The photon production rate of the QGP: an improved estimate from lattice QCD

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Workshop "Canterbury Tales of Hot QFTs in the LHC Era",
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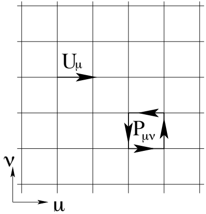
Regularization of QCD on a lattice



Gluons: $U_\mu(x) = e^{iag_0 A_\mu(x)} \in SU(3)$
'link variables'

Quarks: $\psi(x)$ 'on site', Grassmann

Gauge-invariance exactly preserved; no gauge-fixing required.

Imaginary-time path-integral representation of QFT (Matsubara formalism):

- Starting point for **Monte-Carlo simulations** using importance sampling.
- Representation of the Euclidean correlator as a Fourier series:

$$G_E^{AB}(x) = T \sum_{\ell \in \mathbb{Z}} e^{-i\omega_\ell x_0} \; \tilde{G}_E^{AB}(\omega_\ell, \boldsymbol{x}), \qquad \omega_\ell = 2\pi\ell T.$$

Then Wightman correlator for $t^2 - \boldsymbol{x}^2 < 0$ given by

$$G_>^{AB}(t, \boldsymbol{x}) \equiv \frac{1}{Z} \mathrm{Tr}\{e^{-\beta H} A(t) B(0)\} = T \sum_{\ell \in \mathbb{Z}} e^{\omega_\ell t} \; \tilde{G}_E^{AB}(\omega_\ell, \boldsymbol{x}).$$

References & list of coauthors

- Bastian B. Brandt, Anthony Francis, Tim Harris, HM, Aman Steinberg, in preparation
- Harris & Steinberg gave a combined presentation at LAT17.

Lattice papers on the photon rate:

- Karsch, Laermann, Petreczky, Stickan, Wetzorke 2002; S. Gupta 2004; Aarts, Allton, Foley, Hands, Kim 2007: quenched calculations, $k = 0$.
- hep-lat/0610061 (LAT06): Aarts, Allton, Foley, Hands: quenched, $k \neq 0$
- 1012.4963: Ding, Francis, Kaczmarek, Karsch, Laermann, Soeldner, quenched calculation with continuum limit, $k = 0$.
- 1212.4200 (JHEP): Brandt, Francis, HM, Wittig: $N_f = 2$, $N_t = 16$, $k = 0$, $m_\pi = 270$, $T = 250$MeV.
- 1307.6763 (PRL), 1412.6411 (JHEP): Aarts, Allton, Amato,Giudice, Hands, Skullerud: $N_f = 2 + 1$, $k = 0$, anisotropic, fixed-scale temperature scan, $m_\pi = 384$ MeV
- 1512.07249 (PRD): Brandt, Francis, Jäger, HM, $N_f = 2$, $k = 0$, $N_t = 12 \to 24$, $m_\pi = 270$, fixed-scale scan across the phase transition.
- 1604.07544 (PRD): Ghiglieri, Kaczmarek, Laine, F. Meyer: quenched calculation with continuum limit, $k \neq 0$.
- here: $N_f = 2$ calculation with continuum limit at $T = 250$MeV, $k \neq 0$.

Definitions

Euclidean-time vector correlators ($\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} = 2\mathrm{diag}(1,-1,-1,-1)$),

$$G^{\mu\nu}(x_0,\boldsymbol{k}) = \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \Big\langle V^\mu(x)\, V^\nu(y)\Big\rangle, \qquad V^\mu = \sum_f Q_f\, \bar\psi_f \gamma^\mu \psi_f$$

- all diagonal components of $G^{\mu\nu}$ are positive; spectral representation:

$$G^{\mu\nu}(x_0,\boldsymbol{k}) \overset{\mu=\nu}{=} \int_0^\infty \frac{d\omega}{2\pi}\, \rho^{\mu\nu}(\omega,\boldsymbol{k})\, \frac{\cosh[\omega(\beta/2-x_0)]}{\sinh(\beta\omega/2)}.$$

- $\rho^{\mu=\nu}(\omega,\boldsymbol{k})/\omega$ is even in ω and non-negative.
- from current conservation: $\omega^2\rho^{00}(\omega,k) = k^i k^j \rho^{ij}(\omega,k)$.
- consequence: $(\hat k^i \hat k^j \rho^{ij} - \rho^{00})/\omega$ has the same sign as $\mathcal{K}^2 \equiv \omega^2 - k^2$, and vanishes at $\omega = k$.
- consider the linear combination

