Conformal anomaly, entanglement entropy and boundaries

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Plan of the talk:

1. Brief review: local Weyl anomaly, entanglement entropy

2. Integral Weyl anomaly in presence of boundaries
   a) d=4       b) d=6

3. Integral Weyl anomaly in odd dimensions

4. Entanglement entropy and boundaries

5. Some open questions
Based on


3. work in progress with Amin Astaneh, Clement Berthiere
Other recent relevant works:


Earlier relevant works:


Let me first remind you briefly the standard story
Local Weyl anomaly

\[ g^{\mu\nu} \langle T_{\mu\nu} \rangle = \frac{c}{24\pi} R, \quad d = 2 \]

\[ g^{\mu\nu} \langle T_{\mu\nu} \rangle = -\frac{a}{5760\pi^2} E_4 + \frac{b}{1920\pi^2} \text{Tr} W^2, \quad d = 4 \]

\[ \text{Tr} W^2 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \]

\[ E_4 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2. \]

(For scalar field \( a = b = 1 \))

\[ g^{\mu\nu} \langle T_{\mu\nu} \rangle = 0, \quad d = 2n + 1 \]
Entanglement entropy and Weyl anomaly

\( \Sigma \) is compact 2d entangling surface

\[
S_{d=4} = \frac{A(\Sigma)}{4\pi \epsilon^2} + s_0 \ln \epsilon
\]

\[
s_0 = \frac{a}{180} \chi[\Sigma] - \frac{b}{240\pi} \int_\Sigma [W_{abab} - \text{Tr} \hat{k}^2]
\]

\( \chi[\Sigma] \) is Euler number of \( \Sigma \)

\( W_{abab} \) is projection of Weyl tensor on subspace orthogonal to \( \Sigma \), \( n^a, a = 1, 2 \) is a pair of normal vectors

\[
\hat{k}_\mu^a = k_\mu^a - \frac{1}{d-2} \gamma_{\mu\nu}k^a, a = 1, 2 \text{ is trace-free extrinsic curvature of } \Sigma
\]
EE in $d$ dimensions

In $d$ dimensions compact entangling surface $\Sigma$ is $(d - 2)$-dimensional

Logarithmic term $s_0$ in entanglement entropy is given by integral over $\Sigma$ of a polynomial invariant constructed from Weyl tensor $W_{\mu\alpha\nu\beta}$, even number of covariant derivatives of Weyl tensor, extrinsic curvature $\hat{\kappa}_{\mu\nu}^a$ and projections on normal vectors $n_{\mu}^a$.

If $d$ is odd no such invariant exists so that

$$s_0 = 0 \quad i.f \quad d = 2n + 1$$
In this talk:

What changes if manifold has boundaries?
Conformal boundary conditions

General (mixed) boundary condition is a combination of Robin and Dirichlet b.c.

\[(\nabla_n + S)\Pi_+ \varphi|_{\partial M} = 0, \quad \Pi_- \varphi|_{\partial M} = 0, \quad \Pi_+ + \Pi_- = 1\]
Conformal scalar field in $d$ dimensions

Dirichlet b.c. ($\Pi_+ = 0$)

$$\phi|_{\partial M} = 0$$

Conformal Robin b.c. ($\Pi_- = 0$)

$$(\nabla_n + \frac{(d-2)}{2(d-1)}K)\phi|_{\partial M} = 0$$

Remark: in $d = 4$ exists one more (complex) Robin b.c.

$$S = \frac{1}{3}K \pm \frac{i}{10} \sqrt{10 \text{Tr} \hat{K}^2}, \quad \hat{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}K$$

for which (classical and quantum) theory is conformal
Dirac field in \( d = 4 \) dimensions

\[
\Pi^- \psi \big|_{\partial M} = 0, \quad (\nabla_n + K/2) \Pi^+ \psi \big|_{\partial M} = 0
\]

\[
\Pi_\pm = \frac{1}{2}(1 \pm i \gamma^* n^\mu \gamma_\mu), \quad \gamma^* \text{ is chirality gamma function}
\]
Integral Weyl anomaly

