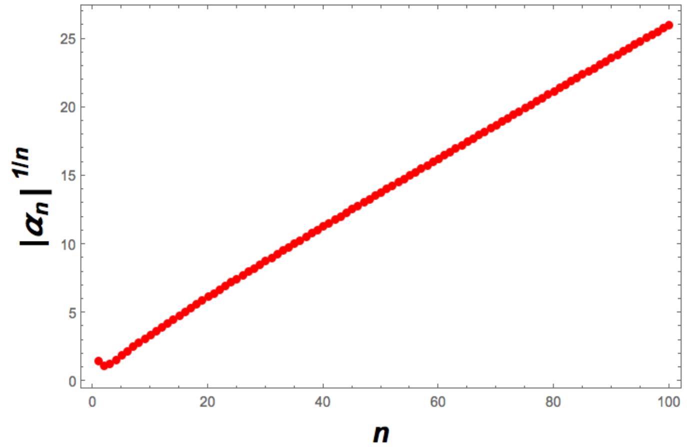
Heller, Spalinski variables

$$\bar{\lambda} w f \frac{df}{dw} + 3\bar{\lambda} f^2 + f\left(w - \frac{14}{3}\bar{\lambda}\right) - \frac{8\bar{\lambda}}{3} - \frac{4}{9}\frac{\eta}{s} - \frac{2w}{3} = 0$$

