Differential Entropy

References:

[1] 1310.4204 "A hole-ographic spacetime"
Balasubramanian, Chowdhury, Czech, de Boer, Heller

[2] 1403.3416 "Holographic Holes in Higher Dimensions"
Myers, Rao, Sugishita

[3] 1408.4770 "Holographic Holes and Differential Entropy"
Headrick, Myers, Wien

[4] 1409.4473 "Nuts and Bolts for Creating Space"
Czech, Lamprou

[5] 1410.1540 "The information Theoretic Interpretation of the Length of a Curve"
Czech, Hayden, Lashkari, Swingle

Outline:

1) Philosophical remarks / motivation

2) A short review of Entanglement Entropy and the Ryu-Takayanagi (RT) formula

3) An example of EE and Differential Entropy (DE)

4) A proof of DE = Length of a bulk curve in AdS_3

5) A proof of DE = Length of a bulk curve in an arbitrary spacetime

6) Subsequent developments

Motivation

- Recall last Fridays colloqium. Hermann Nicolai posed the following kind of question

HN: What happens at singularities in quantum gravity? Is the singularity resolved? Can AdS/CFT as a non-perturbative definition of quantum gravity answer these questions?

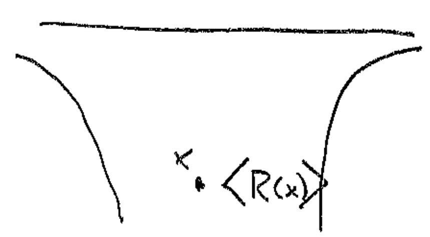
VK: Yes indeed it can in principle, but I don't know how to calculate in a strongly coupled CFT...

HN: Fair enough. So let's imagine someone handed you all the correlators in this strongly coupled CFT. Could you tell what happens to the bulk singularity?

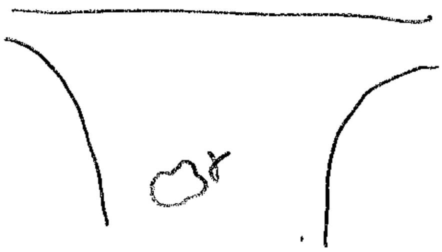
VK: Well... The dictionary between the bulk and the boundary is very indirect so in practice propably not...

HN: Then I guess you should work harder on the dictionary to find CFT observables that correspond to local "observables" in the bulk.

E.g.



The differential entropy is a step in this direction. It calculates lengths of curves in the bulk



$$E = \frac{L_\gamma}{4G_N}$$

Know lengths $\rightarrow$ Know metric

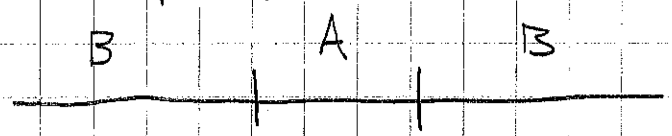
$\rightarrow$ Know curvature !

My goal today is to motivate the CFT expression for E and prove it in some cases.

2. Entanglement Entropy and the RT formula

From now on will specialize to AdS_3/CFT_2.