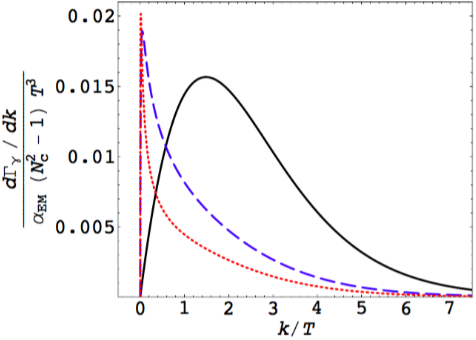
$$\begin{aligned} \rho(\omega,k,\lambda) &= (\delta^{ij} - \hat k^i \hat k^j)\rho^{ij} + \lambda\,(\hat k^i \hat k^j \rho^{ij} - \rho^{00}) \qquad k \equiv |\boldsymbol{k}|, \quad \hat k^i = k^i/k, \\ \text{e.g. } \rho(\omega,k,1) &= \rho^{ii} - \rho^{00} = -\rho^\mu{}_\mu(\omega,k) \quad \rightsquigarrow \quad \text{relevant for the dilepton rate.} \end{aligned}$$

- The differential photon rate per unit volume of plasma:

$$d\Gamma_\lambda(\boldsymbol{k}) = e^2\, \frac{d^3k}{(2\pi)^3\, 2k}\, \frac{\rho(k,k,\lambda)}{e^{\beta k}-1} \quad \text{is independent of } \lambda.$$

Known results in $N = 4$ SYM: weak & strong coupling

$$\frac{d\Gamma_\lambda(\boldsymbol{k})}{dk} = \frac{\alpha}{\pi}\underbrace{\frac{\rho(k,k,\lambda)}{k}}_{\stackrel{k\to 0}{\to} 4\chi_s D}\frac{k^2}{e^{\beta k}-1}$$



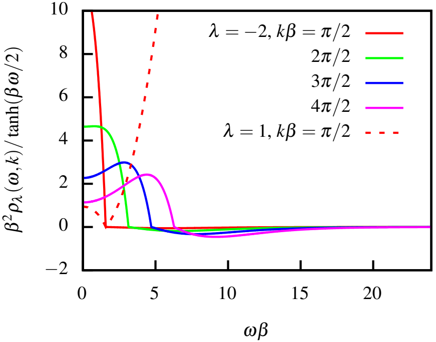
- black: $\lambda_H = \infty$ (AdS/CFT); blue: $\lambda_H = 0.5$; red: $\lambda_H = 0.2$.

From hep-ph/0607237 (JHEP) Caron-Huot, Kovtun, Moore, Starinets, Yaffe.

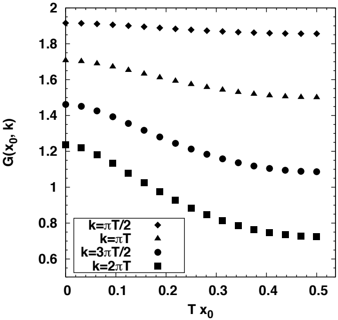
Non-interacting fermions

$$\rho(\omega,k,\lambda) = (\delta^{ij}-\hat{k}^i\hat{k}^j)\rho^{ij}+\lambda\,(\hat{k}^i\hat{k}^j\rho^{ij}-\rho^{00}) = \begin{cases} -\rho^{\mu}{}_{\mu}(\omega,k) & \lambda=1 \\ (\delta^{ij}-3\hat{k}^i\hat{k}^j)\rho^{ij}+2\rho^{00} & \lambda=-2. \end{cases}$$

Spectral function



Euclidean correlator with $\lambda = -2$



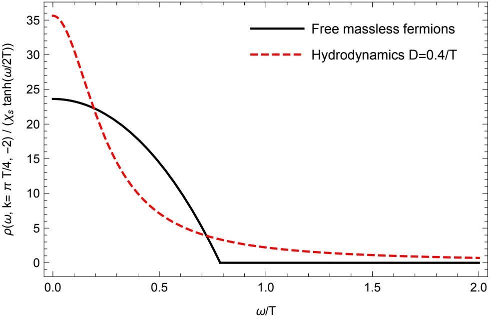
- We choose $\lambda = -2$ from now on: UV-finite correlator even at $x_0 = 0$.
- for $k = \mathrm{O}(\pi T)$, $\rho(k,k,\lambda) = \mathrm{O}(\alpha_s \log \alpha_s)$ in perturbation theory.

The hydrodynamic regime: $k \to 0$

For $k \to 0$, the ρ^{00} contribution parametrically dominates $\rho(\omega, k, -2)$ and the hydrodynamic prediction is

$$\rho(\omega, k, -2)/\omega \approx \frac{4\chi_s\, Dk^2}{\omega^2 + (Dk^2)^2} \qquad \omega, k \ll D^{-1}.$$

- Diffusion pole! $D =$ diffusion coefficient.
- $\chi_s = \int d^4x \langle V^0(x) V^0(0)\rangle$ the static susceptibility.
- $\rightsquigarrow$ for k in the hydrodynamic regime, the spectral weight is concentrated in a region of order Dk^2 around $\omega = 0$, in contrast to the free theory.



