The unreasonable effectiveness of hydrodynamics in describing nuclear collisions

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Canterbury Tales of Hot QFTs in in the LHC Era

St. John's College, Oxford, July 10-14, 2017

Unreasonable Effectiveness of Hydrodynamics

Prehistory 1980-1982

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- How???
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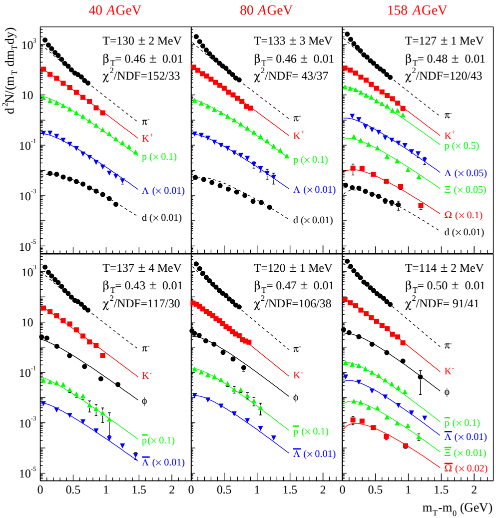
- $\Longrightarrow$ **A decade of frustration!**

Prehistory pre-RHIC (1978-1999):

BUT: There was evidence for flow!
Hydrodynamic flow!
Blast waves!
Mass-splitting of m_T-slopes!
Already at the Bevalac, then again in Si+Si @ AGS and in S+S @ SPS, and finally, stronger, in Au+Au @ AGS and **Pb+Pb @ SPS**:



Prehistory 1990's

- Developed a second, phenomenologically much more successfull research direction: looking for signs of thermal and chemical equilibrium, developing hydrodynamic models

- Mid 1990's: transition from make-believe hydro ("global hydrodynamics") to real hydrodynamic simulations ((1+1)-d $\Longrightarrow$ (2+1)-d $\Longrightarrow$ (3+1)-d), joining other groups (Marburg, Jyväskylä, Frankfurt, Stony Brook, ...)

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- Hydro models yield reasonable qualitative description of p_T-spectra, mass splitting of slopes, HBT radii, but overpredict v_2 @ SPS by factor 2
- But much doubt remained in the community about the meaningfulness of such an approach.

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- Kolb, Sollfrank & Heinz are given the chance for a late entry into the RHIC predictions competition (June 2000):
- "...**If** the QGP medium created at RHIC behaves hydrodynamically, this is what the p_T-spectra and elliptic flow will look like as functions of p_T and $\sqrt{s}$: ..."
  (Prediction based on (2+1)-d ideal fluid dynamics code AZHYDRO)

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- **A bolder prediction would have read:**
- "...We predict that the QGP medium created at RHIC behaves like a fluid, and this is what the p_T-spectra and elliptic flow will look like as functions of p_T and $\sqrt{s}$: ..."

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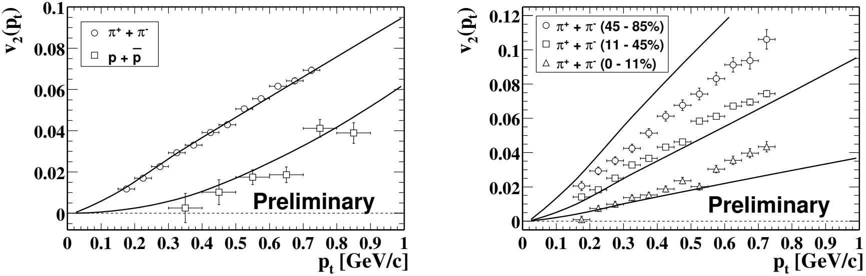
- **Why did we not make this prediction?**
- I thought thermalization was needed for hydro to work, and I knew how hard it was to get rapid thermalization from perturbative QCD, and pQCD was all I knew how to do.

Unreasonable Effectiveness of Hydrodynamics

In June 2000 RHIC turns on, and . . .

Au+Au @ 130 *A* GeV, STAR Collaboration, R. Snellings, Quark Matter 2001
(PRL 87 (2001) 182301; NPA 698 (2002) 193);
curves: AZHYDRO (Kolb, Sollfrank, UH)



VOILA!

Also radial flow: Originally (mis)labelled by Miklos as "the antiproton puzzle", radial flow pushes antiprotons to higher p_T and makes them more abundant than pions at $p_T > 2$ GeV.

Paradigm change

Statistical QCD, Bielefeld, August 2001

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Nuclear Physics A702 (2002) 269c–280c

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Early thermalization at RHIC*

Ulrich Heinz† and Peter Kolb

lower energies it is a bit longer, see Figs. 7, 9 in [6]). At RHIC energies and above, the time it takes the collision zone to dilute from the high initial energy density to the critical value for hadronization is equal to or longer than this saturation time: most or all of the elliptic flow is generated before any hadrons even appear! It thus seems that the only possible conclusion from the successful hydrodynamic description of the observed radial and elliptic flow patterns is that the thermal pressure driving the elliptic flow is partonic pressure, and that the early stage of the collision must have been a thermalized quark-gluon plasma.

Paradigm change

This caused a paradigm change:

- "QGP thermalizes quickly, reaching ≈ local thermal equilibrium after $\tau_{\rm therm} \lesssim 1\,\text{fm}/c$" (not really true!)
- QGP behaves like a liquid, not like a gas (? gasses can behave hydrodynamically!)
- $\Longrightarrow$ (a) The state of matter created at RHIC is actually a QGP, i.e. an approximately thermalized state of quarks and gluons to which one can assign a temperature! (true)
- $\Longrightarrow$ (b) this cannot be understood with perturbative QCD (?)
- $\Longrightarrow$ (c) the QGP must be a strongly coupled plasma (true if correctly interpreted)

IMHO it is still not proven that perturbative QCD calculations, once carried to high enough order, cannot reproduce the strongly coupled collective characteristics of a QGP.