Split space at a constant time slice into two parts



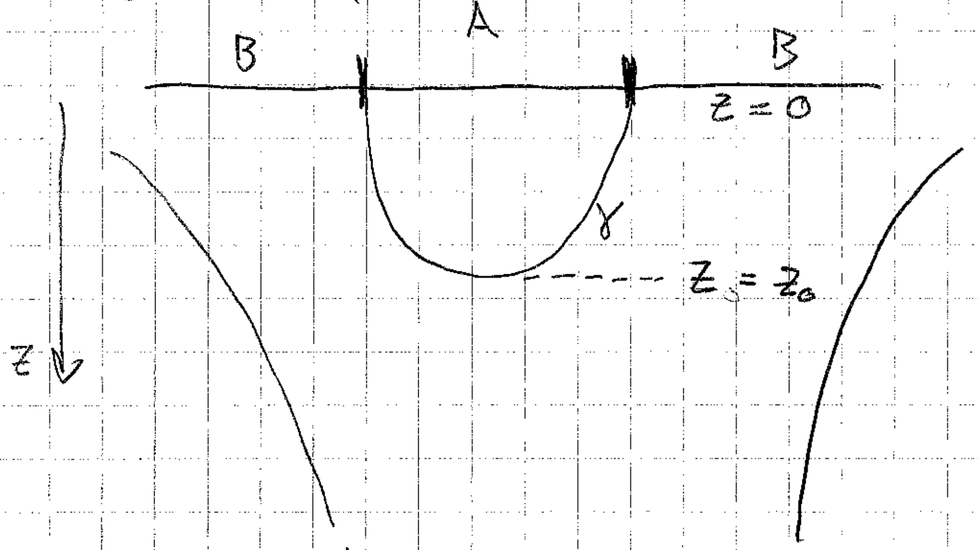
Density matrix of full system

Reduced density matrix: $\rho_A = tr_B\, \rho$

Entanglement entropy: $S_{EE} = -tr_A(\rho_A \log \rho_A)$

Only important thing about S_{EE} is that it is a quantity one can calculate in the QFT_2.

The holographic dual of S_{EE} is known, and is given by the RT formula



$S_{EE} = \dfrac{L_\gamma}{4G_N}$, where L_γ is the minimum length geodesic ending at the boundary at ∂A.

S_{EE} carries information about bulk geometry within the region $z \in (0, z_0)$.

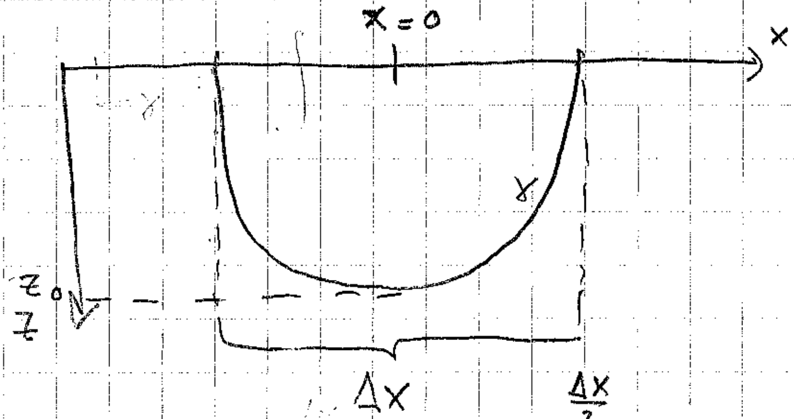
Differential entropy is a quantity carrying information local in the bulk at $z = z_0$

Example RT formula in AdS_3

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2\right)$$

Parametrize geodesic as $z = z(x)$.
Induced metric

$$dh^2 = \frac{1}{z(x)^2}\left(\left(z'(x)\right)^2 + 1\right)dx^2$$



$$L_\gamma = \int dh_\gamma = \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} dx\, \frac{1}{z(x)}\sqrt{1 + \left(z'(x)\right)^2} \equiv \int dx\, \mathcal{L}$$

Minimize $L_\gamma \longrightarrow$ Euler-Lagrange eqns.
Since $\mathcal{L}$ is indep. of x, there is a conserved "Hamiltonian"

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial z'} z' - \mathcal{L} = \frac{(z')^2}{z\sqrt{1+z'^2}} - \frac{\sqrt{1+z'^2}}{z}$$

$$= -\frac{1}{z\sqrt{1+z'^2}}$$

Evaluate at the turning point $z = z_0$, where $z' = 0$

$$\mathcal{H} = -\frac{1}{z_0}$$

$$\sqrt{1+z'^2} = \frac{z_0}{z} \Rightarrow z'^2 = \left(\frac{z_0}{z}\right)^2 - 1$$

$$z' = -\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} = -\frac{1}{z}\sqrt{z_0^2 - z^2}$$

$$-\int_{z_0}^{z(x)} \frac{dz\, z}{\sqrt{z_0^2 - z^2}} = x \Rightarrow \left.\sqrt{z_0^2 - z^2}\right|_{z_0}^{z(x)} = x$$

$$x = \sqrt{z_0^2 - z^2}$$

$$\boxed{x^2 + z^2 = z_0^2}$$

Minimal surface is a semi-circle.