Can interpolate between the two curves using kinetic theory [Hong & Teaney 1003.0699 (PRD)]

A sum rule for $\rho \equiv \rho_{\lambda=-2}$

i. Lorentz invariance and transversity $\Rightarrow \tilde{G}_{\rm E}(\omega_n, k) = 0$ in vacuum and UV finite at $T > 0$

ii. UV finite correlation admits an OPE $\tilde{G}_{\rm E}(\omega_n, k) \sim \frac{\langle \mathcal{O}_4 \rangle}{\omega_n^2}$

Furthermore, charge conservation demands $\tilde{G}_{\rm E}(\omega_n, k) \to 0$ as $k \to 0$ and $\omega_n \neq 0$, so actually

$$\tilde{G}_{\rm E}(\omega_n, k) \sim \frac{k^2 \langle \mathcal{O}_4 \rangle}{\omega_n^4}$$

iii. From the dispersive representation:

$$\tilde{G}_{\rm E}(\omega_n, k) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \omega \frac{\rho(\omega, k)}{\omega^2 + \omega_n^2} \xrightarrow{\omega_n \to \infty} \frac{1}{\pi \omega_n^2} \int_0^\infty \mathrm{d}\omega\, \omega\, \rho(\omega, \mathrm{k})$$

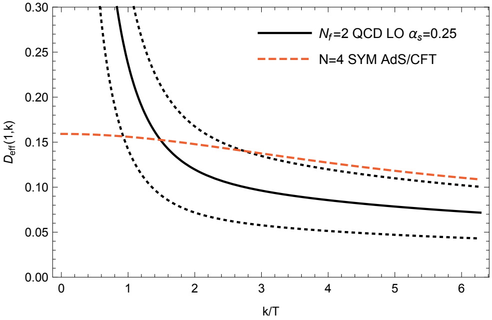
The two expressions are only compatible if the super-convergent sum rule

$$\boxed{\int_0^\infty \mathrm{d}\omega\, \omega \rho(\omega, \mathrm{k}) = 0}$$

holds.

Summary: properties of $\rho(\omega,k) \equiv \rho(\omega,k,-2)$

- non-negative for $\omega \leq k$
- $\rho(\omega,k) \overset{\omega\to\infty}{\sim} k^2/\omega^4$
- sum rule: $\int_0^\infty d\omega\, \omega\rho(\omega,\mathrm{k}) = 0$ (so $\rho(\omega,k)$ must go negative somewhere for $\omega > k$)
- define $D_{\rm eff}(\xi,k) \equiv \frac{\xi\rho(\xi k,k)}{4\chi_s k}$ which tends to D in the limit $k \to 0$ at fixed $\xi = \omega/k$ (inspired by Ghilghieri, Kaczmarek, Laine, F. Meyer 1604.07544).
- $D_{\rm eff}(1,k) \propto$ photon rate.

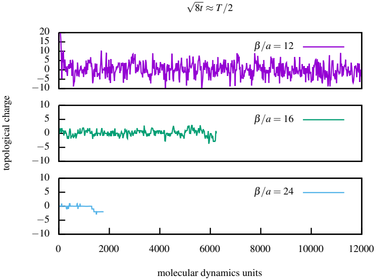


Results from Arnold, Moore, Yaffe hep-ph/0111107 (JHEP); AdS/CFT from hep-ph/0607237.

Lattice set-up with $N_\mathrm{f} = 2$ $\mathrm{O}(a)$-improved Wilson fermions

T (MeV)	T/T_c	β_LAT	β/a	L/a	$m_{\overline{\mathrm{MS}}(2\,\mathrm{GeV})}$ (MeV)	N_meas
250	1.2	5.3	12	48	12	8256
"	"	5.5	16	64	"	4880
"	"	5.83	24	96	"	1680
500	2.4	6.04	16	64	"	8064

- enables continuum limit at $T = 250$ MeV



- further investigation of autocorrelation of topological charge required.

