Tree-level correlations in the strong field regime

François Gelis

St. John's College, Oxford University

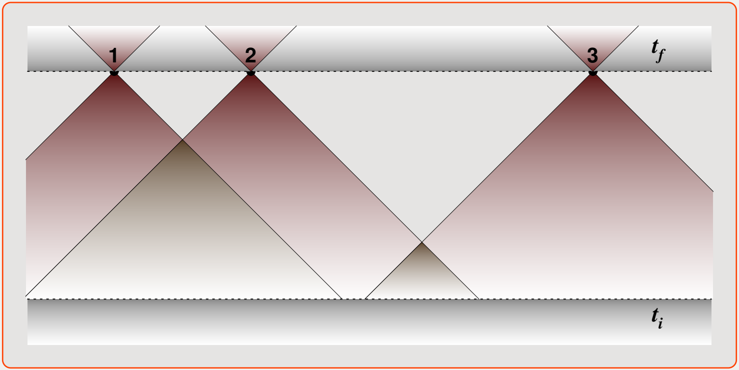
July 10-14, 2017

Institut de
Physique Théorique
CEA/DRF Saclay

Generic problem

- System of fields evolving from some known initial state (pure or mixed state)
- Evolution possibly coupled to a (large) external source
- Perform n local measurements, with no direct causal relation between them (so that the outcome of a measurement does not influence the others)

- **Correlation between these measurements?**
- **In the strong field regime, can it be expressed in terms of a classical field? Which one? How?**



- Even if there is no causal contact at the time of the measurements, correlations exist due to the fact that a common evolution leads to these measurements
- For the correlation to be non-zero, the past light-cones of the measurement events should overlap (at least pairwise)

Introduction

MEASUREMENTS

- Given a local observable $\mathcal{O}(x)$ (e.g., polynomial in the field operator), we wish to calculate :

$$\big\langle \text{in} \big| \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \big| \text{in} \big\rangle_{\text{connected}}$$

- For all pairs of measurements, $(x_i - x_j)^2 < 0$
- For simplicity, assume all times are equal : $x_1^0 = \cdots = x_n^0 = t_f$ (but this can easily be relaxed)

- The final result applies to various types of initial states :
  - Vacuum : $\big|\mathrm{in}\big\rangle \equiv \big|0_{\mathrm{in}}\big\rangle$ (the simplest)
  - Coherent state :

$$\big|\mathrm{in}\big\rangle \equiv \mathcal{N}_\chi \;\exp\Big\{\int_{\mathbf{k}} \chi(\mathbf{k})\, a^\dagger_{\mathrm{in}}(\mathbf{k})\Big\}\,\big|0_{\mathrm{in}}\big\rangle$$

  - Gaussian mixed state :

$$\rho_{\mathrm{in}} \equiv \exp\Big\{-\int_{\mathbf{k}} \beta_{\mathbf{k}} E_{\mathbf{k}}\, a^\dagger_{\mathrm{in}}(\mathbf{k}) a_{\mathrm{in}}(\mathbf{k})\Big\}$$

DYNAMICS

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \underbrace{- V(\phi) + J\phi}_{\mathcal{L}_{\rm int}(\phi)} ,$$

- $V(\phi)$: self-interactions, e.g. $\frac{g^2}{4!}\phi^4$
- $J(x)$: external source

Strong field regime

- Kinetic energy $\sim$ interactions :

$$(\partial_\mu \phi)(\partial^\mu \phi) \sim V(\phi)$$

- For a ϕ^4 theory, occurs when $\phi \sim g^{-1} Q$
- Can be achieved in two ways :
    - Fields are already large in the initial state
    - Large external source $J \sim g^{-1} Q^3$

POWER COUNTING

Order of a connected graph :

$$\mathcal{G} \sim g^{-2} g^{n_E} g^{2n_L} \underbrace{(gJ)^{n_J}}_{g^0}$$

- Usual ordering with the number of loops n_L
- Result non-perturbative in the strong source J
- Likewise, non-perturbative in the initial field if strong ($\Phi_{\text{ini}} \sim g^{-1} Q$)

Example : $\mathcal{O}(x) \equiv g^2\phi^2(x)$

$$\langle\mathcal{O}(x)\rangle \sim 1 \oplus g^2 \oplus \cdots$$

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \underbrace{\langle\mathcal{O}(x)\rangle\,\langle\mathcal{O}(y)\rangle}_{\sim 1\oplus\cdots} + \underbrace{\langle\mathcal{O}(x)\mathcal{O}(y)\rangle_c}_{\sim g^2\oplus\cdots}$$