Unreasonable Effectiveness of Hydrodynamics

Enter viscosity

Questions we started to ask ourselves in the early '00s:

- If an ideal fluid picture describes the RHIC data, how well does this work?
- Can we constrain the transport coefficients from experimental data?

The viscosity bound

2001: Dam Son and friends (Policastro, Kovtun, Starinets) use the AdS/CFT correspondence relating 4d conformal field theories in the strongly coupled limit to classical gravity in 5 dimensions to obtain the KSS bound (2005):

$$\left(\frac{\eta}{s}\right)_{\text{anything}} \gtrsim \frac{1}{4\pi} \approx \frac{1}{12} \approx 0.08$$

But was there any room for viscosity in the data?

- Well, it turns out there was: Some early "setbacks" of the ideal hydro picture (using better EOS and implementing chemical freeze-out at T_c) caused ideal fluid dynamics to overpredict v_2 by about 30%. This created room for viscous damping.

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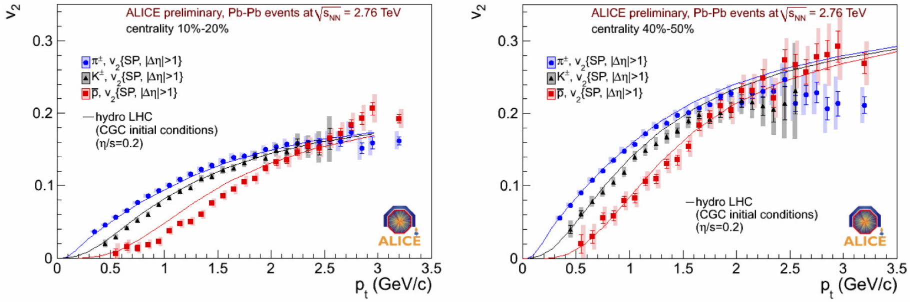
- How to constrain the QGP shear viscosity phenomenologically?
- You need a viscous relativistic hydrodynamic code!
  Such codes started to appear in 2007/2008: UVH2+1, VISH2+1, MUSIC, VISHNU, SONIC, superSONIC, aHydro, vaHydro, GPU-VH, CLVisc, ...

Towards a really predictive theory of relativistic heavy-ion collision dynamics

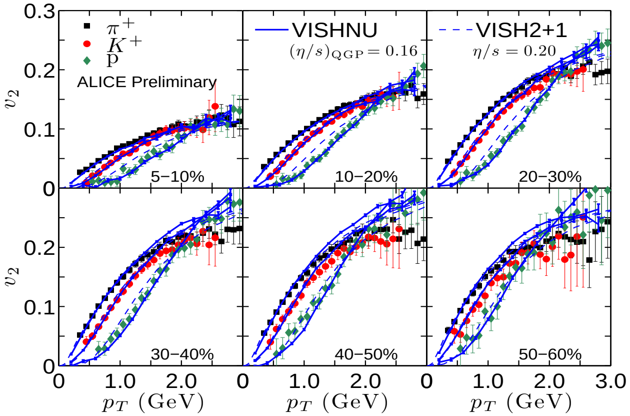
After tuning initial conditions and viscosity at RHIC to obtain a good description of all soft hadron data simultaneously (Song et al. 2010) we successfully predicted the first LHC spectra and elliptic flow measurements:

ALICE, Quark Matter 2011 (VISH2+1 prediction: Shen et al., PRC84 (2011) 044903)



Hybrid (hydro+cascade) approaches work even better:

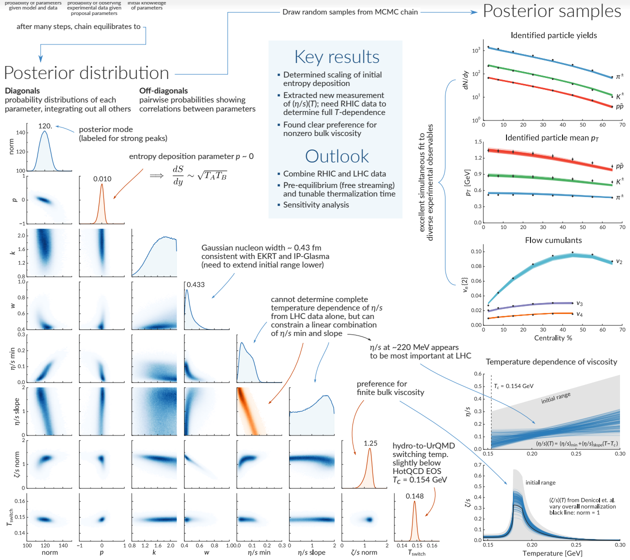
Data: ALICE, preliminary (Snellings, Krzewicki, Quark Matter 2011)
Dashed lines: Shen et al., PRC84 (2011) 044903 (VISH2+1, MC-KLN, $(\eta/s)_{\rm QGP}$=0.2)
Solid lines: Song, Shen, UH 2011 (VISHNU, MC-KLN, $(\eta/s)_{\rm QGP}$=0.16)



VISHNU yields correct magnitude and centrality dependence of $v_2(p_T)$ for pions, kaons **and protons**!