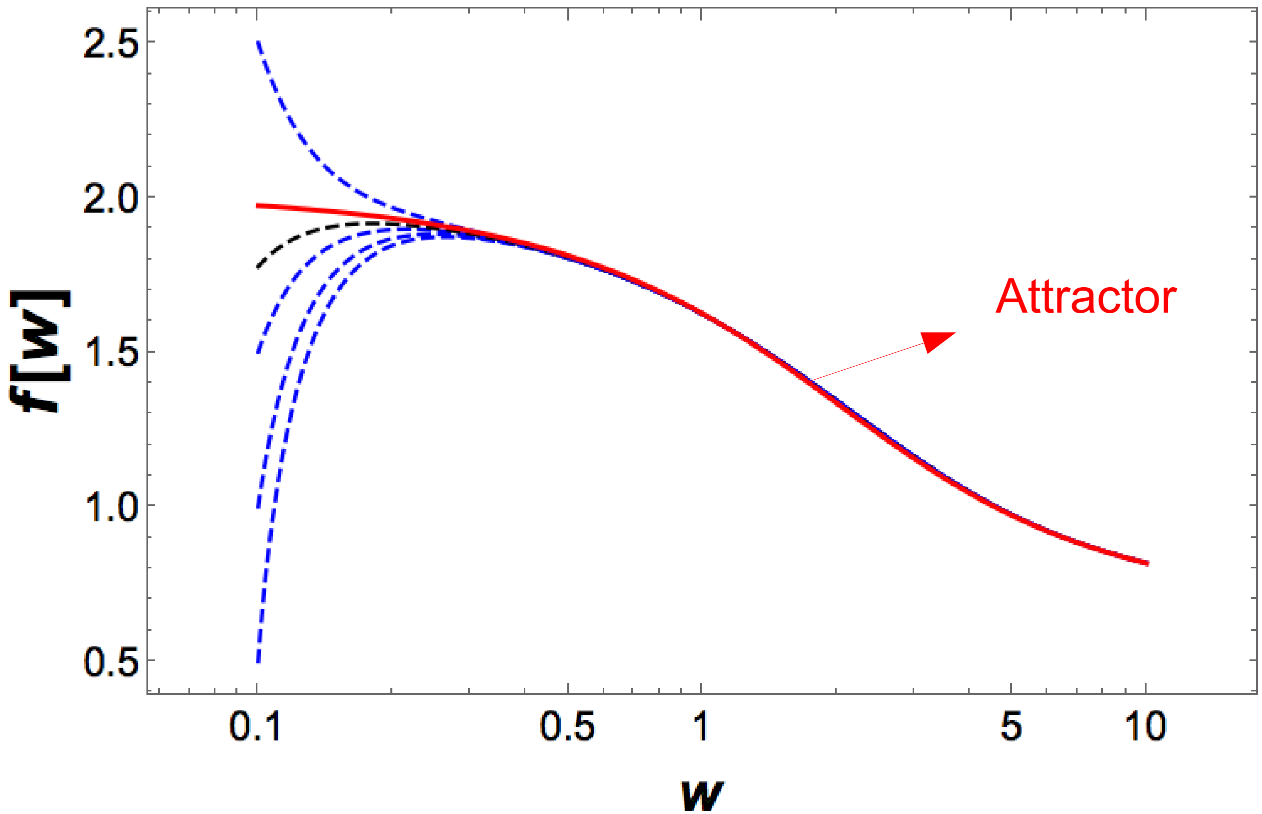
Divergent series

Series

$$f = \sum_{n=0}^{\infty} \frac{\alpha_n}{w^n}$$



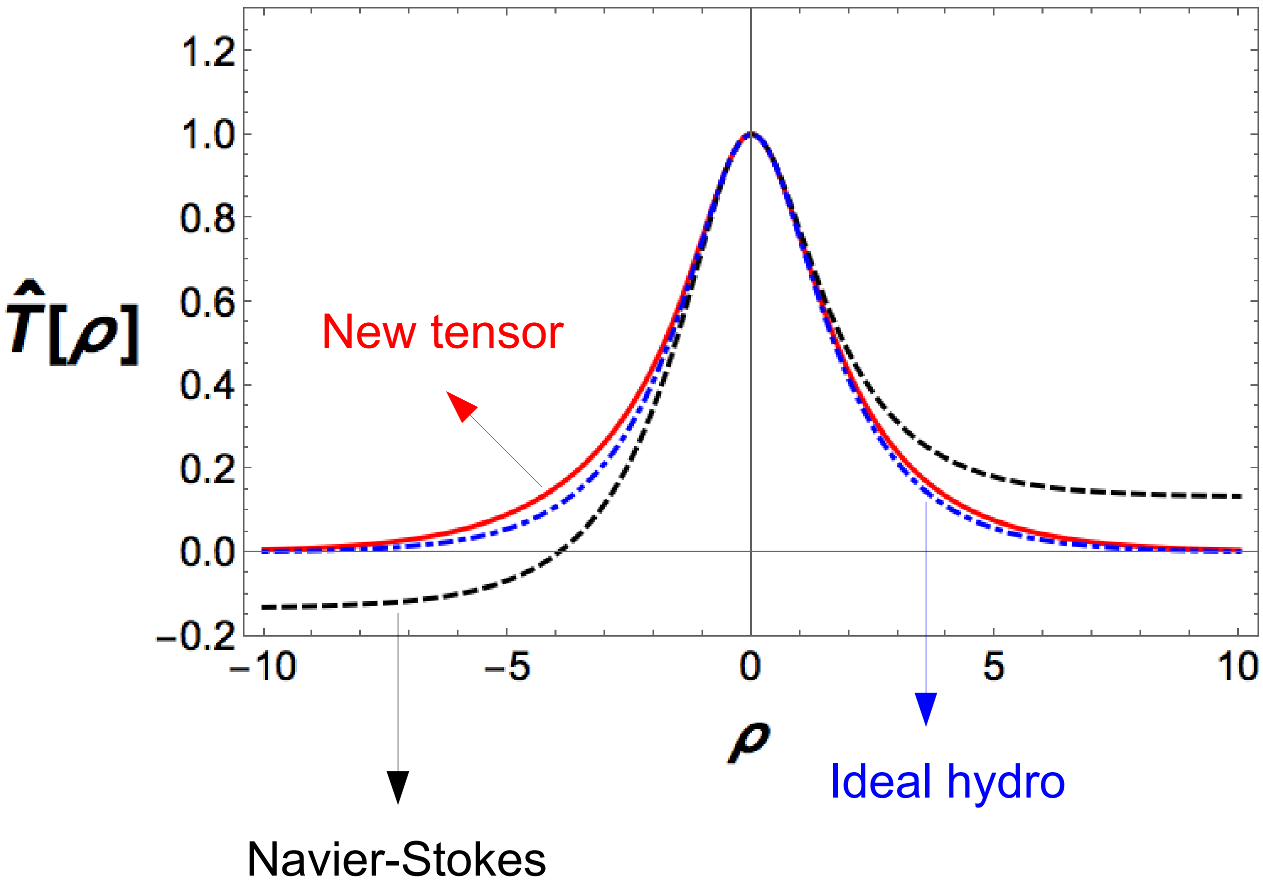
Hydrodynamic attractor (Bjorken flow)



$K_N \gg 1$ $K_N < 1$

Causality → Resummation → Attractor ?

