Relativistic magneto-hydrodynamics

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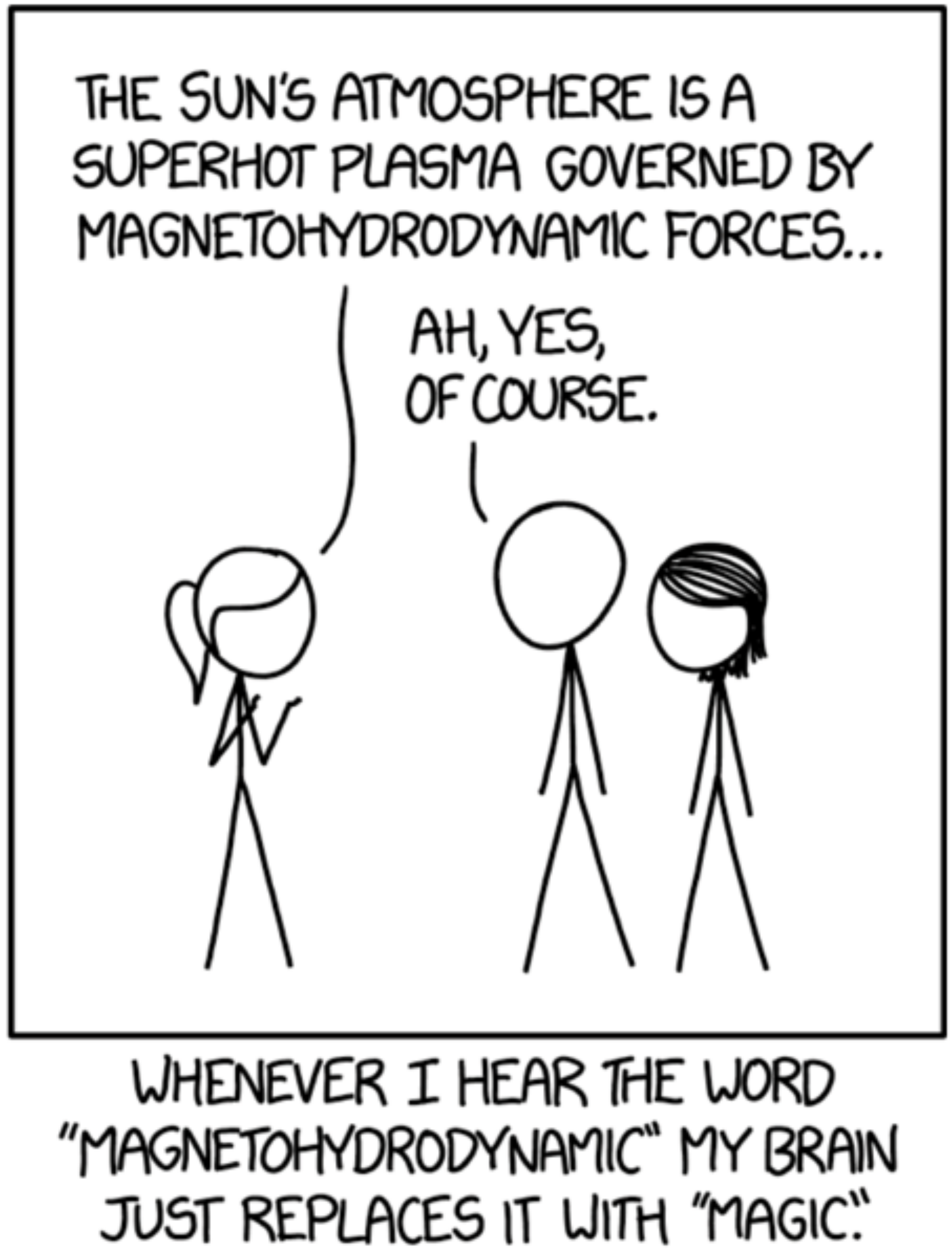
based on arXiv:1606.01226, 1703.08757

July 14, 2017
Canterbury tales of hot QFTs, Oxford

Motivation



https://xkcd.com/1851/

Motivation

To understand things I missed as a student:

What is pressure?