The state of the art

(Bernhard, Moreland, Bass, QM2015)



Questions about the hydrodynamic picture

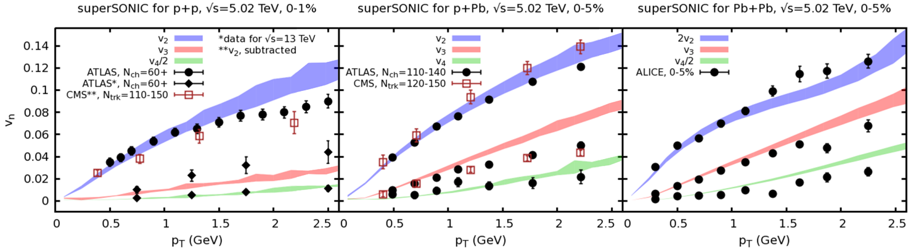
- Why does it work?
- How does it work?
- Where does it stop working?
- What about lower energies? Will it work without creation of a QGP?
- What about smaller collision systems? What is the smallest droplet of strongly interacting matter at a given collision energy that behaves hydrodynamically?
- Can we modify the theory to make it work even better?

Innumerable studies of relativistic viscous fluid dynamics have been made in the last decade; reviewing them and the conclusions they yield would take an entire semester course. Let me pick out a small subset that address the "unreasonable effectiveness" of the hydrodynamic framework that we have witnessed.

Unreasonable Effectiveness of Hydrodynamics

Flow in Pb+Pb, p+Pb and even p+p at the LHC!

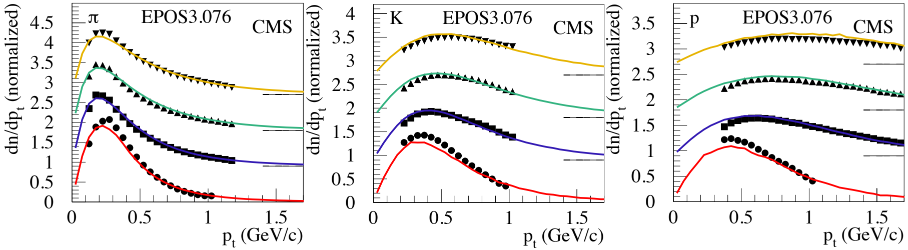
R.D. Weller, P. Romatschke, arXiv:1701.07145



Requires fluctuating proton substructure (gluon clouds clustered around valence quarks (K. Welsh et al. PRC94 (2016) 024919))

Radial flow in pp collisions at the LHC

Werner, Guiot, Karpenko, Pierog (EPOS3), 1312.1233;
Data: CMS Collaboration (8, 84, 160, 235 charged tracks)

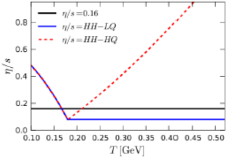


Elliptic flow (double ridge) discovered in high-multiplicity pp by CMS at 7 TeV (and confirmed by ATLAS at 13 TeV) also reproduced by EPOS.

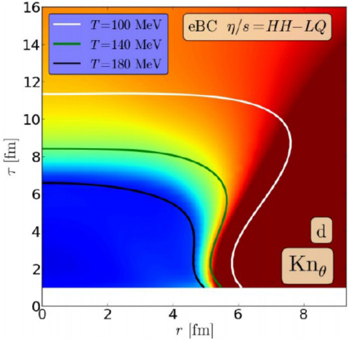
Validity of viscous hydro: Knudsen number check

Niemi & Denicol, arXiv:1404.7327

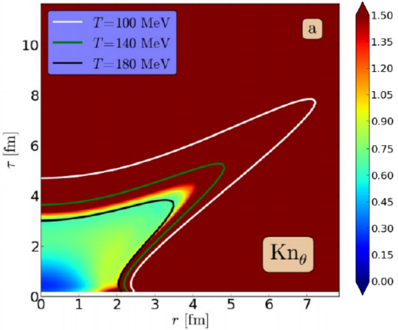
$$\mathrm{Kn} = \tau_{\mathrm{micro}}\,\theta = \tau_{\mathrm{micro}}/\tau_{\mathrm{macro}}$$



Pb+Pb



p+Pb



Earlier freeze-out in p+A than A+A

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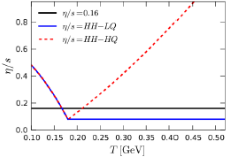

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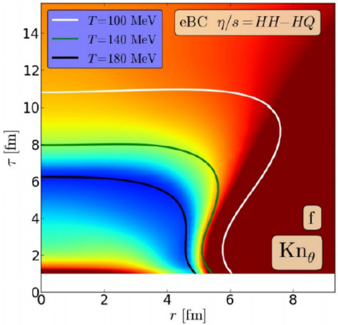

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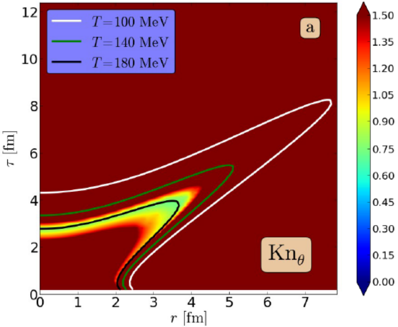
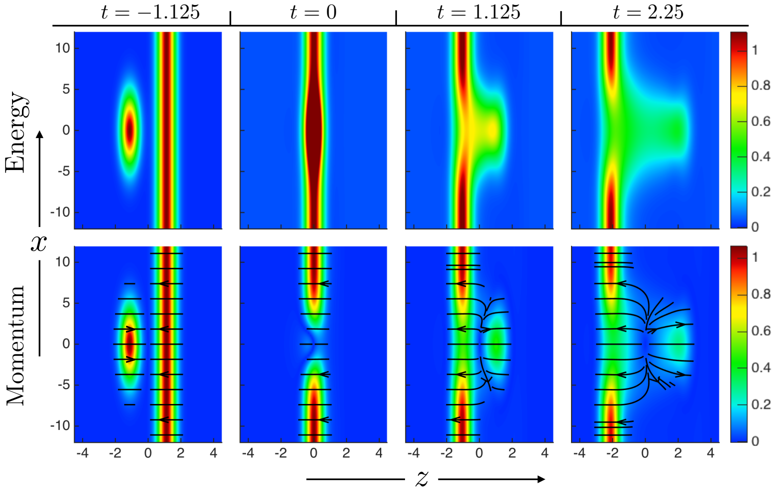


Strong linear rise of η/s above T_c testing the limits of applicability of hydrodynamics in p+A collisions?