Continuum limit 1/3

There are four independent discretizations of the $\lambda = -2$ isovector vector correlator

$$G^{\lambda=-2}(\tau, \boldsymbol{k}) = \left(\delta^{ij} - \frac{3k^i k^j}{k^2}\right) G^{ij}(\tau, \boldsymbol{k}) + 2G^{00}(\tau, \boldsymbol{k})$$

where $G^{\mu\nu}(\tau, \boldsymbol{k}) = \int \mathrm{d^3xe^{-ikx}} \langle \mathrm{V}^\mu(\tau, \mathrm{x}) \mathrm{V}^\nu(0) \rangle$ using both the local or exactly-conserved lattice vector current

In the local-conserved case, there are two discretizations possible by defining the local current on the link, or the conserved current on the site

$$G^{ij}(\tau + a/2, \boldsymbol{k}) = \frac{1}{2}\Big(G^{ij}(\tau, \boldsymbol{k}) + G^{ij}(\tau + a, \boldsymbol{k})\Big)$$
$$G^{00}(\tau, \boldsymbol{k}) = \frac{1}{2}\left(G^{00}(\tau - a/2, \boldsymbol{k}) + G^{00}(\tau + a/2, \boldsymbol{k})\right)$$

Project to all spatial momenta, on and off-axis, with $k\beta \leq 2\pi$

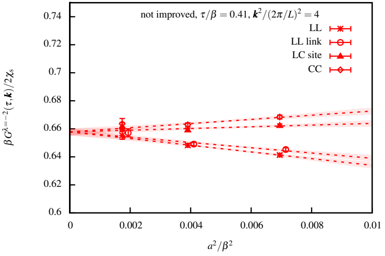
Continuum limit 2/3

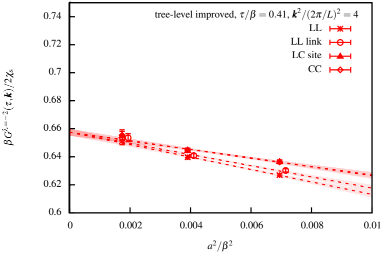
In the chirally-symmetric phase, the matrix-elements of the $\mathrm{O}(a)$-improvement counterterms are suppressed, so we perform a continuum limit in a^2/β^2

Instead we perform tree-level improvement, defined via

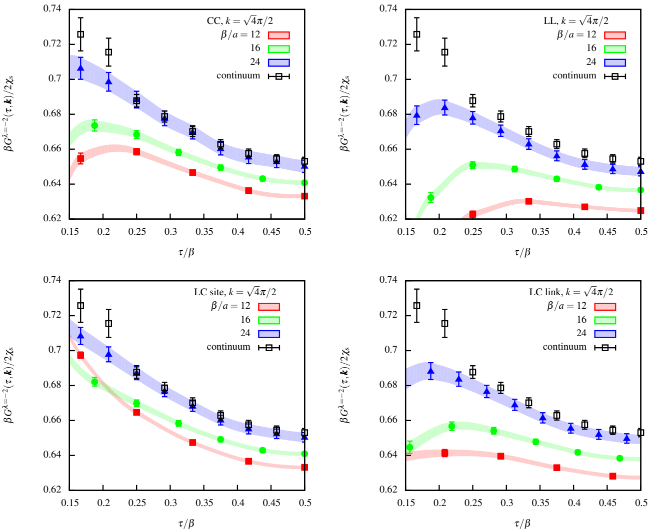
$$G^{\lambda=-2}(\tau,\boldsymbol{k}) \to \frac{G^{\lambda=-2}_{\text{cont.t.l.}}(\tau,\boldsymbol{k})}{G^{\lambda=-2}_{\text{lat.t.l}}(\tau,\boldsymbol{k})} G^{\lambda=-2}(\tau,\boldsymbol{k})$$

A piecewise spline interpolation is used before taking the combined continuum limit of the four discretizations of $\beta G^{\lambda=-2}(\tau,\boldsymbol{k})/\chi_s$





Continuum limit 3/3

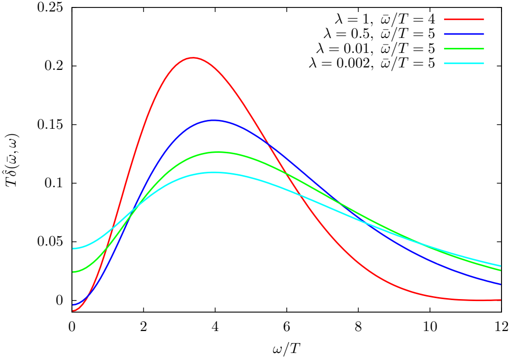


Precision on continuum correlator: about 1%.