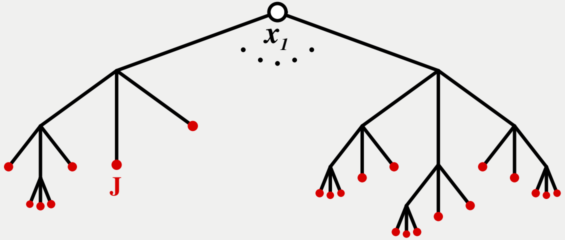
$$\langle\mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z)\rangle = \underbrace{\langle\mathcal{O}(x)\rangle\langle\mathcal{O}(y)\rangle\langle\mathcal{O}(z)\rangle}_{\sim 1\oplus\cdots} + \underbrace{\langle\mathcal{O}(x)\rangle\langle\mathcal{O}(y)\mathcal{O}(z)\rangle_c + \cdots}_{\sim g^2\oplus\cdots} + \underbrace{\langle\mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z)\rangle_c + \cdots}_{\sim g^4\oplus\cdots}$$

- Higher-n correlations are increasingly suppressed
  Expect more complicated expressions

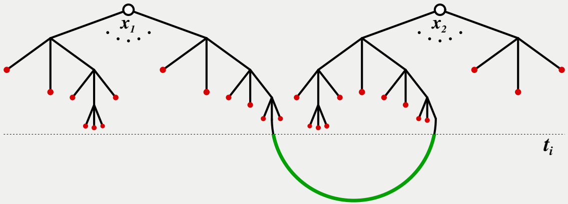
Already known : 1 and 2-point functions [0807.1306]

$$\langle \mathcal{O}(x_1)\rangle_{\text{tree}} = \mathcal{O}(\Phi(x))$$

$$\left(\square + m^2\right)\Phi = \mathcal{L}'_{\text{int}}(\Phi)\ , \quad \Phi_{\text{ini}} \equiv 0$$



$$\left\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\right\rangle_{\substack{\text{connected}\\ \text{tree}}} = \int_{t_i} d^3\mathbf{u}\, d^3\boldsymbol{v} \int_{\mathbf{k}} \frac{1}{2}\left(e^{ik\cdot(u-v)} + \text{c.c.}\right) \times \frac{\delta\mathcal{O}(\Phi(x_1))}{\delta\Phi_{\text{ini}}(\mathbf{u})}\,\frac{\delta\mathcal{O}(\Phi(x_2))}{\delta\Phi_{\text{ini}}(\boldsymbol{v})}\Bigg|_{\Phi_{\text{ini}}=0}$$



- Expressible in terms of the retarded classical field Φ and its derivatives with respect to the initial condition. **Is this true for all n-point functions? If yes, explicit formula?**

Diagrammatic rules

Encapsulate the expectation values in a generating functional :

$$\mathcal{F}[z(\boldsymbol{x})] \equiv \big\langle \text{in} \big| \exp \int_{t_f} d^3\boldsymbol{x}\; z(\boldsymbol{x})\, \mathcal{O}(x) \big| \text{in} \big\rangle$$

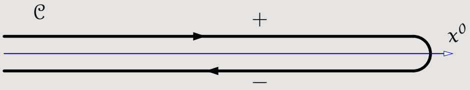
- Correlations are obtained by differentiation of $\ln \mathcal{F}$:

$$\big\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \big\rangle = \frac{\delta^n \ln \mathcal{F}}{\delta z(\boldsymbol{x}_1) \cdots \delta z(\boldsymbol{x}_n)} \Big|_{z \equiv 0}$$

- Note : $\mathcal{F}[z]$ contains disconnected graphs

Feynman rules

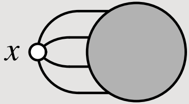
- $\langle \text{in} | \cdots | \text{in} \rangle$ expectation value
  $\Rightarrow$ usual Schwinger-Keldysh rules



- Addition vertex representing $\mathcal{O}(t_f, \boldsymbol{x})$
  - Localized on the surface $x^0 = t_f$
  - As many legs as fields in $\mathcal{O}$
  - Coupling "constant" : $z(\boldsymbol{x})$
  - No need to specify if fields are on the $+$ or $-$ branch

To all orders (diagram for $\mathcal{O} \sim \phi^4$) :

$$\frac{\delta \ln \mathcal{F}}{\delta z(x)} = \sum \left(\begin{array}{c} \text{all connected vacuum graphs} \\ \text{with a } \mathcal{O}\text{-vertex pulled out at } x \end{array} \right)$$

$=$ 

- Introduce half-sum and difference of the fields :

$$\phi_2 \equiv \frac{1}{2}\left(\phi_+ + \phi_-\right)\,, \qquad \phi_1 \equiv \phi_+ - \phi_-$$

- Propagators :

$$\begin{aligned} G^0_{21} &= G^0_{++} - G^0_{+-} \quad \text{(retarded)} \\ G^0_{12} &= G^0_{++} - G^0_{-+} \quad \text{(advanced)} \\ G^0_{22} &= \frac{1}{2}\left[G^0_{+-} + G^0_{-+}\right] \\ G^0_{11} &= 0 \end{aligned}$$