Validity of viscous hydro: Exact solution at ∞ coupling

Chesler, arXiv:1506.02209, colliding shock waves in AdS_5 for p+A



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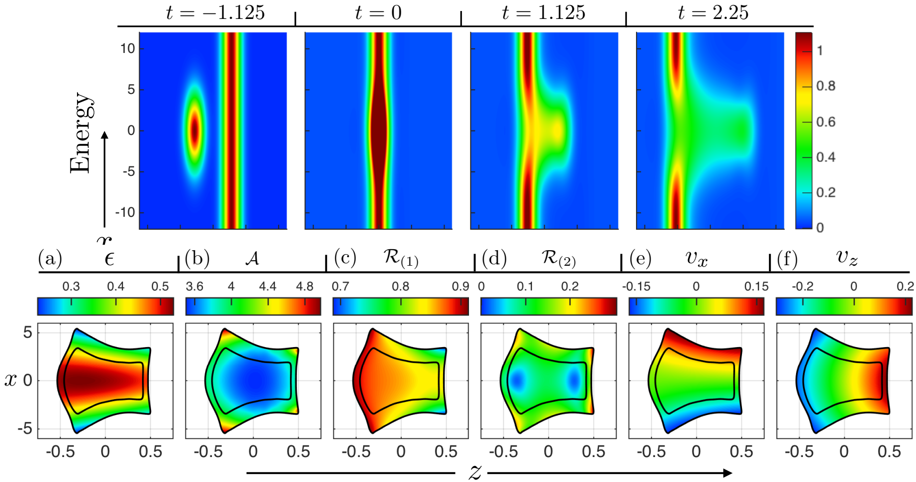

First-order terms in Re^{-1} large, but second-order terms small almost everywhere!

Importance of second-order terms in Kn and Re^{-1} in A+A:

$$\Delta^{\mu\alpha}\Delta^{\nu\beta}u^\lambda d_\lambda \pi_{\alpha\beta} = -\frac{1}{\tau_\pi}(\pi^{\mu\nu} - 2\eta\sigma^{\mu\nu}) - \frac{\delta_{\pi\pi}\pi^{\mu\nu}\theta}{\tau_\pi} + \frac{\varphi_7}{\tau_\pi}\pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} - \frac{\tau_{\pi\pi}}{\tau_\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \frac{\lambda_{\pi\Pi}}{\tau_\pi}\Pi\sigma^{\mu\nu},$$

$$u^\lambda d_\lambda \Pi = -\frac{1}{\tau_\Pi}(\Pi + \zeta\theta) - \frac{\delta_{\Pi\Pi}}{\tau_\Pi}\Pi\theta + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$

with transport coefficients from Boltzmann equation for massless Boltzmann gas.

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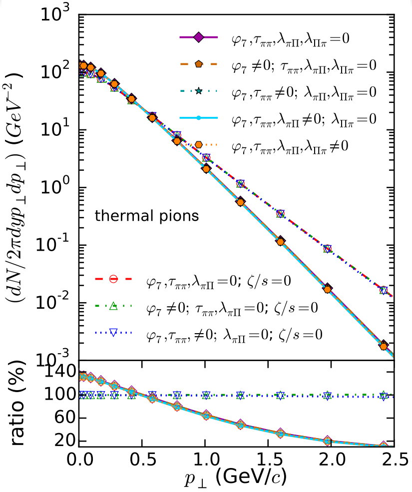

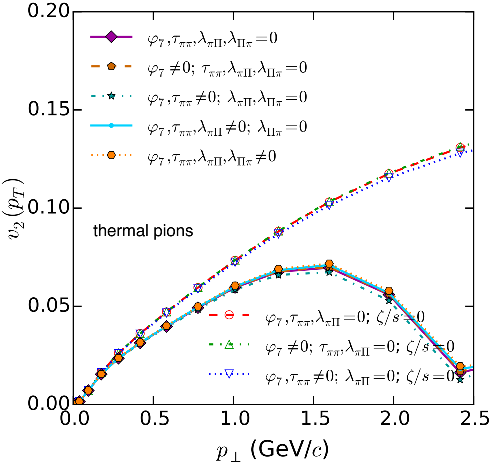

Bulk viscosity matters

Non-linear second-order terms make almost no difference

Unreasonable Effectiveness of Hydrodynamics

Kinetic theory vs. hydrodynamics

Both simultaneously valid if weakly coupled and small pressure gradients.

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Boltzmann Equation in Relaxation Time Approximation (RTA):

$$p^\mu \partial_\mu f(x,p) = C(x,p) = \frac{p\cdot u(x)}{\tau_{\rm rel}(x)}\Big(f_{\rm eq}(x,p) - f(x,p)\Big)$$

For conformal systems $\tau_{\rm rel}(x) = c/T(x) = 5\eta/(sT) \equiv 5\bar{\eta}/T(x)$.