Variation of effective action under constant rescaling of metric

\[ A \equiv \partial_\sigma W[e^{2\sigma}g_{\mu\nu}] = \int_{\mathcal{M}_d} \langle T^\mu_\mu \rangle \]

For free fields integral Weyl anomaly reduces to computation of heat kernel coefficient \( A_d \).
General structure

\[ \int_{\mathcal{M}_d} \sqrt{g} \langle T_{\mu\nu} \rangle g^{\mu\nu} = a \, \chi(\mathcal{M}_d) + b_k \int_{\mathcal{M}_d} \sqrt{\gamma} I_k(W) \]

\[ + b'_k \int_{\partial \mathcal{M}_d} \sqrt{\gamma} J_k(W, \hat{K}) + c_n \int_{\partial \mathcal{M}_d} \sqrt{\gamma} K_n(\hat{K}), \]

\( \chi[\mathcal{M}_d] \) is Euler number of manifold \( \mathcal{M}_d \), \( I_k(W) \) are conformal invariants constructed from the Weyl tensor, \( K_n(\hat{K}) \) are polynomial of degree \((d - 1)\) of the trace-free extrinsic curvature, \( K_{\mu\nu} = K_{\mu\nu} - \frac{1}{d-2} \gamma K \) is trace free extrinsic curvature of boundary; \( \hat{K}_{\mu\nu} \to e^\sigma \hat{K}_{\mu\nu} \) if \( g_{\mu\nu} \to e^\sigma g_{\mu\nu} \).
Q: Does it mean that there are new conformal charges $b'_n, c_n$?

A: we suggest that in appropriate normalization $b'_n = b_n$ and the corresponding boundary term $J_k(W, \hat{K})$ is in fact the Hawking-Gibbons type term for the bulk action $I_k(W)$

$c_n$ are indeed new boundary conformal charges
Gibbons-Hawking type terms

Re-writing functional of curvature in a form linear in Riemann tensor

\[ I_{\text{bulk}} = \int_{\mathcal{M}_d} \left( U^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - U^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu} + F(V) \right) \]

In order to cancel normal derivatives of the metric variation on the boundary one should add a boundary term,

\[ I_{\text{boundary}} = -\int_{\partial\mathcal{M}_d} U^{\alpha\beta\mu\nu} P^{(0)}_{\alpha\beta\mu\nu} \]

\[ P^{(0)}_{\alpha\beta\mu\nu} = n_\alpha n_\nu K_{\beta\mu} - n_\beta n_\nu K_{\alpha\mu} - n_\alpha n_\mu K_{\beta\nu} + n_\beta n_\mu K_{\alpha\nu} \]

\( n^\mu \) is normal vector and \( K_{\mu\nu} \) is extrinsic curvature of \( \partial\mathcal{M}_d \)

Barvinsky-SS (95)
For a bulk invariant expressed in terms of Weyl tensor only,

$$I[W] = \int_{\mathcal{M}_d} \left( U^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu} - U^{\alpha\beta\mu\nu} V_{\alpha\beta\mu\nu} + F(V) \right)$$

$$- \int_{\partial \mathcal{M}_d} U^{\alpha\beta\mu\nu} P_{\alpha\beta\mu\nu}$$

$$P_{\alpha\beta\mu\nu} = P_{\alpha\beta\mu\nu}^{(0)} - \frac{1}{d-2} \left( g_{\alpha\mu} P_{\beta\nu}^{(0)} - g_{\alpha\nu} P_{\beta\mu}^{(0)} - g_{\beta\mu} P_{\alpha\nu}^{(0)} \right)$$

$$+ g_{\beta\nu} P_{\alpha\mu}^{(0)} + \frac{P^{(0)}}{(d-1)(d-2)} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})$$

$$P_{\mu\nu}^{(0)} = n_\mu n^\alpha K_{\alpha\beta} + n_\mu n^\alpha K_{\alpha\nu} - K_{\mu\nu} - n_\mu n_\nu K$$

$$P^{(0)} = -2K$$

$P_{\alpha\beta\mu\nu}$ has same symmetries as the Weyl tensor. In particular, $P_{\mu\alpha\nu}^{\alpha} = 0$.