$$L_\gamma = 2 \int_0^{\frac{\Delta x}{2}} dx \, \frac{1}{z} \underbrace{\sqrt{1 + z'^2}}_{= \frac{z_0}{z}} = 2 \int dx \, \frac{z_0}{z^2}$$

$$= 2 \int_{z_0}^{0} dz \, \frac{x'(z)}{x'(z)} \frac{z_0}{z^2} \qquad , \; x' = - \frac{z}{\sqrt{z_0^2 - z^2}}$$

$$= -2 \int_{z_0}^{0} dz \, \frac{z_0}{z\sqrt{z_0^2 - z^2}}$$

Need a cutoff at $z = \varepsilon$ due to $1/z$ in the integrand.

$$= 2 \int_{\varepsilon}^{z_0} dz \, \frac{z_0}{z\sqrt{z_0^2 - z^2}}$$

$$= 2 \log \left(\frac{z_0 + \sqrt{z_0^2 - \varepsilon^2}}{\varepsilon} \right) \approx 2 \log \left(\frac{2 z_0}{\varepsilon} \right)$$

Since the geodesic is a semi-circle, we know $\Delta x = 2 z_0$.

$$L_\gamma = 2 \log \left(\frac{\Delta x}{\varepsilon} \right)$$

$$S_{EE} = \frac{L_\gamma}{4 G_N} = \frac{1}{2 G_N} \log \left(\frac{\Delta x}{\varepsilon} \right)$$

Use $c = \frac{3}{2 G_N}$

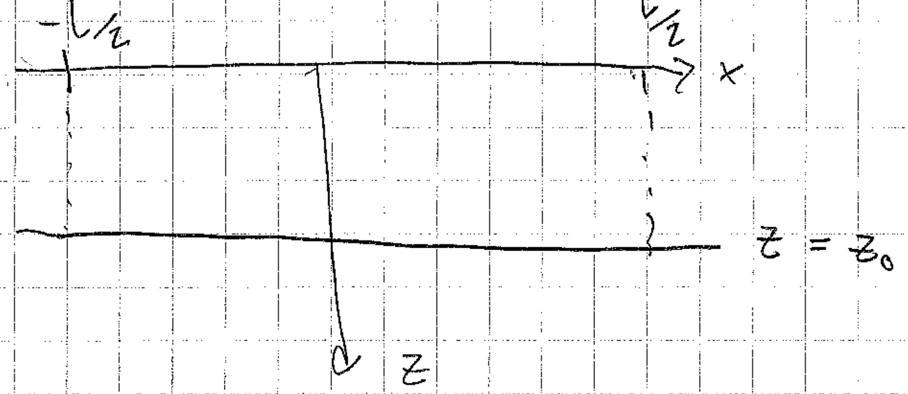
$$S_{EE} = \frac{c}{3} \log \left(\frac{\Delta x}{\varepsilon} \right)$$

Agrees with the CFT result, see e.g. "Entanglement Entropy and Quantum Field Theory" 0405152v3, Cardy & Calabrese

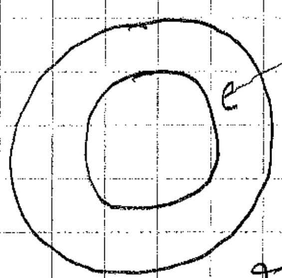
Example of Differential Entropy

S_{EE} knows about bulk geometry in the region $z \in (0, z_0)$, with $z_0 = \frac{\Delta x}{2}$. Can we construct a quantity which knows about the geometry at $z = z_0$?