What are Maxwell’s equations in matter?

Motivation

I was talking about the same subject here at Oxford last year, and may have made some ~~incorrect~~ incomplete statements.

Hopefully, today's talk will be an improvement.

Definitions

Hydrodynamics = whatever happens when stuff flows

Magneto = whatever happens when stuff is placed
in E, B fields

Image: http://sachikokodama.com

What is MHD?

Classical low-energy effective theory for systems with U(1) gauge fields and locally in thermal equilibrium.

Vanilla MHD:

Take Navier-Stokes

Add Maxwell's eq-s in matter

Set E_i to zero, allow B_i non-zero

What is MHD?

Want:

- Thermodynamics in electromagnetic fields
- Navier-Stokes modified by “bound” charges/currents
- Systematic derivative expansion
- Include electric fields
- Classify transport coefficients
- Connect with microscopics through Kubo formulas
- Statistical fluctuations
- …

How hydro equations are solved in practice

1. Write down hydro equations (Eckart, Landau-Lifshitz)

2. Mess with the eq-s to make them causal (Israel-Stewart)

3. Find the right variables suitable for numerics

4. Solve equations numerically

5. Study interesting physics

How hydro equations are solved in practice

1. Write down hydro equations (Eckart, Landau-Lifshitz)
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3. Find the right variables suitable for numerics
4. Solve equations numerically
5. Study interesting physics

Until recently, no consensus on step 1 for relativistic MHD

Outline

I. Thermodynamics

II. Hydro with fixed E & B

III. Hydro with dynamical E & B

Equilibrium in external fields

Add external time-independent $g_{\mu\nu}$, A_μ

Compute $W = -i \ln Z[g_{\mu\nu}, A_\mu]$

Local correlations $\implies$ $W[g,A] = \int d^{d+1}x \sqrt{-g}\, \mathcal{F}(g,A)$

Near-uniform fields $\implies$ expand $\mathcal{F}(g,A)$ in derivatives of g,A

Leading order $\implies$ $\mathcal{F}(g,A) = P + O(\partial)$

BBBJMS arXiv:1203.3544
JKKMRY arXiv:1203.3556

Response to external sources

$$W[g, A] = \int_{\mathcal{M}} d^{d+1}x \, \sqrt{-g} \, \mathcal{F}(g, A) + \int_{\partial\mathcal{M}} d^d x \, \sqrt{\gamma} \, L(g, A, n)$$

electromagnetic fields

gravitational field,
rotation

surface tension,
boundary effects

Thermodynamic variables

Timelike Killing vector V^μ, e.g. $V^\mu = (1,\mathbf{0})$ for matter "at rest"

$$T = \frac{1}{\beta_0\sqrt{-V^2}}, \qquad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}, \qquad \mu = \frac{V^\mu A_\mu + \Lambda_V}{\sqrt{-V^2}}$$

JLY arXiv:1310.7024

Definition of electric and magnetic fields:

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma$$

Equilibrium relations

$$u^\lambda \partial_\lambda T = 0\,, \qquad u^\lambda \partial_\lambda \mu = 0$$

things don't depend on time

$$a_\lambda = -\partial_\lambda T / T$$

gravitational potential induces temperature gradient

$$E^\alpha - T\Delta^{\alpha\beta}\partial_\beta \left(\frac{\mu}{T}\right) = 0$$

electric field induces charge gradient: this is electric screening

$$\nabla_\mu u_\nu = -u_\mu a_\nu - \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\, u^\alpha \Omega^\beta$$

shear and expansion vanish in equilibrium

$$a^\mu \equiv u^\lambda \nabla_\lambda u^\mu$$
$$\Omega^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha u_\beta$$