- Vertices :

$$[1222] = -ig^2\,, \quad [1112] = -ig^2/4\,, \quad \text{all others zero}$$

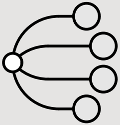
- Observables depend only on ϕ_2

Tree level

FIRST DERIVATIVE OF $\ln \mathcal{F}[z]$

Tree level :

$$\left.\frac{\delta \ln \mathcal{F}}{\delta z(\boldsymbol{x})}\right|_{\text{tree}} = \mathcal{O}(\phi_2(x)) = \; x$$



(each blob is a ϕ_2 at tree level)

Representation by coupled integral equations

$$\phi_1(x) = i\int d^4y\; G^0_{12}(x,y)\,\frac{\partial L_{\rm int}(\phi_1,\phi_2)}{\partial\phi_2(y)} + \int_{t_f} d^3\boldsymbol{y}\; G^0_{12}(x,y)\; z(\boldsymbol{y})\;\mathcal{O}'(\phi_2(y))$$

$$\phi_2(x) = i\int d^4y\; \Big\{G^0_{21}(x,y)\,\frac{\partial L_{\rm int}(\phi_1,\phi_2)}{\partial\phi_1(y)} + G^0_{22}(x,y)\,\frac{\partial L_{\rm int}(\phi_1,\phi_2)}{\partial\phi_2(y)}\Big\} + \int_{t_f} d^3\boldsymbol{y}\; G^0_{22}(x,y)\; z(\boldsymbol{y})\;\mathcal{O}'(\phi_2(y))$$

- $L_{\rm int}(\phi_1,\phi_2) \equiv \mathcal{L}_{\rm int}(\phi_2+\frac{1}{2}\phi_1) - \mathcal{L}_{\rm int}(\phi_2-\frac{1}{2}\phi_1)$
- Contains z to all orders due to non-linearities

Representation by EOM + boundary conditions

Equations of motion (for ϕ^4 interaction + source) :

$$\Big[\square_x + m^2 + \tfrac{g^2}{2}\phi_2^2\Big]\,\phi_1 + \frac{g^2}{4!}\,\phi_1^3 = 0$$

$$(\square_x + m^2)\,\phi_2 + \frac{g^2}{6}\,\phi_2^3 + \frac{g^2}{8}\,\phi_1^2\phi_2 = J$$

$z(x)$ enters only in the boundary conditions :

- At t_f : $\phi_1(t_f, x) = 0\,,\quad \partial_0\phi_1(t_f, x) = i\,z(x)\,\mathcal{O}'(\phi_2(t_f, x))$
- At t_i, relation between the Fourier modes :

$$\widetilde{\phi}_2^{(+)}(k) = -\frac{1}{2}\,\widetilde{\phi}_1^{(+)}(k)\,,\quad \widetilde{\phi}_2^{(-)}(k) = \frac{1}{2}\,\widetilde{\phi}_1^{(-)}(k)$$

(here, written for an empty initial state)

Expansion in z

Write $\phi_{1,2}$ as formal series in z :

$$
\begin{aligned}
\phi_1(x) \equiv \phi_1^{(0)}(x) \quad &+ \quad \int d^3\mathbf{x}_1 \; z(\mathbf{x}_1)\; \phi_1^{(1)}(x;\mathbf{x}_1) \\
&+ \quad \frac{1}{2!}\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \; z(\mathbf{x}_1)z(\mathbf{x}_2)\; \phi_1^{(2)}(x;\mathbf{x}_1,\mathbf{x}_2) \\
&+ \quad \cdots \\
\phi_2(x) \equiv \phi_2^{(0)}(x) \quad &+ \quad \int d^3\mathbf{x}_1 \; z(\mathbf{x}_1)\; \phi_2^{(1)}(x;\mathbf{x}_1) \\
&+ \quad \frac{1}{2!}\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \; z(\mathbf{x}_1)z(\mathbf{x}_2)\; \phi_2^{(2)}(x;\mathbf{x}_1,\mathbf{x}_2) \\
&+ \quad \cdots
\end{aligned}
$$

Order 0 (1-point function)

Simply set $z \equiv 0$ in the boundary conditions :

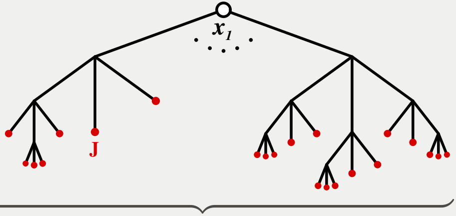
$$\phi_1^{(0)}(t_f, \boldsymbol{x}) = 0 \,, \quad \partial_0 \phi_1^{(0)}(t_f, \boldsymbol{x}) = 0 \quad \Rightarrow \quad \forall x \,, \; \phi_1^{(0)}(x) = 0$$

$$\phi_2^{(0)} = \Phi$$

$$(\square + m^2)\,\Phi = \mathcal{L}'_{\rm int}(\Phi) \,, \quad \Phi_{\rm ini} \equiv 0 \;\; \text{at } t_i$$