Construct a CFT quantity which computes the length of a line at $z = z_0$

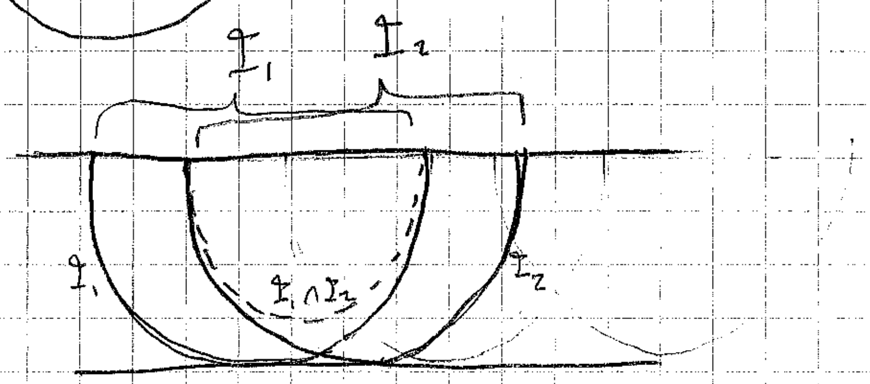


Note, this would be more "natural" in global AdS

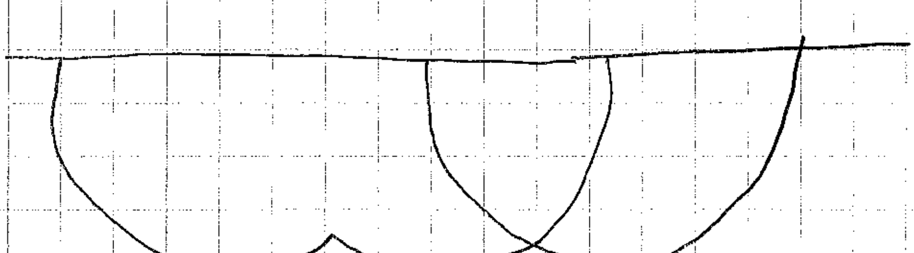


Length of a circle in global AdS.

Idea:

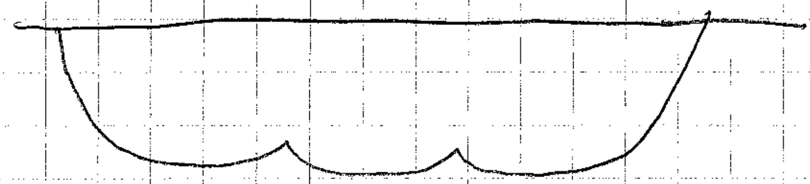


$$S(I_1) + S(I_2) - S(I_1 \cap I_2)$$



Repeat

$$S(I_1)+S(I_2)-S(I_1\cap I_2)+S(I_3)-S(I_2\cap I_3)$$



$\vdots$

Take a limit as $I_i \cap I_j$ shrink towards zero size.

A bit more precisely $I_j = \left(-\frac{\Delta x}{2}+\frac{j}{K}L,\ \frac{\Delta x}{2}+\frac{j}{K}L\right)$

Take x-direction to be compact $x \sim x+L$.
The number of intervals $K \to \infty$

<u>Differential Entropy</u>:

$$\left[\ E = \lim_{K\to\infty}\sum_{j=1}^{K}\left(S(I_j)-S(I_j\cap I_{j+1})\right)\ \right]$$

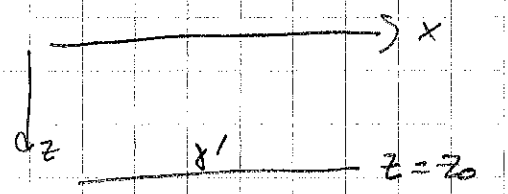
For our example $S(I_j) = \frac{1}{2G_N}\log\left(\frac{\Delta x}{\varepsilon}\right)$

$$S(I_j\cap I_{j+1}) = \frac{1}{2G_N}\log\left(\frac{\Delta x - \frac{L}{K}}{\varepsilon}\right)$$