Polarization vectors

$$\delta_F W[g,A] = \tfrac{1}{2}\int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\,M^{\mu\nu}\,\delta F_{\mu\nu} + \int_{\partial\mathcal{M}} \ldots$$

polarization tensor

Definition of polarization vectors:

$$M_{\mu\nu} = p_\mu u_\nu - p_\nu u_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho m^\sigma$$

electric polarization magnetic polarization

$$\delta_F W[g,A] = \int_{\mathcal{M}} d^{d+1}x\,\sqrt{-g}\,(p_\mu\,\delta E^\mu + m_\mu\,\delta B^\mu) + \int_{\partial\mathcal{M}} \ldots$$

Polarization vectors

If you are lost, all this was just a covariant way of saying

$$\delta F = \int d^3x \; (\mathbf{p}\cdot\delta\mathbf{E} + \mathbf{m}\cdot\delta\mathbf{B})$$

in equilibrium, defining **p** and **m**.

Note the ambiguity:

$$F = F + 0 = F + \int d^3x \; \big(X(\mathbf{x})\,\nabla\cdot\mathbf{B} + \mathbf{Y}(\mathbf{x})\cdot(\nabla\times\mathbf{E})\big)$$

$$\mathbf{p} \to \mathbf{p} - \nabla\times\mathbf{Y} \;, \qquad \mathbf{m} \to \mathbf{m} - \nabla X$$

Thus **p** and **m** not uniquely defined

Bound charges and bound currents

Define charge density and spatial current:

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$\mathcal{N}$: charge density

$\mathcal{J}^\mu$: spatial current, orthogonal to u_μ

Take the variation $\delta_A W[g, A]$:

$$J^\mu = \rho u^\mu - \nabla_\lambda M^{\lambda\mu}$$

ρu^μ: "free charges", $\rho \equiv \partial \mathcal{F} / \partial \mu$

$\nabla_\lambda M^{\lambda\mu}$: "bound charges"

$$\mathcal{N} = \rho - \nabla_\mu p^\mu + p^\mu a_\mu - m_\mu \Omega^\mu$$

$$\mathcal{J}^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho m_\sigma + \epsilon^{\mu\nu\rho\sigma} u_\nu a_\rho m_\sigma$$

a_μ = acceleration
Ω_μ = vorticity

Bound charges and bound currents

Define charge density and spatial current:

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

charge density

spatial current, orthogonal to u_μ

Take the variation $\delta_A W[g, A]$:

$$J^\mu = \rho u^\mu - \nabla_\lambda M^{\lambda\mu}$$

"free charges" $\rho \equiv \partial \mathcal{F}/\partial\mu$

"bound charges"

$$n = \rho - \boldsymbol{\nabla}\cdot\mathbf{p} - \mathbf{p}\cdot\boldsymbol{\nabla}T/T - 2\mathbf{m}\cdot\boldsymbol{\omega}$$

$$\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{m} + \mathbf{m}\times\boldsymbol{\nabla}T/T$$

Bound charges and bound currents

Define charge density and spatial current:

$$J^{\mu} = \mathcal{N} u^{\mu} + \mathcal{J}^{\mu}$$

charge density

spatial current, orthogonal to u_{μ}

Take the variation $\delta_A W[g,A]$:

$$J^{\mu} = \rho u^{\mu} - \nabla_{\lambda} M^{\lambda\mu}$$

"free charges" $\rho \equiv \partial \mathcal{F}/\partial \mu$

"bound charges"

Equilibrium surface charge and current:

$$\sigma_s = \mathbf{p}\cdot\mathbf{n} + O(\partial) \qquad \mathbf{j} = \mathbf{m}\times\mathbf{n} + O(\partial)$$

These were equilibrium charges and currents.

Now need to find equilibrium $T^{\mu\nu}$.

For that, need the derivative expansion.

Derivative expansion

$$W[g, A] = \int \sqrt{-g}\, p + O(\partial)$$

How do we count derivatives?

Clearly, $g_{\mu\nu}$, $T \sim O(1)$

In equilibrium, $E^\alpha - T\Delta^{\alpha\beta}\partial_\beta \left(\frac{\mu}{T}\right) = 0$

So if $\mu \sim O(1)$, then $E \sim O(\partial)$. This is screening.