Analysis 1: the Backus-Gilbert method

Linearity:
$$\sum_{i=1}^{n} c_i(\bar\omega)\, G(t_i) = \int_0^\infty \frac{d\omega}{2\pi}\,\rho(\omega)\,\underbrace{\sum_{i=1}^{n} c_i(\bar\omega)\,\frac{\cosh[\omega(\beta/2-t_i)]}{\sinh[\omega\beta/2]}}_{\widehat{\delta}(\bar\omega,\omega)}$$

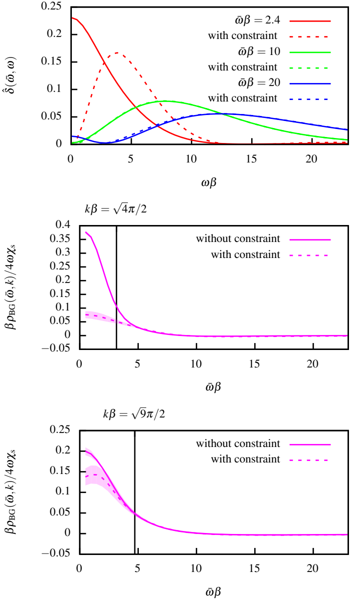
- choose the coefficients $c_i(\bar\omega)$ so that the 'resolution function' $\widehat{\delta}(\bar\omega,\omega)$ is as narrowly peaked around a given frequency $\bar\omega$ as possible
  (idea behind the Backus-Gilbert method, [used in Robaina et al. PRD 92 (2015) 094510.])



Resolution function at $\bar\omega = 4T$ for $N_t = 24$, $t_i/a = 5, \dots 12$.

- Resolution only improves slowly with increasing n
- Large, sign-alternating coefficients $\Rightarrow$ need for ultra-precise input data.

Backus-Gilbert method 2/3



← resolution function $\hat{\delta}(\bar{\omega},\omega)$

acts like a smearing kernel

a linear constraint $\hat{\delta}(\bar{\omega}=0,\omega)=0$ removes contributions from $\rho(\omega=0,k)$

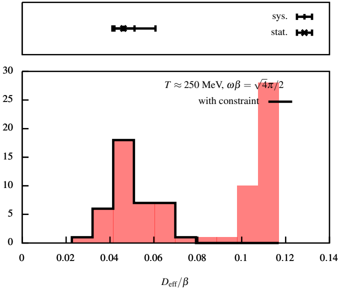
← spectral function $\rho_{\rm BG}(\bar{\omega},k)$

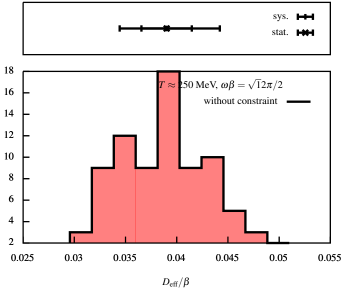
at $k\beta \gtrapprox \pi$, the photon rate is consistent with or without the constraint

Backus-Gilbert method 3/3

Estimate a systematic uncertainty by repeating with many variations.

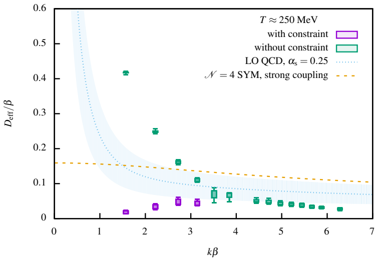
variation	values
τ_{min}/β	$\{0.1, \ldots, 0.25\}$
extra constraint	{yes, no}
α (regularization)	$\{10^{-2}, \cdots, 10^{-4}\}$
tree-level improved	{yes, no}
discretization (@ $T = 500$ MeV)	{LL, LC site, LC link, CC}