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Macroscopic currents:

$$j^\mu(x) = \int_p p^\mu\, f(x,p) \equiv \langle p^\mu \rangle; \quad T^{\mu\nu}(x) = \int_p p^\mu\, p^\nu\, f(x,p) \equiv \langle p^\mu p^\nu \rangle$$

where $$\int_p \cdots \equiv \frac{g}{(2\pi)^3} \int \frac{d^3p}{E_p} \cdots \equiv \langle \dots \rangle$$

Hydrodynamics for strongly anisotropic expansion (I)

Account for large viscous flows by including their effect already at leading order in the Chapman-Enskog expansion:

Expand the solution $f(x,p)$ of the Boltzmann equation as

$$f(x,p) = f_0(x,p) + \delta f(x,p) \qquad \left(\left|\delta f/f_0\right| \ll 1\right)$$

where f_0 is parametrized through macroscopic observables as

$$f_0(x,p) = f_0\left(\frac{\sqrt{p_\mu \Xi^{\mu\nu}(x) p_\nu} - \tilde{\mu}(x)}{\tilde{T}(x)}\right)$$

where $\Xi^{\mu\nu}(x) = u^\mu(x) u^\nu(x) - \Phi(x)\Delta^{\mu\nu}(x) + \xi^{\mu\nu}(x).$

$u^\mu(x)$ defines the local fluid rest frame (LRF).
$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the spatial projector in the LRF.
$\tilde{T}(x)$, $\tilde{\mu}(x)$ are the effective temperature and chem. potential in the LRF.
$\Phi(x)$ accounts for bulk viscous effects in expanding systems.
$\xi^{\mu\nu}(x)$ describes deviations from local momentum isotropy in anisotropically expanding systems due to shear viscosity.

Hydrodynamics for strongly anisotropic expansion (II)

$u^\mu(x),\ \tilde{T}(x),\ \tilde{\mu}(x)$ are fixed by the Landau matching conditions:

$$T^\mu_{\ \nu} u^\nu = \mathcal{E}(\tilde{T}, \tilde{\mu}; \xi, \Phi) u^\mu; \qquad \Big\langle u{\cdot}p \Big\rangle_{\delta f} = \Big\langle (u{\cdot}p)^2 \Big\rangle_{\delta f} = 0$$

$\mathcal{E}$ is the LRF energy density. We introduce the true local temperature $T(\tilde{T}, \tilde{\mu}; \xi, \Phi)$ and chemical potential $\mu(\tilde{T}, \tilde{\mu}; \xi, \Phi)$ by demanding $\mathcal{E}(\tilde{T}, \tilde{\mu}; \xi, \Phi) {=} \mathcal{E}_{\rm eq}(T, \mu)$ and $\mathcal{N}(\tilde{T}, \tilde{\mu}; \xi, \Phi) {\equiv} \langle u{\cdot}p\rangle_{f_0} {=} \mathcal{R}_0(\xi, \Phi)\mathcal{N}_{\rm eq}(T, \mu)$ (see cited literature for $\mathcal{R}$ functions).

Writing

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} \equiv T_0^{\mu\nu} + \Pi^{\mu\nu}, \qquad j^\mu = j_0^\mu + \delta j^\mu \equiv j_0^\mu + V^\mu,$$

the conservation laws

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad \partial_\mu j^\mu(x) = \frac{\mathcal{N}(x) - \mathcal{N}_{\rm eq}(x)}{\tau_{\rm rel}(x)}$$

are sufficient to determine $u^\mu(x)$, $T(x)$, $\mu(x)$, but not the dissipative corrections $\xi^{\mu\nu}$, Φ, $\Pi^{\mu\nu}$, and V^μ whose evolution is controlled by microscopic physics.

Hydrodynamics for strongly anisotropic expansion (III)

Different hydrodynamic approaches can be characterized by the different assumptions they make about the dissipative corrections and/or the different approximations they use to derive their dynamics from the underlying Boltzmann equation:

- **Ideal hydro:** local momentum isotropy ($\xi^{\mu\nu} = 0$), $\Phi = \Pi^{\mu\nu} = V^{\mu} = 0$.

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- **Anisotropic hydrodynamics (aHydro):** allows for leading-order local momentum anisotropy ($\xi^{\mu\nu}$, $\Phi \neq 0$), evolved according to equations obtained from low-order moments of BE, but ignores residual dissipative flows: $\Pi^{\mu\nu} = V^\mu = 0$.

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Different hydrodynamic approaches can be characterized by the different assumptions they make about the dissipative corrections and/or the different approximations they use to derive their dynamics from the underlying Boltzmann equation:

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- **Navier-Stokes (NS) theory:** local momentum isotropy ($\xi^{\mu\nu}=0$), $\Phi=0$, ignores microscopic relaxation time by postulating instantaneous constituent relations for $\Pi^{\mu\nu}$, V^μ.
- **Israel-Stewart (IS) theory:** local momentum isotropy ($\xi^{\mu\nu}=0$), $\Phi=0$, evolves $\Pi^{\mu\nu}$, V^μ dynamically, keeping only terms linear in $\mathrm{Kn}=\lambda_{\mathrm{mfp}}/\lambda_{\mathrm{macro}}$
- **Denicol-Niemi-Molnar-Rischke (DNMR) theory:** improved **IS theory** that keeps nonlinear terms up to order Kn^2, $\mathrm{Kn}\cdot\mathrm{Re}^{-1}$ when evolving $\Pi^{\mu\nu}$, V^μ.
- **Anisotropic hydrodynamics (aHydro):** allows for leading-order local momentum anisotropy ($\xi^{\mu\nu}$, $\Phi\neq 0$), evolved according to equations obtained from low-order moments of BE, but ignores residual dissipative flows: $\Pi^{\mu\nu}=V^\mu=0$.
- **Viscous anisotropic hydrodynamics (vaHydro):** improved **aHydro** that additionally evolves residual dissipative flows $\Pi^{\mu\nu}$, V^μ with **IS** or **DNMR theory**.