- $\phi_1^{(0)}$ is zero everywhere
- $\phi_2^{(0)} = \Phi$ is the classical solution with (null) retarded boundary condition. Straightforward to obtain numerically
- $\big\langle \mathcal{O}(x) \big\rangle_{\rm tree} = \mathcal{O}(\Phi(x))$

$$\left\langle \mathcal{O}(x_1) \right\rangle_{\text{tree}} = \sum_{\text{trees}}$$



compact notation : ①

Equations of motion :

$$\Big[\square + m^2 - \mathcal{L}''_{\rm int}(\Phi)\Big]\,\phi_1^{(1)} = 0$$
$$\Big[\square + m^2 - \mathcal{L}''_{\rm int}(\Phi)\Big]\,\phi_2^{(1)} = 0$$

Boundary conditions :

$$t_f\ :\qquad \phi_1^{(1)}(x;x_1) = 0\,,\quad \partial_0\phi_1^{(1)}(x;x_1) = i\,\delta(x-x_1)\,\mathcal{O}'(\Phi(x_1))$$
$$t_i\ :\qquad \widetilde{\phi}_2^{(1+)}(\boldsymbol{k}) = -\frac{1}{2}\,\widetilde{\phi}_1^{(1+)}(\boldsymbol{k})\,,\quad \widetilde{\phi}_2^{(1-)}(\boldsymbol{k}) = \frac{1}{2}\,\widetilde{\phi}_1^{(1-)}(\boldsymbol{k})$$

Order 1 (2-point function)

Solution :

$$\phi_1^{(1)}(x;x_1) = G_{12}(x,x_1)\,\mathcal{O}'(\Phi(x_1))$$
$$\phi_2^{(1)}(x;x_1) = G_{22}(x,x_1)\,\mathcal{O}'(\Phi(x_1))$$

(G_{12}, G_{22} = propagators dressed by the background field Φ)

$$\big\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\big\rangle_{\rm tree} = \mathcal{O}'(\Phi(x_1))\,G_{22}(x_1,x_2)\,\mathcal{O}'(\Phi(x_2))$$

Expression in terms of ***mode functions*** **:**

$$\left[\square + m^2 - \mathcal{L}''_{\rm int}(\Phi(x))\right] a_{\pm\mathbf{k}}(x) = 0$$

$$a_{\pm\mathbf{k}}(x) \underset{x^0 \to t_i}{\longrightarrow} e^{\mp i k \cdot x}$$

$$G_{22}(x,y) = \int_{\mathbf{k}} \frac{1}{2}\Big(a_{+\mathbf{k}}(x)a_{-\mathbf{k}}(y) + a_{-\mathbf{k}}(x)a_{+\mathbf{k}}(y)\Big)$$

with :

$$a_{\pm\mathbf{k}}(x) = \mathbf{T}_{\pm\mathbf{k}}\; \Phi(x)\Big|_{\Phi_{\rm ini}=0}$$

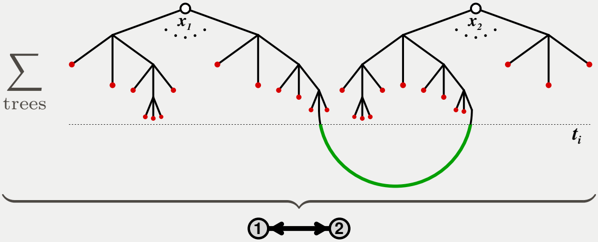
$$\mathbf{T}_{\pm\mathbf{k}} \equiv \int_{t_i} d^3\mathbf{y}\; e^{\mp i k \cdot y} \frac{\delta}{\delta \Phi_{\rm ini}(y)}$$

ORDER 1 (2-POINT FUNCTION)

$$\mathcal{O}'(\Phi(x_1))\, G_{22}(x_1,x_2)\, \mathcal{O}'(\Phi(x_2))$$
$$= \mathcal{O}(\Phi(x_1)) \Big[\underbrace{\int_k \frac{1}{2}\Big(\overleftarrow{\mathbf{T}}_{+k}\overrightarrow{\mathbf{T}}_{-k} + \overleftarrow{\mathbf{T}}_{-k}\overrightarrow{\mathbf{T}}_{+k} \Big)}_{\otimes} \Big] \mathcal{O}(\Phi(x_2))$$

- Tree-level 2-point correlations are obtained from classical fields, by differentiation w.r.t. the initial condition

$$\left\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\right\rangle_{\rm tree} = \sum_{\rm trees}$$



- Can one generalize the z-expansion to obtain higher correlations? **YES, but very painful combinatorics**
- Is the result expressible in terms of derivatives of Φ with respect to $\Phi_{\rm ini}$? **NO, at 3-point and beyond**