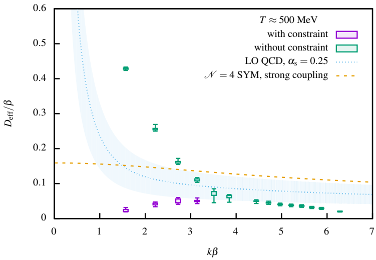


Preliminary results from the BG method

T=250MeV



T=500MeV



Results display virtually no visible temperature effects

Inverse problem appears to be controlled when $k\beta > \pi$

Improved momentum resolution using on- and off-axis momenta

Analysis 2: Padé fit ansatz

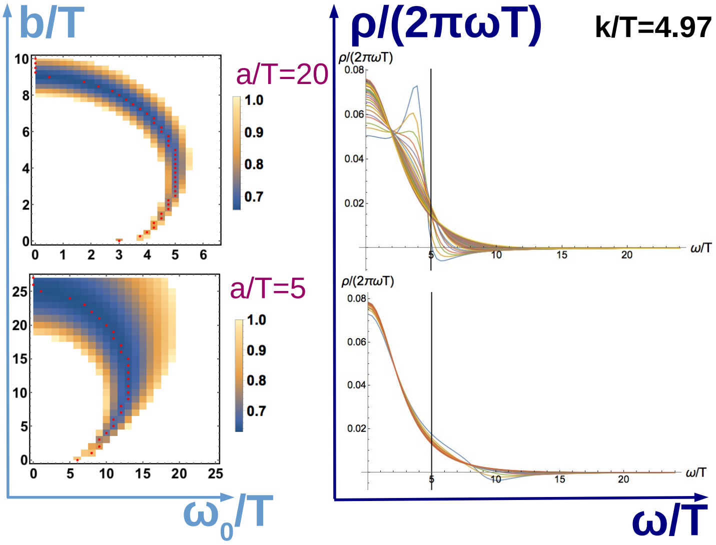
$$\frac{\rho(\omega,k)}{\tanh[\omega\beta/2]} = \frac{A(1+B\omega^2)}{[(\omega-\omega_0)^2+b^2][(\omega+\omega_0)^2+b^2][\omega^2+a^2]},$$

- $\rho(\omega,k) \sim 1/\omega^4$ at large ω (consistent with OPE);
- sum rule $\Rightarrow B = B(a,b,\omega_0)$;
- four-parameter fit (one linear, three non-linear);
- at small k, expect $a = Dk^2$ and $(\omega_0, b) = \mathrm{O}(T)$;
- it turns out that the χ^2 has a flat valley $\Rightarrow$ scan in the non-linear parameters (a, b, ω_0).
- accept all solutions that satisfy:
  1. $\rho(\omega,k) \geq 0$ for $\omega \leq k$;
  2. $\chi^2/\mathrm{d.o.f.} \leq 1$ (keeping only diagonal part of covariance matrix)
  3. "there can be no arbitrarily long relaxation times": $\min(a,b) > \min(D_{\mathrm{AdS/CFT}}k^2, D_{\mathrm{pert}}^{-1})$

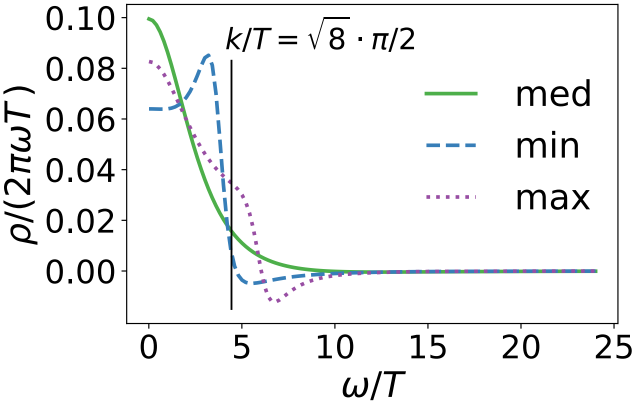
$$D_{\mathrm{AdS/CFT}} = \tfrac{1}{2\pi T}, \qquad D_{\mathrm{pert}}^{-1} = \mathrm{O}(\alpha_s^2)T = 0.46T, \qquad \alpha_s = 0.25.$$

Arnold, Moore, Yaffe hep-ph/0302165

Padé fit to the spectral function - uncorrelated $\chi^2(\omega_0, b)$-landscape

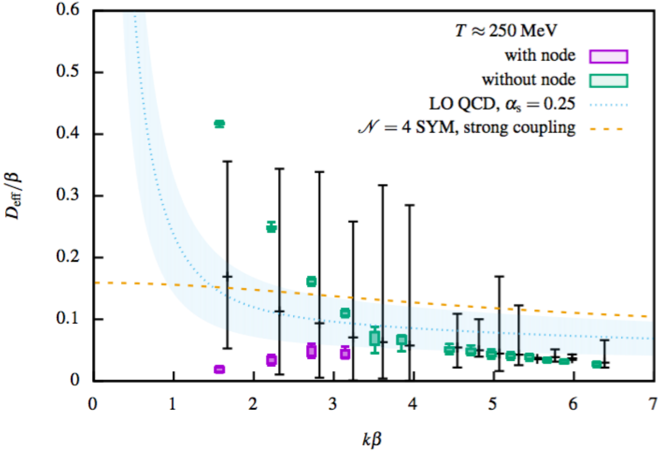


Spectral functions from Padé fit



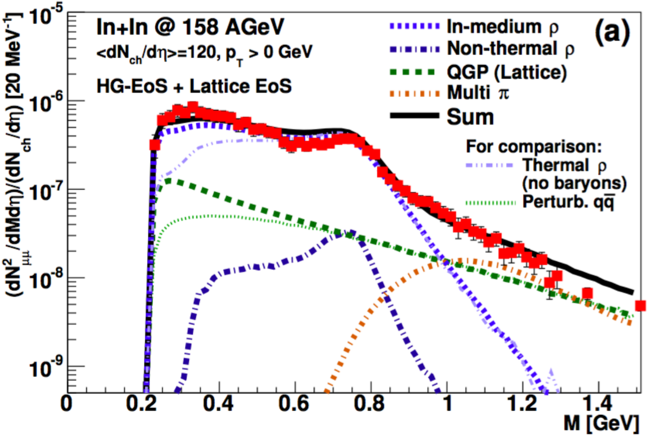
- all three describe the lattice data, fullfill the positivity requirement and do not have singularities too close to the real axis.