$P_{\alpha\beta\mu\nu}$ can be expressed in terms of $\hat{K}_{\mu\nu}$.
Examples

1. \[ \int_{\mathcal{M}_d} \text{Tr} (W^n) - \int_{\partial \mathcal{M}_d} n \text{Tr} (PW^{n-1}) \]

2. \[ \int_{\mathcal{M}_d} \text{Tr} (W \nabla^2 W) - 2 \int_{\partial \mathcal{M}_d} \text{Tr} (P \nabla^2 W) \]
Integral Weyl anomaly in $d = 4$: anomaly of type A

First of all, bulk integral of $E_4$ is supplemented by some boundary terms to form a topological invariant, the Euler number,

$$\chi[\mathcal{M}_4] = \frac{1}{32\pi^2} \int_{\mathcal{M}_4} E_4$$

$$- \frac{1}{4\pi^2} \int_{\partial \mathcal{M}_4} \left( K^{\mu\nu} R_{n\mu n\nu} - K^{\mu\nu} R_{\mu\nu} - K R_{nn} + \frac{1}{2} K R \right. $$

$$\left. - \frac{1}{3} K^3 + K Tr K^2 + \frac{2}{3} Tr K^3 \right)$$

$R_{\mu\nu n n} = R_{\mu\nu\alpha\beta} n^\alpha n^\beta$ and $R_{nn} = R_{\mu\nu} n^\mu n^\nu$

Dowker-Schofield (90)

Herzog-Huang-Jensen (2015)
Integral Weyl anomaly in $d = 4$: anomaly of type B

Gibbons-Hawking type boundary term:

$$\int_{\mathcal{M}_4} \text{Tr} W^2 - 2 \int_{\partial \mathcal{M}_4} \text{Tr} (WP)$$

Due to properties of Weyl tensor:

$$\text{Tr} (WP) = \text{Tr} (WP^{(0)}) = 4W^{\mu\nu\alpha\beta}n_\mu n_\beta \tilde{K}_{\nu\alpha}$$
Integral Weyl anomaly in $d = 4$

\[
\int_{\mathcal{M}_4} \langle T \rangle = - \frac{a}{180} \chi[\mathcal{M}_4]
\]

\[
+ \frac{b}{1920\pi^2} \left( \int_{\mathcal{M}_4} \text{Tr} W^2 - 8 \int_{\partial \mathcal{M}_4} W^\mu{}_{\nu\alpha\beta} n^\mu n_\beta \hat{K}^\nu\alpha \right)
\]

\[
+ \frac{c}{280\pi^2} \int_{\partial \mathcal{M}_4} \text{Tr} \hat{K}^3
\]

For B-anomaly balance between bulk and boundary terms agrees with calculation for free fields of spin $s=0,1/2,1$

Fursaev (2015)

also Herzog-Huang-Jensen (2015)
Values of boundary charge $c$:

(Malmed (88), Dowker-Schofield (95), Fursaev (2015))

$c = 1$ for $s = 0$ (Dirichlet b.c.)

$c = 7/9$ for $s = 0$ (Robin b.c.)

$c = 5$ for $s = 1/2$ (mixed b.c.)