No similar constraint on B, can take $B \sim O(1)$

Derivative expansion

$$W[g,A] = \int \sqrt{-g}\, p + O(\partial)$$

Insulator: p=p(T, E^2, B^2, E·B)

Conductor: p=p(T, μ, B^2)

In between: p=p(T, μ, E^2, B^2, E·B)

Example: $T^{\mu\nu}$ in external E,B fields

Leading order: $\mathcal{F} = p(T, \mu, E^2, B^2, E{\cdot}B)$

Calculate $T^{\mu\nu}$:

$$T^{\mu\nu} = p\, g^{\mu\nu} + (Ts + \mu\rho)u^\mu u^\nu + T^{\mu\nu}_{\rm EM}$$

$$T^{\mu\nu}_{\rm EM} = M^{\mu\alpha} g_{\alpha\beta} F^{\beta\nu} + u^\mu u^\alpha \left(M_{\alpha\beta} F^{\beta\nu} - F_{\alpha\beta} M^{\beta\nu}\right)$$

This $T^{\mu\nu}$ is symmetric, first derived for ideal gas — W.Israel, Gen.Rel.Grav. 1978

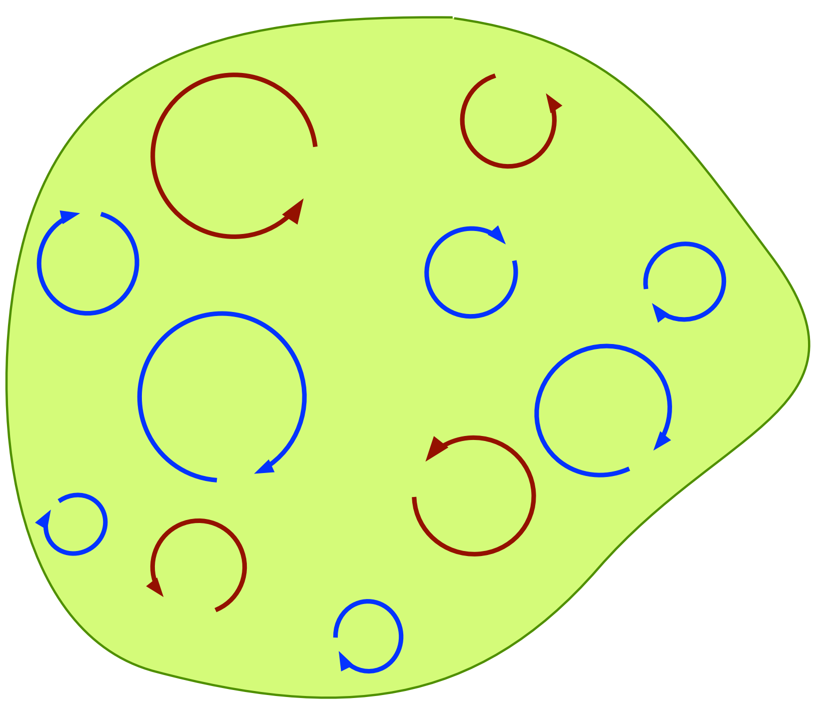
Does not assume any particular microscopic model of matter — PK arXiv:1606.01226

P-invariant conducting fluid in 3+1 dim

Free energy: $\mathcal{F}(g,A) = p(T,\mu,B^2) + M_\Omega(T,\mu,B^2)\, \mathbf{B}\cdot\Omega + O(\partial^2)$

Vary $W[g,A] = \int d^{d+1}x \, \sqrt{-g}\, \mathcal{F}(g,A)$ to find $T^{\mu\nu}{}_{eq}$, $J^{\mu}{}_{eq}$

In constant B-field: $T_s^{\mu\nu} = Q_s^\mu u^\nu + Q_s^\nu u^\mu$, $Q_s^\alpha = M_\Omega \epsilon^{\alpha\mu\nu\rho} u_\mu B_\nu n_\rho$