Gubser flow



New causal tensor solution → Well defined temperature ~ Israel-Stewart

Conclusions

- Exact solutions of kinetic theory can be used to reveal nontrivial dynamics of non-hydro modes (thermalization mapped onto complex plane)

- Hydrodynamic series diverges for simple flows (Bjorken, Gubser)

- Attractor solutions may be useful to define hydrodynamics in the large Kn regime → Extension to nontrivial flows such as ebe IC?

- Consequence of hydrodynamic attractors to HIC phenomenology???

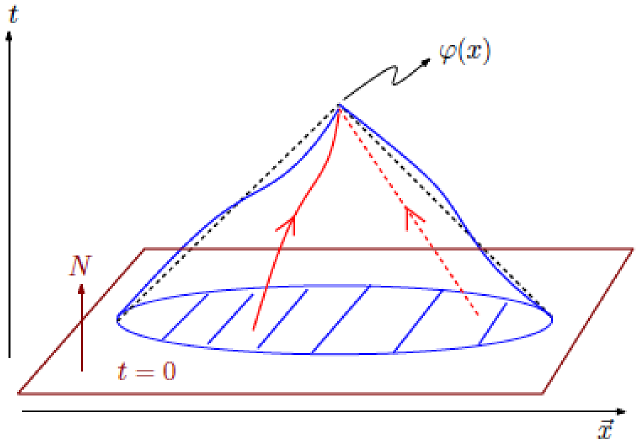
- A causal, stable, “GR compatible” 2nd-like theory can be written down (at least in the conformal case)

EXTRA SLIDES

Causality in general relativity is formulated in the same terms as in special relativity: the four-velocity v of any physical entity satisfies

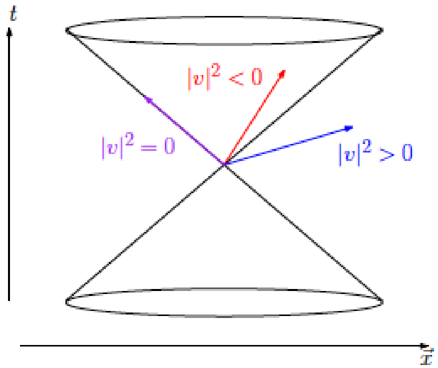
$$|v|^2 = g_{\alpha\beta} v^\alpha v^\beta \leq 0.$$

Note that that the causal structure is far more complicated than in Minkowski space since $g_{\alpha\beta} = g_{\alpha\beta}(x)$. One can better formulate causality in terms of the domain of dependence of solutions to Einstein's equations:



Causality in GR.

A theory is causal if for any field φ its value at x depends only on the "past domain of dependence of x."



Light-cone.

Causality: The four-velocity of any physical entity satisfies $|v|^2 = \eta_{\alpha\beta} v^\alpha v^\beta \leq 0$. "Nothing propagates faster than the speed of light."

We recall the definition of Gevrey spaces. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Let $f : \Omega \to \mathbb{R}^n$ and

$$\|f\|_s^2 = \sum_{\beta \leq s} \int_\Omega |\partial^\alpha f|^2$$

be the s-Sobolev norm on Ω, where $s > \frac{n}{2}$. For $\sigma \geq 1$ and $\delta > 0$, set

$$\|f\|_{G_{\sigma,\delta}} = \sum_{\beta \geq 0} \frac{\delta^{|\beta|}}{|\beta|!^\sigma} \|\partial^\alpha f\|_s .$$

We say that f belongs to the Gevrey space $G_{\sigma,\delta}(\Omega)$ if this last norm is finite. We have $C^\omega(\Omega) \subseteq G_{\sigma,\delta}(\Omega) \subseteq C^\infty(\Omega)$. The basic idea is the following: an analytic function satisfies inequalities of the type $|\partial^\beta f| \leq C^{|\alpha|+1}|\beta|!$. A Gevrey function satisfies the weaker inequality $|\partial^\beta f| \leq C^{|\alpha|+1}|\beta|!^\sigma$.