Strong field approximation

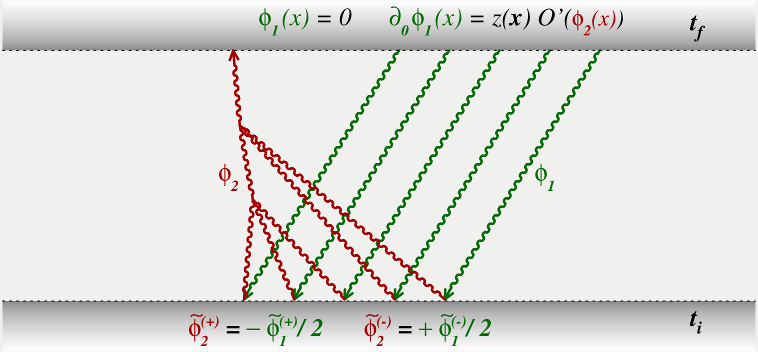
Approximation :

$$\phi_1 \ll \phi_2\ , \quad \text{i.e.} \quad \phi_+ - \phi_- \ll \phi_+ + \phi_-$$

Equations of motion :

$$(\square + m^2)\,\phi_2 - \mathcal{L}'_{\rm int}(\phi_2) = 0 \qquad \text{(no mixing with } \phi_1\text{)}$$

$$\Big[\square + m^2 - \mathcal{L}''_{\rm int}(\phi_2)\Big]\,\phi_1 = 0 \qquad \text{(linear, with } \phi_2 \text{ background)}$$



- Intricate mixing via the boundary conditions
- ϕ_2 is a strong field, and its non-linearities cannot be neglected
- Admits a formal solution to all orders in $z(\mathbf{x})$

All-orders solution

Solution of the EOM for ϕ_1 :

$$\phi_1(x) = \int_{t_f} d^3\mathbf{u}\; G_{12}(x, u)\; z(\mathbf{u})\; \mathcal{O}'(\phi_2(u))$$

Boundary condition at t_i :

$$\phi_2(t_i, \mathbf{x}) = \int_{t_f} d^3\mathbf{u}\; G_{22}(x, u)\; z(\mathbf{u})\; \mathcal{O}'(\phi_2(u))$$

- Propagators G_{12} and G_{22} dressed by ϕ_2
- Solution for ϕ_1 valid everywhere
- Solution for ϕ_2 valid only at t_i (before non-linearities set in)
  Can be used as initial condition for the nonlinear evolution

Formal solution in the bulk :

$$\phi_2(x) = \underbrace{\exp\left\{\int_{t_i} d^3\mathbf{y}\; \phi_2(t_i,\mathbf{y}) \frac{\delta}{\delta\Phi_{\rm ini}(t_i,\mathbf{y})}\right\}}_{\text{translation operator of } \Phi_{\rm ini}} \Phi(x)\Big|_{\Phi_{\rm ini}\equiv 0}$$

- All the non-linear dynamics already encoded in $\Phi[\Phi_{\rm ini}]$

Formal solution in the bulk :

$$\phi_2(x) = \underbrace{\exp\left\{\int_{t_i} d^3y\ \phi_2(t_i,y)\frac{\delta}{\delta\Phi_{ini}(t_i,y)}\right\}}_{\text{translation operator of } \Phi_{ini}}\ \Phi(x)\Big|_{\Phi_{ini}\equiv 0}$$

- All the non-linear dynamics already encoded in $\Phi[\Phi_{ini}]$

- Also valid for $\mathcal{O}(\phi_2)$:

$$\mathcal{O}(\phi_2(x)) = \exp\left\{\int_{t_i} d^3y\ \phi_2(t_i,y)\frac{\delta}{\delta\Phi_{ini}(t_i,y)}\right\}\mathcal{O}(\Phi(x))\Big|_{\Phi_{ini}\equiv 0}$$

Rewrite $\phi_2(t_i, \mathbf{y})$ as follows :

$$\phi_2(t_i, \mathbf{y}) = \frac{1}{2} \int_{\mathbf{k}} \int_{t_f} d^3\mathbf{u}\; z(\mathbf{u})\; \mathcal{O}(\phi_2(\mathbf{u})) \left\{ \overleftarrow{\mathbb{T}}_{+\mathbf{k}}\; e^{+i\mathbf{k}\cdot\mathbf{y}} + \overleftarrow{\mathbb{T}}_{-\mathbf{k}}\; e^{-i\mathbf{k}\cdot\mathbf{y}} \right\}$$

$$\mathcal{O}\big(\phi_2(x)\big) = \exp\left\{ \int_{t_f} d^3\mathbf{u}\; z(\mathbf{u})\, \mathcal{O}(\phi_2(t_f, \mathbf{u})) \times \underbrace{\frac{1}{2} \int_{\mathbf{k}} \Big[\overleftarrow{\mathbb{T}}_{+\mathbf{k}} \overrightarrow{\mathbb{T}}_{-\mathbf{k}} + \overleftarrow{\mathbb{T}}_{-\mathbf{k}} \overrightarrow{\mathbb{T}}_{+\mathbf{k}} \Big]}_{\otimes} \right\} \mathcal{O}\big(\Phi(x)\big) \Big|_{\Phi_{ini} \equiv 0}$$