Angular momentum:

$$\frac{\mathbf{L}}{V} = 2M_\Omega \mathbf{B}$$

Magneto-vortical susceptibility

System at rest
in flat space,
constant B-field:

$$\frac{\mathbf{L}}{V} = 2M_\Omega \mathbf{B}$$

System rotating
in flat space,
no B-field:

$$\mathbf{m} = 2M_\Omega\, \boldsymbol{\omega}$$

Static (zero frequency)
correlation functions:

$$\langle T^{tx} J^z \rangle = -k_x k_z M_\Omega \quad \text{etc.}$$

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I. Thermodynamics

II. Hydro with fixed E & B

III. Hydro with dynamical E & B

Hydro equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$$

$$\nabla_\mu J^\mu = 0$$

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + T^{\mu\nu}_{non\text{-}eq} \ , \qquad J^\mu = J^\mu_{eq} + J^\mu_{non\text{-}eq}$$

get from equilibrium $W[g,A]=\int p + O(\partial)$

e.g. $J^\mu_{eq} = \rho u^\mu - \nabla_\lambda M^{\lambda\mu}$

Hydro equations

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$$

$$\nabla_\mu J^\mu = 0$$

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + T^{\mu\nu}_{non\text{-}eq}\,, \qquad J^\mu = J^\mu_{eq} + J^\mu_{non\text{-}eq}$$

vanish in equilibrium, depend on ∂_μ, B_μ, E_μ, η, ζ, …

Transport coefficients

For P-invariant conducting fluid in 3+1dim:

- one thermodynamic susceptibility M_Ω
- two shear viscosities ($\perp$ and $\|$ to B)
- three bulk viscosities
- two electrical conductivities ($\perp$ and $\|$ to B)
- two Hall viscosities ($\perp$ and $\|$ to B)
- one Hall conductivity

Eleven coefficients total:

1 thermodynamic, non-dissipative
3 non-equilibrium, non-dissipative
7 non-equilibrium, dissipative

Constitutive relations*

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\mathcal{E} = -p + T\,p_{,T} + \mu\,p_{,\mu} + \big(TM_{\Omega,T} + \mu M_{\Omega,\mu} - 2M_\Omega\big)\,B\cdot\Omega\,,$$

$$\mathcal{P} = p - \tfrac{4}{3}\,p_{,B^2}B^2 - \tfrac{1}{3}(M_\Omega + 4M_{\Omega,B^2}B^2)B\cdot\Omega - \zeta_1\nabla\cdot u - \zeta_2 b^\mu b^\nu \nabla_\mu u_\nu\,,$$

$$\begin{aligned}\mathcal{Q}^\mu = &-M_\Omega\epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\sigma B_\rho + (2M_\Omega - TM_{\Omega,T} - \mu M_{\Omega,\mu})\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma T/T \\ &- M_{\Omega,B^2}\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho\partial_\sigma B^2 + (-2p_{,B^2} + M_{\Omega,\mu} - 2M_{\Omega,B^2}B\cdot\Omega)\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma \\ &+ M_\Omega\epsilon^{\mu\nu\rho\sigma}\Omega_\nu E_\rho u_\sigma\,,\end{aligned}$$

$$\begin{aligned}\mathcal{T}^{\mu\nu} = &\,2p_{,B^2}\big(B^\mu B^\nu - \tfrac{1}{3}\Delta^{\mu\nu}B^2\big) + M_{\Omega,B^2}B^{\langle\mu}B^{\nu\rangle}B\cdot\Omega + M_\Omega B^{\langle\mu}\Omega^{\nu\rangle} \\ &- \eta_\perp\sigma_\perp^{\mu\nu} - \eta_\parallel(b^\mu\Sigma^\nu + b^\nu\Sigma^\mu) - b^{\langle\mu}b^{\nu\rangle}\big(\eta_1\nabla\cdot u + \eta_2 b^\alpha b^\beta\nabla_\alpha u_\beta\big) \\ &- \tilde\eta_\perp\tilde\sigma_\perp^{\mu\nu} - \tilde\eta_\parallel(b^\mu\tilde\Sigma^\nu + b^\nu\tilde\Sigma^\mu)\,,\end{aligned}$$