Result at $T = 250\,$MeV (preliminary)



- black: from Padé fit; purple, green: Backus-Gilbert method.
- to do: influence of using $\chi^2_{\rm correlated}$
- to do: global fit for all k.

Outlook: dilepton spectrum in heavy-ion collisions



NA60, AIP Conf.Proc. 1322, 1 (2010); model by Endres et al. PhysRevC 91 (2015) 054911.

Bridging the gap btw lattice & heavy-ion collisions? (my theorist's view)

Production rate of dileptons (invariant mass $M = \sqrt{\mathcal{K}^2}$) per unit volume of fluid at rest: ($\mathcal{K}^2 = \omega^2 - \boldsymbol{k}^2$; neglecting m_ℓ)

$$\frac{d\Gamma_{\ell^-\ell^+}(T,\mathcal{K}^2,\omega)}{d^4\mathcal{K}} = \frac{\alpha^2\,(-\rho^\mu{}_\mu(T,\omega,k))}{6\pi^3\mathcal{K}^2(e^{\beta\omega}-1)}$$

Integrate over all possible energies ω of the timelike photon (relative to fluid rest-frame) at fixed $\mathcal{K}^2 = M^2$:

$$\frac{d\Gamma_{\ell^-\ell^+}(T,M^2)}{dM} = \frac{2M\alpha^2}{3\pi^2}\int_0^\infty d\eta\,\frac{\sinh^2\eta}{e^{\beta M\cosh\eta}-1}(-\rho^\mu{}_\mu)(T, M\cosh\eta, M\sinh\eta).$$

Experimental dilepton spectrum (after subtraction of non-thermal sources):

$$\frac{dN}{dM}(\sqrt{s},M) = \int_{T_{\rm min}}^{T_{\rm max}} dT\,\underbrace{w(\sqrt{s},T)}_{\text{from hydro}}\,\frac{d\Gamma_{\ell^-\ell^+}(T,M^2)}{dM}$$

where $dT\,w(\sqrt{s},T)$ is the volume of fluid at temperature T created during the collision, integrated over time.

Making contact with HIC dilepton spectra

Weighted integral over the experimental dilepton spectrum (e.g. $f(M) \sim \exp(-(M-\bar{M})^2/(2\Delta^2))$), corresponds to (returning to the (ω, k) variables):

$$\int_0^\infty \frac{dN}{dM}(\sqrt{s}, M)\, f(M)\, dM$$
$$= \frac{2\alpha^2}{3\pi^2} \int_{T_{\rm min}}^{T_{\rm max}} dT\, w(\sqrt{s}, T) \int_0^\infty dk\, k^2 \int_k^\infty \frac{d\omega}{e^{\beta\omega}-1}\, \frac{f(\sqrt{\omega^2-k^2})}{\omega^2-k^2}\, (-\rho^\mu{}_\mu)(T,\omega,k).$$

Determine coefficients $c_i(T,k)$ so as to minimize the $L^2[0,\infty]$ norm of

$$\frac{\theta(\omega^2-k^2)}{e^{\beta\omega}-1}\, \frac{f(\sqrt{\omega^2-k^2})}{\omega^2-k^2} - \sum_{i=1}^n c_i(T,k)\, \frac{\cosh[\omega(\beta/2 - x_0^{(i)})]}{\sinh[\omega\beta/2]}\;;$$

then

$$\int_0^\infty \underbrace{\frac{dN}{dM}(\sqrt{s}, M)}_{\rm experiment}\, f(M)\, dM$$
$$\simeq \frac{2\alpha^2}{3\pi^2} \int_{T_{\rm min}}^{T_{\rm max}} dT\, \underbrace{w(\sqrt{s}, T)}_{\rm hydro} \int_0^\infty dk\, k^2 \sum_{i=1}^n c_i(T,k)\, \underbrace{(-G^\mu{}_\mu)(T, x_0^{(i)}, k)}_{\rm lattice}.$$

Conclusion

- Photon rate: first lattice calculation in dynamical QCD with continuum limit.
- Chose a favorable linear combination; super-convergent sum rule.
- Result stable for $k > \pi T$, compatible with weak-coupling prediction.
- Dilepton rate: the additional variable M^2 offers some flexibility to make contact with experiment.