$c = 8$ for $s = 1$ (absolute or relative b.c.)
Local Weyl anomaly in $d = 6$

$$\langle T \rangle = A = aE_6 + b_1 I_1 + b_2 I_2 + b_3 I_3 + TD$$

where $E_6$ is the Euler density in $d = 6$ and we defined

$$I_1 = \text{Tr}_1(W^3) = W_{\alpha \mu \nu \beta} W^{\mu \sigma \rho \nu} W_{\sigma}^{\alpha \beta \rho}$$

$$I_2 = \text{Tr}_2(W^3) = W_{\alpha \beta}^{\mu \nu} W_{\mu \nu}^{\sigma \rho} W_{\sigma \rho}^{\alpha \beta}$$

$$I_3 = \text{Tr} (W \nabla^2 W) + \text{Tr}_2(WXW)$$

$$X_{\alpha \beta}^{\mu \nu} = X^{[\mu \delta^\nu]}_{[\alpha \beta]}, \quad X_\nu^\mu = 4R_\nu^\mu - \frac{6}{5}R \delta_\nu^\mu$$
Integral Weyl anomaly in $d = 6$

$$\int_{\mathcal{M}_6} \langle T \rangle = a' \chi[\mathcal{M}_6]$$

$$+ b_1 \left( \int_{\mathcal{M}_6} \text{Tr}_1 W^3 - 3 \int_{\partial\mathcal{M}_6} \text{Tr}_1 (PW^2) \right)$$

$$+ b_2 \left( \int_{\mathcal{M}_6} \text{Tr}_2 W^3 - 3 \int_{\partial\mathcal{M}_6} \text{Tr}_2 (PW^2) \right)$$

$$+ b_3 \left[ \int_{\mathcal{M}_6} \text{Tr} (W \nabla^2 W) - 2 \int_{\partial\mathcal{M}_6} \text{Tr} (P \nabla^2 W) \right.$$  

$$+ \int_{\mathcal{M}_6} \text{Tr}_2 (WXW) - \int_{\partial\mathcal{M}_6} \text{Tr}_2 (WQW)$$

$$+ \int_{\partial\mathcal{M}_6} \left( c_1 \text{Tr} \hat{K}^2 \text{Tr} \hat{K}^3 + c_2 \text{Tr} \hat{K}^5 \right)$$

two new boundary charges $c_1$ and $c_2$

there may exist additional invariant with derivatives of extrinsic curvature
Integral Weyl anomaly in odd dimensions

Euler number of $\mathcal{M}_d$ vanishes if $d$ is odd

Euler number of boundary $\partial \mathcal{M}_d$ may appear in integral anomaly

$d = 3$ :

$$\int_{\mathcal{M}_3} \langle T \rangle = \frac{c_1}{96} \chi[\partial \mathcal{M}_3] + \frac{c_2}{256\pi} \int_{\partial \mathcal{M}_3} \text{Tr} \hat{R}^2$$

$(c_1, c_2)$:

$(-1, 1)$ for scalar field (Dirichlet b.c.)

$(1, 1)$ for scalar field (conformal Robin b.c.)

$(0, 2)$ for Dirac field (mixed b.c.)

Remark: similar anomaly for defects Jensen-O’Bannon (2015)
Integral Weyl anomaly in $d = 5$

$$\int_{\mathcal{M}_5} \langle T \rangle = c_1 \chi[\partial\mathcal{M}_5]$$

$$+ \int_{\partial\mathcal{M}_5} \left[ c_2 \text{Tr} \, W^2 + c_3 W_{\alpha n\beta n} W^{\alpha \beta} + c_4 W_{n\alpha\beta\mu} W^{\alpha\beta\mu} 
+ c_5 W^{\alpha\mu\beta\nu} \hat{K}_{\alpha\beta} \hat{K}_{\mu\nu} + c_6 W^{\alpha \beta}_{\ n\ n} \hat{K}_{\alpha\sigma} \hat{K}^{\sigma}_{\beta} 
+ c_7 (\text{Tr} \, \hat{K}^2)^2 + c_8 \text{Tr} \, \hat{K}^4 + c_9 \text{Tr} \, (\hat{K} D \hat{K}) \right]$$

$\mathcal{D}$ is conformal operator acting on trace free symmetric tensor in 4 dimensions

values of $c_k$ for conformal scalar field: work in progress with Clement Berthiere
Entanglement entropy: $d = 3$
(recent work with Fursaev)