Implicit functional identity for $\mathcal{O}(\phi_2)$:

$$\underline{\mathcal{O}\big(\phi_2(x)\big)} = \exp\left\{ \int_{t_f} d^3\mathbf{u}\; z(\mathbf{u})\; \underline{\mathcal{O}(\phi_2(t_f, \mathbf{u}))}\; \otimes \right\} \mathcal{O}\big(\Phi(x)\big) \Big|_{\Phi_{ini} \equiv 0}$$

Diagrammatic representation :

$$\textcircled{i} \quad \equiv \quad \mathcal{O}\big(\Phi(t_f, \mathbf{x}_i)\big)$$

$$\textcircled{A} \longleftrightarrow \textcircled{B} \quad \equiv \quad A \otimes B$$

$$\underline{\frac{\delta \ln \mathcal{F}}{\delta z(\mathbf{x}_1)}} = \exp\left\{ \left(\int_{t_f} d^3\mathbf{u}\; z(\mathbf{u})\, \underline{\frac{\delta \ln \mathcal{F}}{\delta z(\mathbf{u})}} \right) \longleftrightarrow \right\} \textcircled{1} \,\Bigg|_{\Phi_{\rm ini} \equiv 0}$$

(Reminder : $\mathcal{O}(\phi_2)$ is the first derivative of $\ln \mathcal{F}$)

Generating function for labeled trees :

- **P. Flageolet, R. Sedgewick** : *Analytic Combinatorics*, p 127

$$w(z) = e^{z\,w(z)} \quad \Rightarrow \quad w(z) = \sum_{n\geq 0} (n+1)^{n-1}\,\frac{z^n}{n!}$$

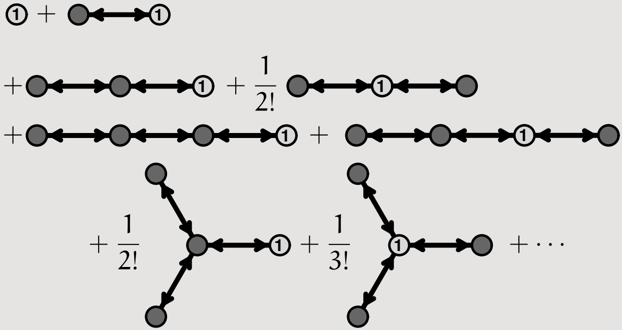
- **Cayley's formula** :

$(n+1)^{n-1} \;=\; \#$ of connected trees with $n+1$ *labeled* nodes

Introduce :  $\equiv \int d^3\mathbf{u}\; z(\mathbf{u})\; \mathcal{O}\big(\Phi(t_f, \mathbf{u})\big)$

Solution : sum of all trees with one labeled node

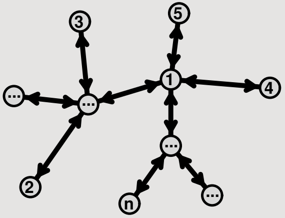
$$\frac{\delta \ln \mathcal{F}}{\delta z(\mathbf{x}_1)} =$$



Note : each blob is itself an infinite sum of tree Feynman diagrams

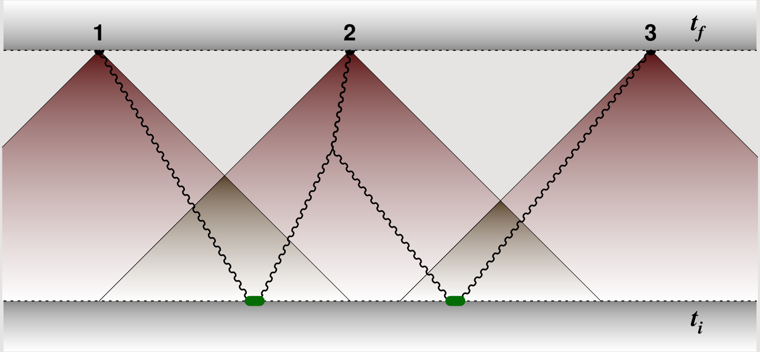
- Differentiating with respect to $z(x_2)\cdots z(x_n)$:
  - selects trees with exactly n nodes
  - puts labels onto the remaining nodes
  - removes the symmetry factors

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle_{\text{tree level strong fields}} = \sum_{\text{trees with } n \text{ labeled nodes}}$$