Theorem 4.5. *Let T be given by (3.1) with g being the Minkowski metric. Suppose that χ and λ satisfy (3.3), with $a_1 = 4$, $a_2 \geq 4$, or or $a_1 \geq 4$ and $a_2 = \frac{a_1}{a_1-1}$, where $\eta : (0,\infty) \to (0,\infty)$ is a given analytic function. Let $\epsilon_0, \epsilon_1 : \mathbb{R}^3 \to \mathbb{R}$ and $v_0, v_2 : \mathbb{R}^3 \to \mathbb{R}^3$ belong to $G^{(s)}(\mathbb{R}^3)$ for some $1 \leq s < \frac{7}{6}$, and assume that $\epsilon_0 \geq C_0 > 0$, where C_0 is a constant.*

Then, there exists a $\mathcal{T} > 0$, a function $\epsilon : [0,\mathcal{T}) \times \mathbb{R}^3 \to (0,\infty)$ and a vector field $u : [0,\mathcal{T}) \times \mathbb{R}^3 \to \mathbb{R}^4$, such (ϵ, u) satisfy equations (3.2) and (3.5) in $[0,\mathcal{T}) \times \mathbb{R}^3$, $\epsilon(0,\cdot) = \epsilon_0$, $\partial_0 \epsilon(0,\cdot) = \epsilon_1$, $u(0,\cdot) = u_0$, and $\partial_0 u(0,\cdot) = u_1$, where ∂_0 is the derivative with respect to the first coordinate in $[0,\mathcal{T}) \times \mathbb{R}^3$. This solution belongs to $G^{(s)}([0,\mathcal{T}) \times \mathbb{R}^3)$ and is unique in this class. In particular, (ϵ, u) it is smooth. Finally, the solution is causal, in the following sense. For any $q \in [0,T) \times \mathbb{R}^3$, $(\epsilon(q), u(q))$ depends only on $(\epsilon_0, \epsilon_1, v_0, v_1)|_{\mathbb{R}^3 \cap J^-(q)}$, where $J^-(q)$ is the causal past of q (with respect to the Minkowski metric).

Definition B.3. We say a real valued function $f \in G^{(s)}(U)$ if $f \in C^\infty(U)$ and for every compact set $K \subset U$ there exists a positive constant C such that

$$\left| \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| \leq C^{|\alpha|+1} (\alpha!)^s,$$

for all $x \in K$ and all $\alpha = (\alpha_1, \dots, \alpha_n)$ $(\alpha_i \in \{0,1,2,\dots\})$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

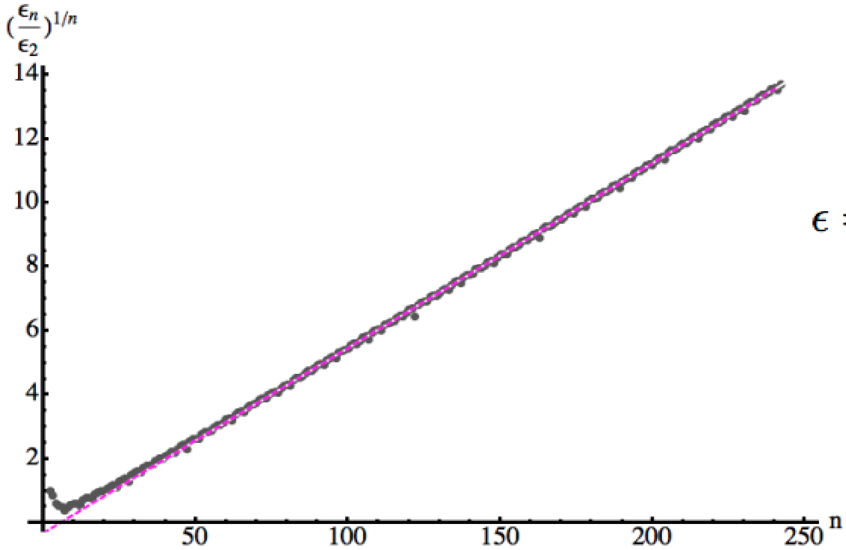
Thus, Gevrey functions are smooth functions whose derivatives grow at most as $(\alpha!)^s$. The larger the s, the faster the function and its derivatives are allowed to grow, with $s = 1$ corresponding to analytic functions. Differently than analytic functions, however, the power series for f will be only an asymptotic series when $s > 1$. Another crucial difference between $s = 1$ and $s > 1$ is that there exist compactly supported functions in $G^{(s)}$, $s > 1$. This is important for applications in general relativity, where one usually constructs global in space solutions upon gluing locally defined solutions with the help of functions with compact support. This cannot be done in the analytic class since an analytic function that vanishes on an open set vanishes everywhere.