Unreasonable Effectiveness of Hydrodynamics

BE for systems with highly symmetric flows: I. Bjorken flow

- Longitudinal boost invariance, transverse homogeneity ("physics on the light cone", no transverse flow) $\Longrightarrow$ $\boldsymbol{u^\mu = (1,0,0,0)}$ in Milne coordinates (τ, r, ϕ, η) where $\tau = (t^2{-}z^2)^{1/2}$ and $\eta = \frac{1}{2}\ln[(t{-}z)/(t{+}z)]$ $\Longrightarrow$ $\boldsymbol{v_z = z/t}$

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- Symmetry restricts possible dependence of distribution function $f(x,p)$ (Baym '84, Florkowski et al. '13, '14):

$$f(x,p) = f(\tau; p_\perp, w) \qquad \text{where} \qquad w = tp_z - zE = \tau m_\perp \sinh(y{-}\eta).$$

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$$\partial_\tau f(\tau; p_\perp, w) = -\frac{f(\tau; p_\perp, w) - f_{\rm eq}(\tau; p_\perp, w)}{\tau_{\rm rel}(\tau)}.$$

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- **Solution:**

$$f(\tau; p_\perp, w) = D(\tau,\tau_0) f_0(p_\perp, w) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm rel}(\tau')} D(\tau,\tau')\, f_{\rm eq}(\tau'; p_\perp, w)$$

where $$D(\tau_2,\tau_1) = \exp\left(-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm rel}(\tau'')}\right).$$

BE for systems with highly symmetric flows: II. Gubser flow

- Longitudinal boost invariance, azimuthally symmetric radial dependence ("physics on the light cone" with azimuthally symmetric transverse flow)
  (Gubser '10, Gubser & Yarom '11)
  $\Longrightarrow$ $u^\mu = (1,0,0,0)$ in de Sitter coordinates (ρ,θ,ϕ,η) where
  $\rho(\tau,r) = -\sinh^{-1}\left(\frac{1-q^2\tau^2+q^2r^2}{2q\tau}\right)$ and $\theta(\tau,r) = \tan^{-1}\left(\frac{2qr}{1+q^2\tau^2-q^2r^2}\right)$.
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- With $T(\tau, r) = \hat{T}(\rho(\tau, r))/\tau$ RTA BE simplifies to the ODE

$$\frac{\partial}{\partial\rho} f(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma) = -\frac{\hat{T}(\rho)}{c}\left[f\left(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma\right) - f_{\mathrm{eq}}\left(\hat{p}^\rho/\hat{T}(\rho)\right)\right].$$

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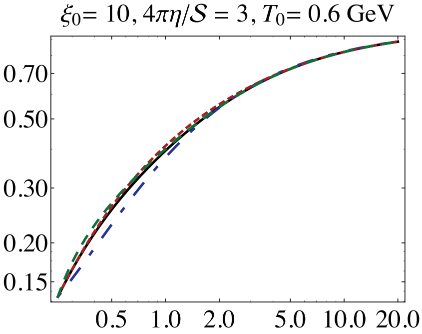
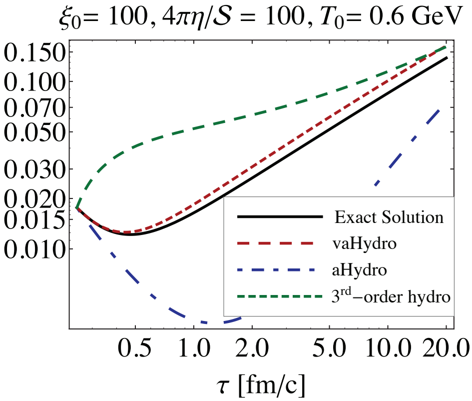
Exact BE vs. hydrodynamic approximations: Bjorken flow

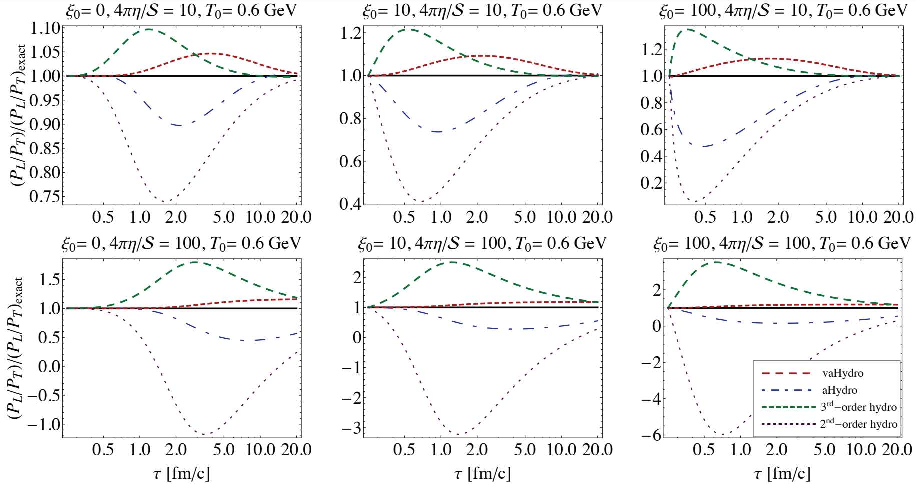
Pressure anisotropy P_L/P_T vs. τ:





In the right plot, IS theory yields negative $P_L/P_T < 0$!

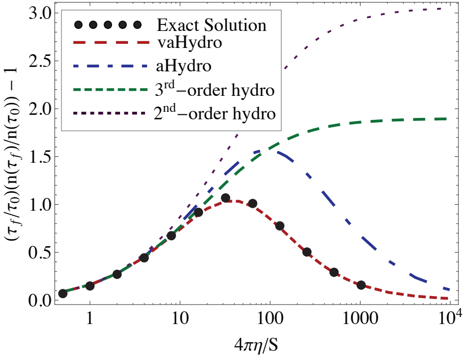
Exact BE vs. hydrodynamic approximations: Bjorken flow



vaHydro agrees within a few % with exact result, even for very large $\eta/\mathcal{S}$!

Exact BE vs. hydrodynamic approximations: Bjorken flow

Total entropy (particle) production $\frac{n(\tau_f)\cdot\tau_f}{n(\tau_0)\cdot\tau_0} - 1$



VAHYDRO gets both the the ideal fluid and free-streaming limits right (!)