$$\mathcal{N} = p_{,\mu} + M_{\Omega,\mu}B\cdot\Omega - m\cdot\Omega\,,$$

$$\mathcal{J}^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu\nabla_\rho m_\sigma + \epsilon^{\mu\nu\rho\sigma}u_\nu a_\rho m_\sigma + \left(\sigma_\perp\mathbb{B}^{\mu\nu} + \sigma_\parallel\frac{B^\mu B^\nu}{B^2}\right)V_\nu + \tilde\sigma\,\tilde V^\mu$$

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \qquad b^\mu \equiv B^\mu/B$$

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \tfrac{2}{3}\Delta_{\alpha\beta}\nabla\cdot u)$$

$$\tilde\sigma^{\mu\nu} \equiv \frac{1}{2B}\left(\epsilon^{\mu\lambda\alpha\beta}u_\lambda B_\alpha\sigma_\beta{}^\nu + \epsilon^{\nu\lambda\alpha\beta}u_\lambda B_\alpha\sigma_\beta{}^\mu\right)$$

$$\mathbb{B}^{\mu\nu} \equiv \Delta^{\mu\nu} - b^\mu b^\nu \qquad \Sigma^\mu \equiv \mathbb{B}^{\mu\lambda}\sigma_{\lambda\rho}b^\rho$$

$$V^\mu \equiv E^\mu - T\Delta^{\mu\nu}\partial_\nu(\mu/T)$$

$$\tilde v^\mu \equiv \epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho v_\sigma/B$$

$$m^\mu = \big(2\,p_{,B^2} + 2M_{\Omega,B^2}B\cdot\Omega\big)\,B^\mu + M_\Omega\Omega^\mu$$

* In thermodynamic frame, up to O(∂)

Other things

Inequality constraints on η's, ζ's, σ's from 2-nd law

Equality constraints on η's, ζ's, σ's from Onsager relations

Eigenmodes: collective cyclotron modes, sound, diffusion,…

Express η's, ζ's, σ's in terms of $\langle T_{\mu\nu} T_{\alpha\beta} \rangle$, $\langle T_{\mu\nu} J_{\alpha} \rangle$, $\langle J_{\mu} J_{\alpha} \rangle$

Transport coefficients for P-violating fluids

Hernandez, PK, arXiv:1703.08757

Is any of this near-equilibrium relativistic MHD business actually physically relevant?

I don't know. But the following story comes to mind.

In 1948, A.I.Akhiezer and his student L.E.Pargamanik worked out the kinetic waves of plasma in a magnetic field.

Akhiezer showed the results to L.D.Landau and was told: "Where have you seen plasma, moreover in magnetic field?"

As Landau disapproved, the results could only be published in an insignificant journal *Notes of Kharkov University* in 1948.

In 1958, I.B.Bernstein in Princeton didn't ask Landau's opinion, and independently worked out the same waves in Phys.Rev.

These are now called Bernstein waves in all plasma literature.

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I. Thermodynamics

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III. Hydro with dynamical E & B

What are the equations?

E, B external: $\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$, $\nabla_\mu J^\mu = 0$

E, B dynamical: $\nabla_\mu T^{\mu\nu} = 0$, $J^\mu = 0$

these are Maxwell's equations

Generating functional W[g,A] = effective action S[g,A]

$$W[g,A] = \int d^{d+1}x \sqrt{-g}\, \mathcal{F}(g,A)$$

$$\mathcal{F} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{F}_m$$

definition of $\mathcal{F}_m$: includes pressure, polarization, derivative expansion