Divergence of the gradient expansion at strong coupling

$$K_N \sim \frac{\ell}{L} \ll 1$$

Fluid/gravity correspondence (aka Chapman-Enskog at strong coupling) developed by Minwalla, Hubeny, Rangamani, and etc

1st study of gradient expansion at large orders at strong coupling by Heller et al., PRL (2013), for SYM with Bjorken expansion



$$\epsilon = \frac{3}{8} N_c^2 \pi^2 \frac{1}{\tau^{4/3}} \left(\epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \ldots \right)$$

Given that heavy ion data indicates that T ~ QCD transition the QGP is a nearly perfect fluid …

There must have been nearly perfect fluidity in the early universe

Experimental consequences of that are not yet known (are there any??)

Given that around those temperatures QCD is not conformal, we would like to use a nonconformal gravity dual in a FLRW spacetime

This was done by A. Buchel, M. Heller, JN in arXiv:1603.05344 [hep-th] PRD (2016)

Toy model for QCD: N=2* gauge theory

Pilch, Warner, Buchel, Peet, Polchinski, 2000
A. Buchel, S. Deakin, P. Kerner and J. T. Liu, NPB 784 (2007) 72

A relevant deformation of SYM: Breaking of SUSY

N = 4 SYM theory + $\delta\mathcal{L} = -2\int d^4x\,\left[\, m_b^2\,\mathcal{O}_b + m_f\,\mathcal{O}_f \,\right]$

$$\mathcal{O}_b = \frac{1}{3}\mathrm{Tr}\left(\,|\phi_1|^2 + |\phi_2|^2 - 2\,|\phi_3|^2\,\right)\,,$$

$$\mathcal{O}_f = -\mathrm{Tr}\Big(i\,\psi_1\psi_2 - \sqrt{2}g_{\rm YM}\,\phi_3[\phi_1,\phi_1^\dagger] + \sqrt{2}g_{\rm YM}\,\phi_3[\phi_2^\dagger,\phi_2]$$

$$+\mathrm{h.c.}\Big) + \frac{2}{3}m_f\,\mathrm{Tr}\left(\,|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\,\right)$$

Bosonic mass

Fermionic mass

C. Hoyos, S. Paik, and L. G. Yaffe, JHEP 10, 062 (2011)

Toy model for QCD: N=2* gauge theory

Pilch, Warner, Buchel, Peet, Polchinski, 2000

Classical gravity dual action:

$$S = \frac{1}{16\pi G_5} \int \mathrm{d}^5 x \sqrt{-g} \left(R - 12(\partial\alpha)^2 - 4(\partial\chi)^2 - V \right),$$

Scalar potential

$$V = -e^{-4\alpha} - 2e^{2\alpha} \cosh 2\chi + \frac{1}{4} e^{8\alpha} \sinh^2 2\chi.$$

Bulk viscosity

$$\frac{\zeta}{\eta} \sim \mathcal{O}(1) \left(\frac{1}{3} - c_s^2 \right)$$

- Well defined stringy origin

- Non-conformal strongly interacting plasma: $\varepsilon \neq 3p$

- Used in tests of holography in non-conformal setting

N=2* gauge theory in a FLRW Universe

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Characteristic formulation of gravitational dynamics in asymptoticaly AdS5 spacetimes

Chesler,Yaffe, 2013

Assuming spatial isotropy and homogeneity $\boldsymbol{x} = \{x, y, z\}$ leads to

$$ds_5^2 = 2dt\,(dr - Adt) + \Sigma^2\,d\boldsymbol{x}^2,$$

$$\Sigma = \frac{a}{r} + \mathcal{O}(r^{-1})\,,\; A = \frac{r^2}{8} - \frac{\dot{a}r}{a} + \mathcal{O}(r^0)$$

$$\alpha = -\frac{8m_b^2 \ln r}{3r^2} + \mathcal{O}(r^{-2})\,,\; \chi = \frac{2m_f}{r} + \mathcal{O}(r^{-2}).$$

→ Encode non-equilibrium dynamics in an expanding Universe !!!