Screening masses: static and non-static

Consider perturbating the Hamiltonian,

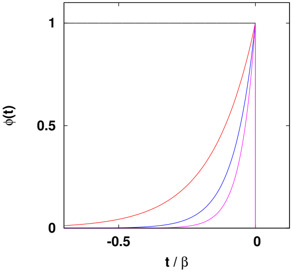
$$\hat{H}_\phi(t) = \hat{H} - \int d^3y \, \phi(t,\boldsymbol{y})\hat{J}(t,\boldsymbol{y}),$$

with the external perturbation given by

$$\phi(t,\boldsymbol{y}) = \delta(\boldsymbol{y})e^{\omega t}\theta(-t), \qquad \omega \geq 0.$$

Linear response $\Rightarrow$

$$\delta\langle J(t=0,\boldsymbol{x})\rangle = \underbrace{G_E^{JJ}(\omega_n,\boldsymbol{x})}_{\text{Euclidean corr.}}, \qquad \text{for } \omega = \omega_n = 2\pi T n.$$



Correlation length in Matsubara sector ω_n = length scale over which a perturbation with the time dependence $e^{\omega_n t}$ is screened $(n \geq 0)$.

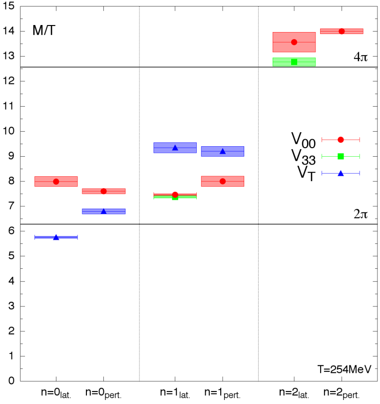
Screening masses at high temperatures

Weak-coupling picture of flavor-non-singlet screening masses:

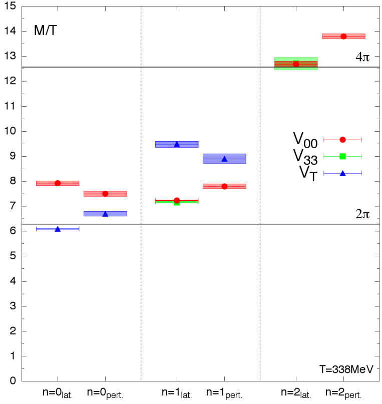
- fermions have an effective mass of at least $\pi T \Rightarrow$ dimensional reduction
- they form non-relativistic, 2+1d bound states of size $\mathrm{O}(m_E^{-1})$
  Laine, Vepsalainen hep-ph/0311268
- expect bound state to be described by a Schrödinger equation in 2+1d.
- Non-static sector: potential has a connection with an effective potential used in the calculation of the dilepton production rate

  [Aurenche, Gelis, Moore, Zakaret hep-ph/0211036; Caron-Huot 0811.1603; Panero, Rummukainen, Schäfer 1307.5850].

Vector screening masses: lattice vs. EFT



$T = 254$ MeV



$T = 340$ MeV

Satisfactory agreement between lattice QCD and the EFT predictions.

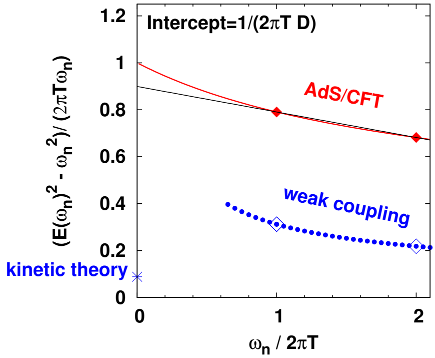
Brandt et al. 1404.2404; $N_t = 16$ and $N_t = 12$, $N_s = 64$; $m_\pi|_T = 0 = 270$MeV

Non-static screening masses and transport coefficients

Linear response along with a constitutive equation for the vector current $\boldsymbol{J} \Rightarrow$

$$G_E^{J_0 J_0}(\omega_n, k) \overset{\omega_n, k\to 0}{=\!=\!=} \frac{\chi_s D k^2}{\omega_n + D k^2} \qquad \Rightarrow \qquad E(\omega_n)^2 \overset{\omega_n \to 0}{\sim} \frac{\omega_n}{D}.$$

χ_s = static susceptibility, D = diffusion coefficient, $E(\omega_n)$ = screening mass in sector ω_n



In the limit $T \to \infty$, extrapolating the screening masses in the lowest Matsubara sectors to $\omega_n = 0$ gives the correct result, $1/(T\,D) = 0$.

Brandt, Francis, Laine, HM 1408.5917; Kinetic theory: Arnold, Moore & Yaffe hep-ph/0111107