CAUSAL STRUCTURE IN THE STRONG FIELD REGIME

- Correlations entirely due to initial state fluctuations



Other initial states

$$\big|\mathrm{in}\big\rangle \equiv \mathcal{N}_\chi \;\exp\left\{\int_{\mathbf{k}} \chi(\mathbf{k})\; a^\dagger_{\mathrm{in}}(\mathbf{k})\right\}\big|0_{\mathrm{in}}\big\rangle$$

$$a_{\mathrm{in}}(\mathbf{p})\,\big|\chi\big\rangle = \chi(\mathbf{p})\,\big|\chi\big\rangle$$

$$\big|\mathcal{N}_\chi\big|^2 = \exp\left\{-\int_{\mathbf{k}}\big|\chi(\mathbf{k})\big|^2\right\}$$

$$\mathcal{O}\big(\phi_2(x)\big) = \exp\left\{\int_{t_f} d^3\mathbf{u}\; z(\mathbf{u})\,\mathcal{O}(\phi_2(u)) \;\otimes\right\}\mathcal{O}\big(\Phi(x)\big)\Big|_{\Phi_{\mathrm{ini}}\equiv\Phi_\chi}$$

$$\Phi_\chi(x) \equiv \int_k \Big(\chi(\mathbf{k})\,e^{-ik\cdot x} + \chi^*(\mathbf{k})\,e^{+ik\cdot x}\Big)$$

Gaussian mixed state

- Equation of motion and boundary condition at t_f unchanged
- Boundary condition at t_i $\quad (f_k = \text{initial occupation number})$:

$$\widetilde{\phi}_2^{(+)}(k) = -\left(\frac{1}{2} + f_k\right)\widetilde{\phi}_1^{(+)}(k)\,, \quad \widetilde{\phi}_2^{(-)}(k) = \left(\frac{1}{2} + f_k\right)\widetilde{\phi}_1^{(-)}(k)$$

- Correlations have the same diagrammatic representation, with :

$$\otimes \quad \rightarrow \quad \int_k \left(\frac{1}{2} + f_k\right)\left[\overleftarrow{T}_{+k}\overrightarrow{T}_{-k} + \overleftarrow{T}_{-k}\overrightarrow{T}_{+k}\right]$$

Beyond strong fields

When is $\phi_1 \ll \phi_2$ satisfied ?

- Highly occupied initial state :
  - Coherent state

$$\big|\text{in}\big\rangle \equiv \mathcal{N}_\chi \;\exp\Big\{\int_{\mathbf{k}} \chi(\mathbf{k})\, a^\dagger_{\text{in}}(\mathbf{k})\Big\}\,\big|0_{\text{in}}\big\rangle$$

  with $\chi(\mathbf{k}) \gg 1$

  - Gaussian mixed state

$$\rho_{\text{in}} \equiv \exp\Big\{-\int_{\mathbf{k}} \beta_{\mathbf{k}} E_{\mathbf{k}}\, a^\dagger_{\text{in}}(\mathbf{k}) a_{\text{in}}(\mathbf{k})\Big\}$$

  with $f_{\mathbf{k}} \equiv (e^{\beta_{\mathbf{k}} E_{\mathbf{k}}} - 1)^{-1} \gg 1$

When is $\phi_1 \ll \phi_2$ satisfied ?

- Empty (or lowly occupied) initial state, and unstable classical dynamics :
  - Backward evolution of ϕ_1 :

$$\phi_1(x^0) \sim \phi_1(t_f)\; e^{\mu(t_f - x^0)} \qquad (\mu > 0 \;:\; \text{Lyapunov exponent})$$

  - Boundary condition at t_i :

$$\phi_2(t_i) \sim \phi_1(t_i) \sim \phi_1(t_f)\; e^{\mu(t_f - t_i)}$$

  - Forward evolution of ϕ_2 :

$$\phi_2(x^0) \sim \phi_1(t_f)\; e^{\mu(t_f - t_i)}\; e^{\mu(x^0 - t_i)}$$

$$\frac{\phi_1(x^0)}{\phi_2(x^0)} \sim e^{-2\mu(x^0 - t_i)} \ll 1$$

When is $\phi_1 \ll \phi_2$ satisfied ?