N=2* gauge theory in a FLRW Universe

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Conformal limit When $m_b = m_f = 0$,

Analytical solution for SYM in FLRW spacetime

$$\alpha = \chi = 0\,,\ \Sigma = \frac{ar}{2}\,,\ A = \frac{r^2}{8}\left(1 - \frac{\mu^4}{r^4 a^4}\right) - \frac{\dot{a}}{a}\, r,$$

First studied by P. S. Apostolopoulos, G. Siopsis, and N. Tetradis, PRL, (2009)

Temperature

$$T = \frac{\mu}{4\pi\, a}.$$

Energy density

$$\epsilon = \frac{3}{8}\pi^2 N^2 T^4 + \frac{3N^2(\dot{a})^4}{32\pi^2 a^4}$$

Pressure

$$P = \frac{1}{3}\epsilon - \frac{N^2(\dot{a})^2\ddot{a}}{8\pi^2 a^3}$$

Conformal anomaly!!!! $-\epsilon + 3P = \frac{N^2}{32\pi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$

Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

In our FLRW case, the gradient expansion corresponds to

Energy-momentum tensor

$$T_{\mu\nu} = T_{\mu\nu}^{eq} + \Pi_{\mu\nu}(\dot{a}, \{\dot{a}^2, \ddot{a}\}, \cdots),$$

$T_{\mu\nu}^{eq}$ → equilibrium

$\Pi_{\mu\nu}$ → dissipation

In terms of the energy density and pressure out-of-equilibrium

$$\epsilon = \epsilon^{eq} + \mathcal{O}\left(\dot{a}^2, \ddot{a}\right) \ , \ P = P^{eq} - \zeta(\nabla \cdot u) + \mathcal{O}\left(\dot{a}^2, \ddot{a}\right) ,$$

ζ → Bulk viscosity

Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Entropy production

$$\frac{d(a^3 s)}{dt} = \frac{N^2}{16\pi} a^{7-2\Delta} \mu^2 \delta_\Delta^2 (4-\Delta)^2 s_\Delta \times \Omega_\Delta^2,$$

Apparent horizon: $a^3 s = N^2 \Sigma^3/(16\pi)|_{r=r_h}$

$$\Omega_\Delta \equiv \sum_{n=0}^{\infty} \mathcal{T}_{\Delta,n+1}[a] \, \frac{F_{\Delta,n}(1)}{\mu^n}.$$

For single component cosmologies

$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3+3\omega}}$ with constant ω and $H \equiv \dot{a}/a$

$$\mathcal{T}_{\Delta,n}[a] = \left(-\frac{1}{2} - \frac{3\omega}{2}\right)^n \Gamma\left(n + \frac{2(\Delta-4)}{1+3\omega}\right) a^n H^n.$$ → **Factorial growth!!!**

Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

1st analytical proof of the divergence of gradient expansion:

→ Knudsen gradient series has zero radius of convergence

→ Knudsen series leads to acausal and unstable dynamics

→ There must be a new way to define hydrodynamics
beyond the gradient expansion

→ A recent way to understand that involves <u>resurgence.</u>

Divergence of the hydrodynamic series

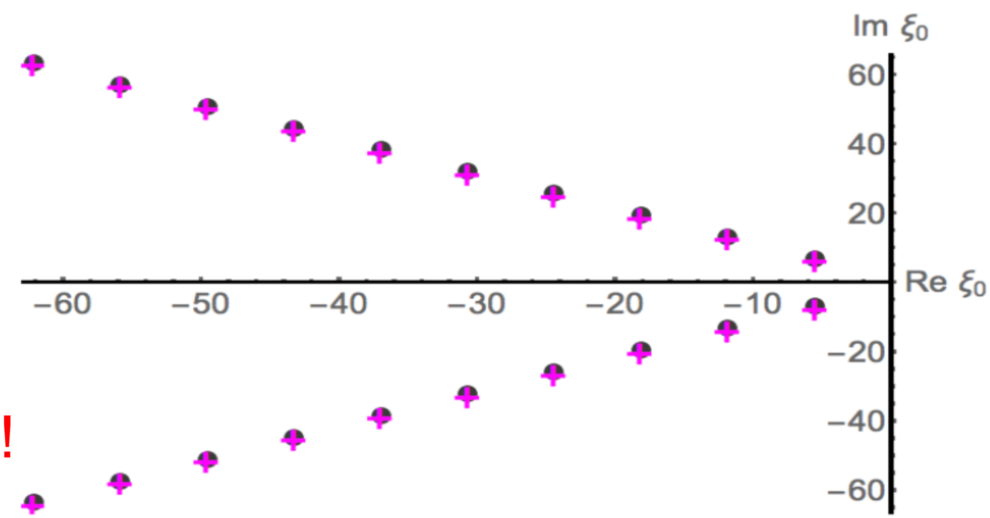
Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Hydrodynamic series