Hydrodynamic equations for systems with Gubser flow*:

- The exact solution for f can be worked out for any "initial" condition $f_0(\hat{p}^2_\Omega, w) \equiv f(\rho_0; \hat{p}^2_\Omega, w)$. We here use equilibrium initial conditions, $f_0 = f_{\rm eq}$.

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- By taking hydrodynamic moments, the exact f yields the exact evolution of all components of $T^{\mu\nu}$. Here, $\Pi^{\mu\nu}$ has only one independent component, $\pi^{\eta\eta}$.

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- By taking hydrodynamic moments, the exact f yields the exact evolution of all components of $T^{\mu\nu}$. Here, $\Pi^{\mu\nu}$ has only one independent component, $\pi^{\eta\eta}$.
- This exact solution of the BE can be compared to solutions of the various hydrodynamic equations in de Sitter coordinates, using identical initial conditions.
  - **Ideal:** $\hat{T}_{\rm ideal}(\rho) = \frac{\hat{T}_0}{\cosh^{2/3}(\rho)}$
  - **NS:** $\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^\eta_\eta(\rho)\tanh\rho$ (viscous T-evolution) with $\bar{\pi}^\eta_\eta \equiv \hat{\pi}^\eta_\eta/(\hat{T}\hat{s})$ and $\hat{\pi}^{\eta\eta}_{NS} = \frac{4}{3}\hat{\eta}\tanh\rho$ where $\frac{\hat{\eta}}{\hat{s}} \equiv \bar{\eta} = \frac{1}{5}\hat{T}\hat{\tau}_{\rm rel}$
  - **IS:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho$
  - **DNMR:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho + \frac{10}{21}\bar{\pi}^\eta_\eta\tanh\rho$
  - **aHydro:** see M. Nopoush et al., PRD 91 (2015) 045007
  - **vaHydro:** see M. Martinez et al., PRC95 (2017) 054907

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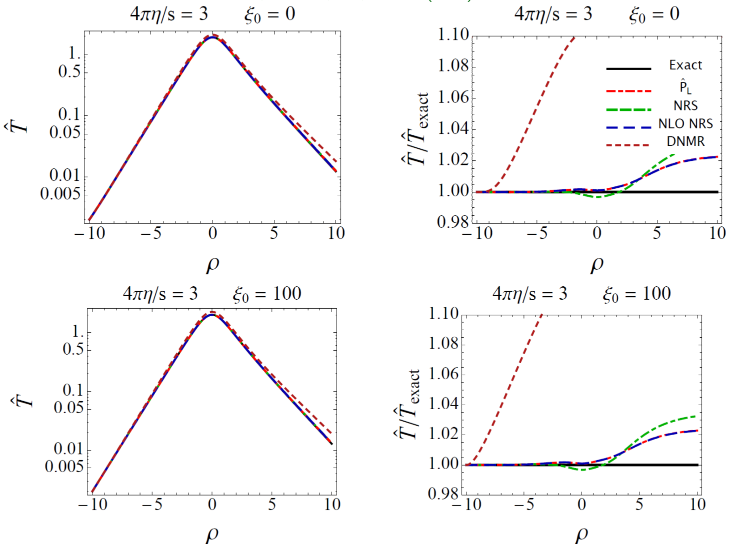
Exact BE vs. hydrodynamic approximations: Gubser flow

Optimal evolution of the momentum deformation parameter ξ?

- **"Standard" viscous hydrodynamics (IS or DNMR)**:
  expansion around local equilibrium $\Longrightarrow \boldsymbol{\xi \equiv 0}$
- **Anisotropic hydrodynamics:**
  expansion around a locally momentum-anisotropic state $\Longrightarrow \boldsymbol{\xi \neq 0}$
  - ***P_L*-matching** (Tinti 2015; Molnar, Niemi, Rischke, 2016):
    Additional Landau matching condition that matches ξ evolution to that of the longitudinal pressure $P_L \Longrightarrow$ no $\delta\tilde{f}$ corrections to P_L.
    In this case ξ can be eliminated, and the evolution equations can be written entirely in terms of macroscopic variables, as in standard viscous hydrodynamics
  - **NSR approach** (Nopoush, Strickland, Ryblewski 2015):
    obtain ξ evolution equation from second moments of the BE
    $\Longrightarrow P_L$ evolution not fully captured by ξ evolution.
  - **NLO-NSR approach** (Martinez, McNelis, UH 2017):
    Same ξ evolution but includes residual $\delta\tilde{f}$ contribution to P_L
    This captures the missing part of the pressure anisotropy.

Exact BE vs. hydrodynamic approximations: Gubser flow

Martinez, McNelis, UH, PRC95 (2017) 054907



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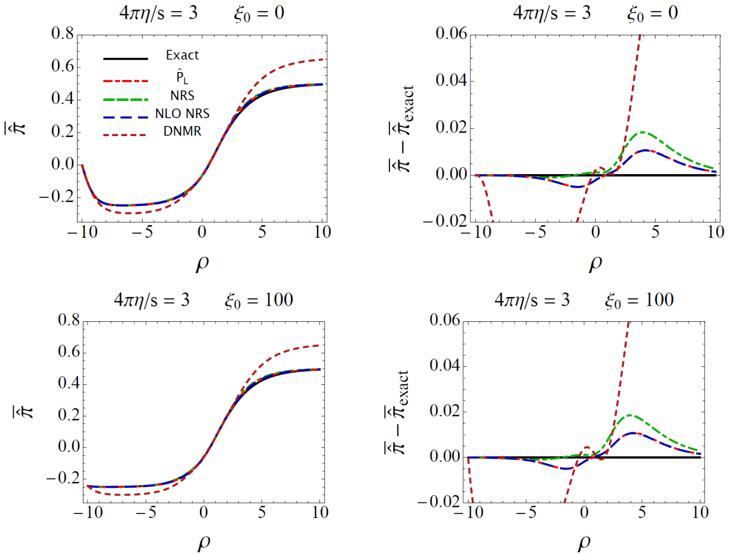