Note : late time evolution

- Non-linear dynamics leads to $\phi_1 \sim \phi_2$ when $t \to \infty$
- Thermalization : occupation $\lesssim 1$ for most modes
- Correlations are those of a thermal system,
  Remembers very little of the initial state

- If $\phi_1 \sim \phi_2$, there are other tree level contributions
- Example of the 3-point function :

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_3)\rangle_{\text{tree}} =$$

$+$

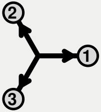
$+$

$+\xi(x_{1,2,3})$

strong field regime

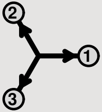
- The pedestrian z-expansion gives :

$$\begin{aligned}\xi(x_{1,2,3}) &= \frac{ig^2}{4}\,\mathcal{O}'(\Phi(x_1))\mathcal{O}'(\Phi(x_2))\mathcal{O}'(\Phi(x_3)) \\ &\quad\times\int d^4y\; G_R(x_1,y)G_R(x_2,y)G_R(x_3,y)\,\Phi(y) \\ &=\end{aligned}$$

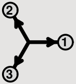


- Retarded propagator : $G_R(x_1, y) \sim e^{\mu(t_f - y^0)}$

$\sim g^4 \, e^{4\mu(t_f - t_i)}$



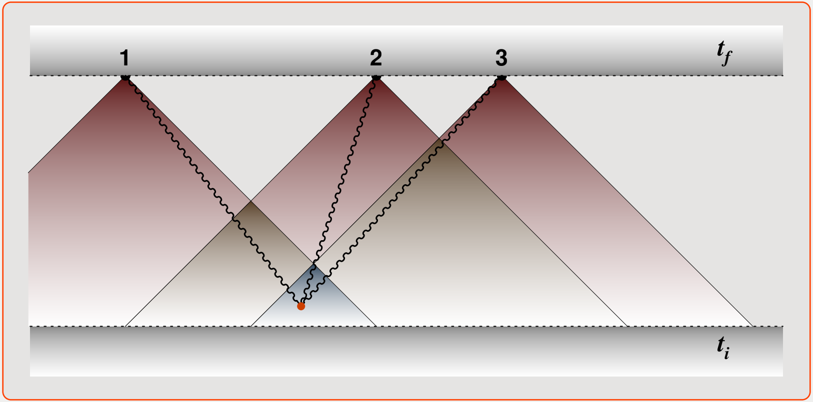
$\sim g^4 \int_{t_i}^{t_f} dy^0 \, e^{3\mu(t_f - y^0)} \sim g^4 \, e^{3\mu(t_f - t_i)}$



$\sim e^{-\mu(t_f - t_i)} \ll 1$

Causal structure

- Beyond the strong field regime, correlations are also created in the bulk ($y^0 > t_i$) by the interactions



Conclusions

- In the strong field regime :
  - all correlations at tree-level depend on the retarded classical field and its derivatives with respect to initial value
  - all correlations are created by initial state fluctuations
  - explicit dependence given by a formula that sums over all trees with n labeled nodes
- Beyond the strong field regime :
  - additional correlations created in the bulk

Mode functions

Definition

$$\left[\square + m^2 - \mathcal{L}''_{\text{int}}(\Phi(x))\right] a_{\pm\mathbf{k}}(x) = 0$$

$$a_{\pm\mathbf{k}}(x) \underset{x^0 \to t_i}{\to} e^{\mp i k \cdot x}$$

- Basis of the linear space of small perturbations around a classical solution

Inner product

- Define : $\big|a\big) \equiv \begin{pmatrix} a \\ \dot{a} \end{pmatrix}$, $\big(a\big| \equiv i \begin{pmatrix} -\dot{a}^* & a^* \end{pmatrix}$

$$\big(a_1\big|a_2\big) \equiv i\int d^3\mathbf{x}\,\Big[a_1^*(x)\,\dot{a}_2(x) - \dot{a}_1^*(x)\,a_2(x)\Big]$$

Properties :

Hermitean : $\big(a_2\big|a_1\big) = \big(a_1\big|a_2\big)^*$

Constant : $\partial_0\big(a_1\big|a_2\big) = 0$

$$\big(a_{+\mathbf{k}}\big|a_{+\mathbf{k}'}\big) = (2\pi)^3\,2E_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{k}')$$
$$\big(a_{-\mathbf{k}}\big|a_{-\mathbf{k}'}\big) = -(2\pi)^3\,2E_{\mathbf{k}}\,\delta(\mathbf{k}-\mathbf{k}')$$
$$\big(a_{+\mathbf{k}}\big|a_{-\mathbf{k}'}\big) = 0$$

Completeness

- A generic perturbation can be decomposed as :

$$\left|a\right) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left[\gamma_{+k} \left|a_{+k}\right) + \gamma_{-k} \left|a_{-k}\right)\right]$$

$$\text{with} \quad \gamma_{+k} = \left(a_{+k}\middle|a\right) \, , \quad \gamma_{-k} = -\left(a_{-k}\middle|a\right)$$

- Equivalently :

$$\left|a\right) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left[\left|a_{+k}\right)\left(a_{+k}\middle|a\right) - \left|a_{-k}\right)\left(a_{-k}\middle|a\right)\right]$$

Completeness of the mode functions :

$$\int \frac{d^3k}{(2\pi)^3 2E_k} \left[\left|a_{+k}\right)\left(a_{+k}\right| - \left|a_{-k}\right)\left(a_{-k}\right|\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$