$$\Omega_\Delta = \sum_{n=0}^{\infty} c_n g^n, \qquad c_n \equiv \frac{\Gamma(n+4-\Delta)F_{\Delta,n}(1)}{(4\pi)^n} \text{ and } g \equiv \frac{H}{T} = \frac{4\pi}{\mu} aH.$$

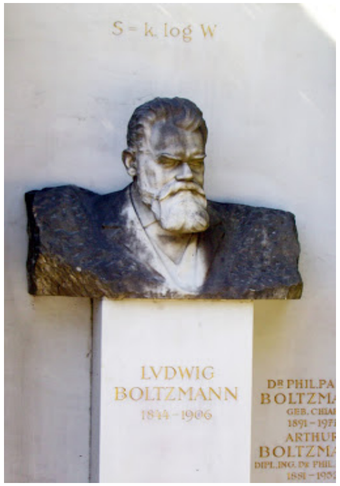
Borel sum

$$\Omega_\Delta^{(B)}(\xi) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \xi^n$$



Borel singularities = are the black hole quasinormal modes !!!

Non-equilibrium entropy



See arXiv:1607.05245

$$S^\mu = -\int_k k^\mu f_k (\ln f_k - 1)$$

One can prove that H-theorem is valid here. Entropy production solely from non-hydrodynamic modes (hydro modes have decoupled).

$$\nabla_\mu S^\mu = \frac{1}{4}\int_{kk'pp'} W_{kk\leftrightarrow pp'} \left[\frac{f_p f_{p'}}{f_k f_{k'}} - \ln\left(\frac{f_p f_{p'}}{f_k f_{k'}}\right) - 1\right] f_k f_{k'} \geq 0$$

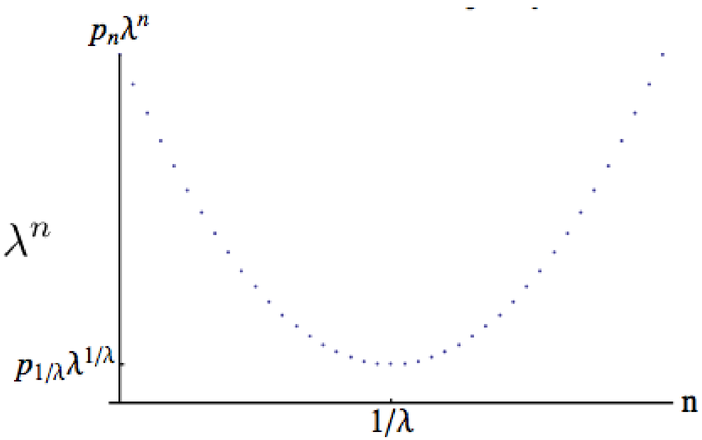
Even though energy-momentum tensor always the same as in equilibrium.

Expansion is never truly adiabatic in this toy Universe.

Resurgence

Recent works by Dunne, Unsal, Cherman, Heller, Janik ...

$$\sum_{n=1}^{\infty} p_n \lambda^n$$



$$\langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n}\lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n}\lambda^n$$

$$\sum_{n=0,k=0}^{\infty} \sum_{q=1}^{k-1} c_{n,k,q}\, g^{2n} \left[e^{-S/g^2}\right]^k \left[\log\left(\frac{1}{g^2}\right)\right]^q$$

Heller, Spalinski, PRL 2015

Hydro expansion

via resurgence

$$f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n}$$

$$\Omega \equiv w^{-\gamma} \exp(-w\xi_0)$$