$\bar{\hat{\pi}} \equiv \hat{\pi}/(4\hat{P})$

Exact BE vs. hydrodynamic approximations: Gubser flow

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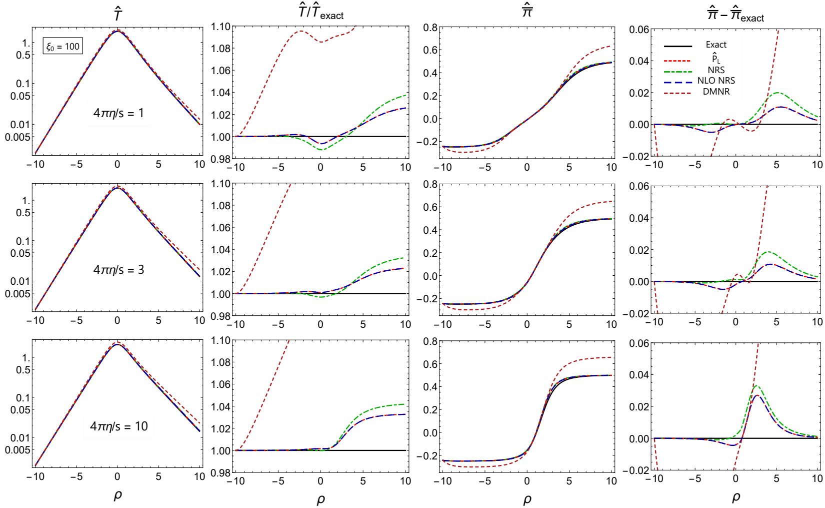


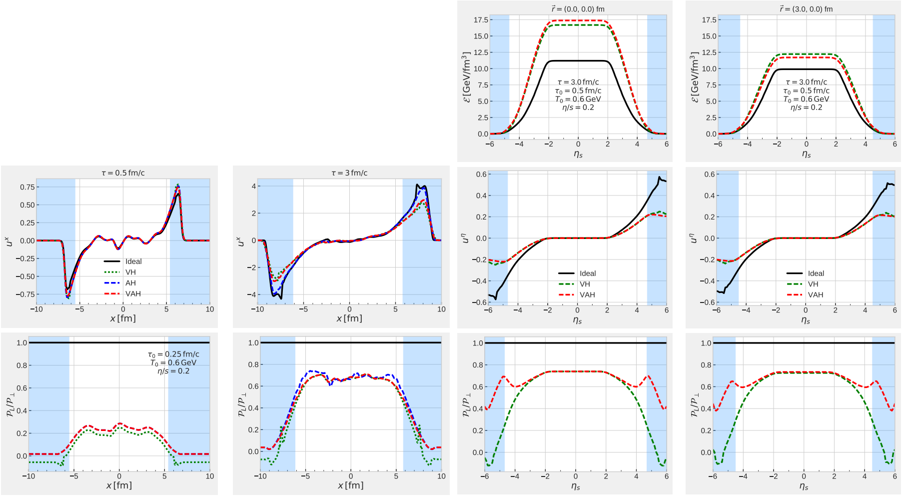
Unreasonable Effectiveness of Hydrodynamics

Summary

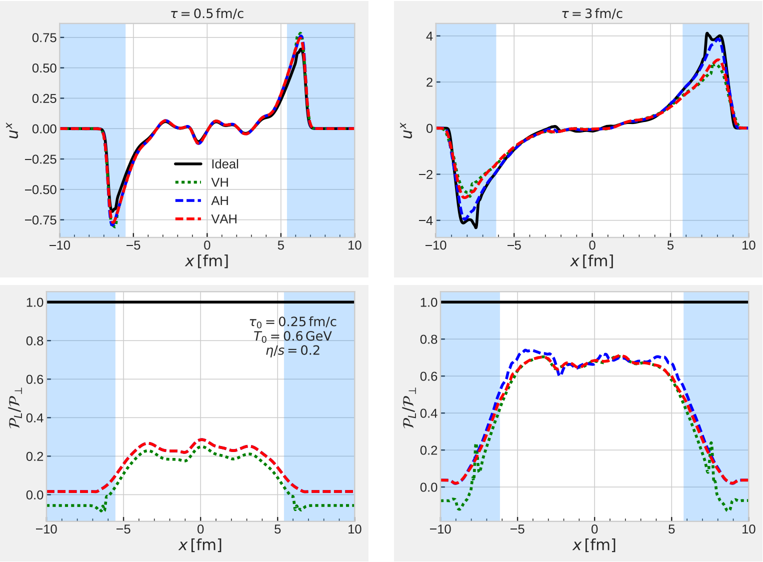
- Viscous relativistic hydrodynamics provides a **robust, reliable, efficient and accurate** description of QGP evolution in heavy-ion collisions.
- It is valid even when the expansion is fast and highly anisotropic, causing large local momentum anisotropies $\Longrightarrow$ **local thermalization not strictly required**.
- While first-order viscous corrections are large in nuclear collisions, especially in small systems, they can be handled efficiently in an **optimized anisotropic hydrodynamic approach** that accounts for local momentum anisotropies at leading order; residual dissipative flows remain small.
- **New exact solutions of the Boltzmann equation** enable powerful tests of the efficiency and accuracy of various hydrodynamic expansion schemes, providing strong support for the **validity and robustness** of second-order viscous hydrodynamics (especially their anisotropic variants).

Extra slides

(3+1)-d vHydro and vaHydro – a comparison



(3+1)-d vHydro and vaHydro – a comparison



(3+1)-d vHydro and vaHydro – a comparison