Maxwell's equations in matter

$J^\mu = 0$ is same eqn as $\nabla_\nu H^{\mu\nu} = \rho u^\mu$

$$H^{\mu\nu} \equiv F^{\mu\nu} - M_m^{\mu\nu} = u^\mu D^\nu - u^\nu D^\mu - \epsilon^{\mu\nu\rho\sigma} u_\rho H_\sigma$$

defines D^μ, H^μ

Example: $\mathcal{F}_{\rm m} = p_{\rm m}(T, \mu, E^2, B^2, E{\cdot}B)$ gives the standard

$$D^\mu = \varepsilon_{\rm m} E^\mu + \beta_{\rm m} H^\mu$$

$$B^\mu = \beta_{\rm m} E^\mu + \mu_{\rm m} H^\mu$$

$\varepsilon_{\rm m} \equiv 1 + \chi_{\rm EE} + \chi_{\rm EB}^2/(1 - \chi_{\rm BB})$

$\mu_{\rm m} \equiv 1/(1 - \chi_{\rm BB})$

$\beta_{\rm m} \equiv \chi_{\rm EB}/(1 - \chi_{\rm BB})$

$\chi_{\rm EE} \equiv 2\partial p_{\rm m}/\partial E^2$

$\chi_{\rm EB} \equiv \partial p_{\rm m}/\partial(E{\cdot}B)$

$\chi_{\rm BB} \equiv 2\partial p_{\rm m}/\partial B^2$

MHD equations

Assume that E & B change slowly so that $T^{\mu\nu}$ and J^{μ} keep the same form as for external E & B

Adopt derivative counting B~O(1), E~O(∂)

Equilibrium action $W[g,A] = \int \sqrt{-g}\,\left(-\frac{1}{2}B^2 + p_{\rm m}(T,\mu,B^2) + M_\Omega(T,\mu,B^2)B\cdot\Omega\right)$

$$\nabla_\mu(T^{\mu\nu}_{eq} + T^{\mu\nu}_{non\text{-}eq}) = 0, \quad J^\mu_{eq} + J^\mu_{non\text{-}eq} = 0, \quad \varepsilon^{\mu\nu\alpha\beta}\nabla_\nu F_{\alpha\beta} = 0$$

Just for fun, add $\frac{1}{2}\varepsilon_{\rm e}E^2$ to the action

This gives MHD eqs that we can do something with

MHD transport coefficients

Compared to hydro in fixed, non-dynamical B-field:

- MHD has the same 11 transport coefficients
- MHD has the same entropy current
- MHD has the same Kubo formulas for viscosities
- MHD has *different* Kubo formulas for conductivities

$$\frac{1}{\omega}\,\mathrm{Im}\,G^{\text{ret.}}_{E_z E_z}(\omega,\mathbf{k}{=}0) = \rho_{\parallel}$$

$$\frac{1}{\omega}\,\mathrm{Im}\,G^{\text{ret.}}_{E_x E_x}(\omega,\mathbf{k}{=}0) = \rho_{\perp}$$

$$\frac{1}{\omega}\,\mathrm{Im}\,G^{\text{ret.}}_{E_x E_y}(\omega,\mathbf{k}{=}0) = -\tilde{\rho}_{\perp}\mathrm{sign}(B_0)$$

$$\sigma_{ab} \equiv \sigma_{\perp}\delta_{ab} + \tilde{\sigma}\epsilon_{ab}$$

$$(\sigma^{-1})_{ab} = \rho_{\perp}\delta_{ab} + \tilde{\rho}_{\perp}\epsilon_{ab}$$

$$\rho_{\parallel} \equiv 1/\sigma_{\parallel}$$

Eigenmodes: $n_0=0$, $B_0\neq0$

Gapped modes: $$\omega = -\frac{i\sigma_\parallel}{\varepsilon_\mathrm{e}} + O(k^2)\,, \qquad \omega = -\frac{i\sigma_\perp \pm \tilde\sigma}{\varepsilon_\mathrm{e}} + O(k^2)$$

Alfvén waves: $$\omega = \pm v_\mathrm{A} k \cos\theta - \frac{i\Gamma_\mathrm{A}}{2}k^2$$

$$v_\mathrm{A}^2 = \frac{B_0^2}{\mu_\mathrm{m}(\epsilon_0{+}p_0) + B_0^2}$$

$$\Gamma_\mathrm{A} = \frac{1}{\epsilon_0{+}p_0}\left(\eta_\perp \sin^2\theta + \eta_\parallel \cos^2\theta\right) + \frac{1}{\mu_\mathrm{m}}\Big(\rho_\perp \cos^2\theta + \rho_\parallel \sin^2\theta\Big)$$