Renyi entropy

$$S^{(n)} \simeq c(n) L/\epsilon - \ln(\epsilon) s^{(n)}$$

$$s^{(n)} = \eta \frac{nA_3(1) - A_3(n)}{n - 1}$$

$A_3(n)$ is heat kernel coefficient on replica manifold $\mathcal{M}_n$
Consider $\mathcal{M} = R^2 \times L$, $L$ is an interval with 2 end points $P_1$ and $P_2$

Entangling surface $\Sigma = L$, replica space $\mathcal{M}_n = C_n \times L$

$$A_3(n) = A_2(C_n) \times A_1(L),$$

$$A_2(C_n) = \frac{1}{12n}(1 - n^2)$$

is the heat kernel coefficient on two-dimensional cone, and

$$A_1(L) = \frac{1}{4} \sum_{P_k} \text{tr} \chi, \quad \chi = \Pi_+ - \Pi_-$$
for scalar field

\[ s^{(n)} = \frac{c_1}{48} \frac{n+1}{n} \sum_P, \quad s^{(n=1)} = \frac{c_1}{24} \sum_P \]

for Dirac field

\[ s^{(n)} = 0, \quad s^{(n=1)} = 0 \]
INTERESTING PREDICTION:

dependence on angle between entangling surface $\Sigma$ and boundary $\partial M$

$$\cos \alpha = (n, t),$$

$n^\mu$ normal vector to $\partial \mathcal{M}_3$, 

t$^\mu$ tangent vector to $\Sigma$.

Assume that the bulk $\mathcal{M}_n$ contains a conical singularity then:

scalar curvature of the boundary

$$\int_{\partial \mathcal{M}_n} \hat{R} \simeq 4\pi \cos \alpha (1 - n), \quad n \to 1$$

and extrinsic curvature of the boundary

$$\int_{\partial \mathcal{M}_n} K^2 \simeq \int_{\partial \mathcal{M}_n} \text{Tr} K^2 \simeq 8\pi (1 - n) f(\alpha) ,$$

$$f(\alpha) = -\frac{1}{32} \frac{\sin^2 \alpha}{\cos \alpha} (1 + 2 \cos^2 \alpha + 5 \cos^4 \alpha)$$
OTHER DIMENSIONS

\[ P = \Sigma \cap \partial M_d \quad dim(P) = d - 3 \]

\[ p^\mu_a, a = 1, 2 \] normal vectors to \( P \) in \( \partial M_d \)

\( \hat{K}^a_{\mu\nu} \) is respective extrinsic curvature of \( P \)

\[ \hat{K}_{ab} = p^\alpha_a p^\beta_b \hat{K}_{\alpha\beta} \]

\( d = 3 : \) \( dim(P) = 0 \) \( s_0(P) \sim \sum_P \)

\( d = 4 : \) \( dim(P) = 1 \) \( s_0(P) \sim \int_P \hat{K}_{aa} \)

\( d = 5 : \) \( dim(P) = 2 \)

possible terms in \( s_0(P) \): \( \chi(P) \), \( W_{nana} \), \( W_{abab} \), \( (\hat{K}_{aa})^2 \), \( \hat{K}_{ab}\hat{K}_{ab} \), \( tr \hat{k}^2 \) and terms with two derivatives of extrinsic curvature
RESUME

in presence of boundaries integral Weyl anomaly is modified by boundary terms

boundary terms for B-anomaly are of Gibbons-Hawking type

additional new boundary charges

in odd dimensions integral Weyl anomaly is non-vanishing (!) and is entirely due to boundary terms

if intersection of entangling surface and boundary is $P$ then there appear new contributions to EE (and RE) due to $P$

in odd dimensions log term in EE (and RE) is non-vanishing (!) and is entirely due to $P$
SOME OPEN QUESTIONS

1. how derive boundary charges from \( n \)-point correlation functions in CFT?

2. what is holographic description of boundary terms in anomaly and in EE?
   (work in progress with Amin Astaneh)
THANK YOU!