Magnetosonic waves, two branches: $$\omega = \pm v_\mathrm{ms} k - \frac{i\Gamma_\mathrm{ms}}{2}k^2$$

$$(v_\mathrm{ms}^2)^2 - v_\mathrm{ms}^2(v_A^2 + v_s^2 - v_A^2 v_s^2 \sin^2\theta) + v_A^2 v_s^2 \cos^2\theta = 0\,, \qquad v_s^2 = \partial p/\partial\epsilon$$

$$\text{slow: } \Gamma_\mathrm{ms} = \frac{\eta}{\epsilon_0{+}p_0} + \frac{1}{\sigma\mu_\mathrm{m}}\,,$$

$$\text{fast: } \Gamma_\mathrm{ms} = \frac{1}{\epsilon_0{+}p_0}\left(\tfrac{4}{3}\eta + \zeta\right)$$

Eigenmodes: $n_0 \neq 0$, $B_0 = 0$

Set σ=η=ζ=0:

$$\omega^2 = \Omega_p^2 + v_s^2 k^2$$

Relativistic Langmuir oscillations

$$\omega^2 = \Omega_p^2 + \frac{k^2}{\varepsilon_\mathrm{e}\mu_\mathrm{m}}$$

$$\Omega_p^2 = \frac{n_0^2}{(\epsilon + p)\varepsilon_e}$$

relativistic "plasma frequency"

Turn on σ, η, ζ:

$$\omega\left(\omega + \frac{i\sigma_\parallel}{\varepsilon_\mathrm{e}}\right) = \Omega_p^2$$

Damped Langmuir oscillations

$$\omega\left(\omega + \frac{i(\sigma_\perp \pm i\tilde{\sigma})}{\varepsilon_\mathrm{e}}\right) = \Omega_p^2$$

Damped transverse waves

$$\omega = -\frac{i\eta k^4}{n_0^2 \mu_\mathrm{m}}$$

Shear modes have $\omega \sim -i\eta k^4$, not $-i\eta k^2$

$$\omega = -iDk^2$$

Charge diffusion

Eigenmodes: $n_0 \neq 0$, $B_0 \neq 0$

Gapped modes	$$\omega = \Omega_{p,i}(B_0)$$	six of them, magnetosonic waves gapped out by n_0
Diffusion mode:	$$\omega = -iDk^2$$	D depends on θ, $\sigma_\perp$, $\sigma_\parallel$
Transverse waves:	$$\omega = \pm \frac{B_0 \cos\theta}{n_0 \mu_{\rm m}}\, k^2$$	similar to Alfvén waves, but at non-zero n_0

Conclusions

- Thermodynamics in external fields can be done with W[g,A]

- At leading order in derivatives, simple equilibrium $T^{\mu\nu}$:

$$T^{\mu\nu}_{\rm EM} = M^{\mu\alpha} g_{\alpha\beta} F^{\beta\nu} + u^\mu u^\alpha \left(M_{\alpha\beta} F^{\beta\nu} - F_{\alpha\beta} M^{\beta\nu} \right)$$

- At one-derivative order, get gyromagnetic physics:

$$\frac{\mathbf{L}}{V} = 2M_\Omega \mathbf{B}\,, \qquad \mathbf{m} = 2M_\Omega\, \boldsymbol{\omega}$$

- Screening does not mean E=0, it means E~O(∂)

- MHD has 11 transport coefficients, of which 7 are dissipative

What we haven’t done

Well-posedness of the PDE problem a la Israel-Stewart

Evaluate the full set of transport coefficients in a given model (kinetic theory, holography)

Statistical fluctuations are aggravated by the B field in 2+1 dim.

Better connection with “dual” formulation of MHD (Sašo’s talk)

Implications of the full set of transport coefs for real systems

Thank you!