$$E = \lim_{K\to\infty} K\left(\frac{1}{2G_N}\log\left(\frac{\Delta x}{\varepsilon}\right)-\frac{1}{2G_N}\log\left(\frac{\Delta x-\frac{L}{K}}{\varepsilon}\right)\right)$$

$$= \frac{L}{2G_N}\frac{1}{\Delta x} = \frac{L}{4G_N z_0}$$

Compare to bulk $dh^2 = \frac{1}{z_0^2}dx^2$

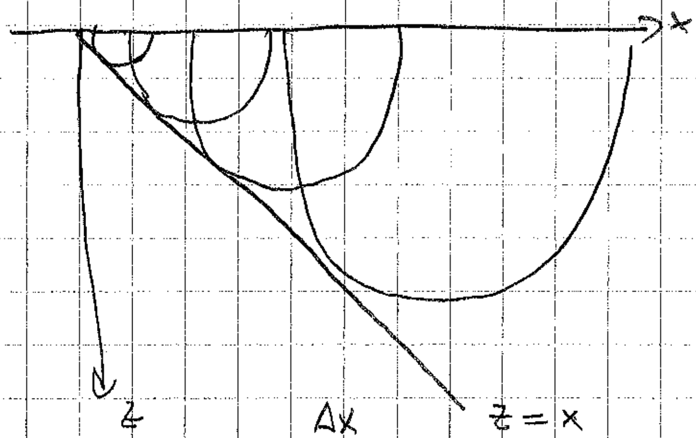


$$\frac{L_{\gamma'}}{4G_N} = \frac{1}{4G_N}\int dh = \frac{1}{4G_N}\int_0^L dx\,\frac{1}{z_0}$$

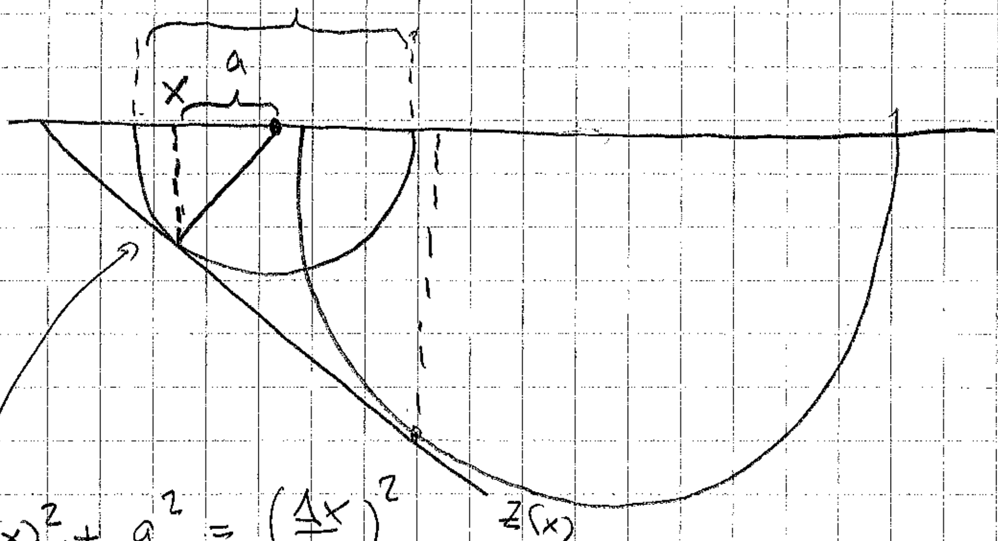
$$= \frac{L}{4G_N z_0}.$$

More general bulk curves

Let's say we want to compute the length of the curve $z = x$.



← Cover the curve with geodesics tangent to that curve. Need circles with increasing size. $\Delta x\,(x)$



$$\begin{cases} z(x)^2 + a^2 = \left(\frac{\Delta x}{2}\right)^2 \\ \frac{a}{z(x)} = z'(x) \end{cases}$$

$z(x) = x$, $x^2 + a^2 = \left(\frac{\Delta x}{2}\right)^2 \Rightarrow \Delta x = 2\sqrt{x^2 + a^2}$

$z(x) = a = x$

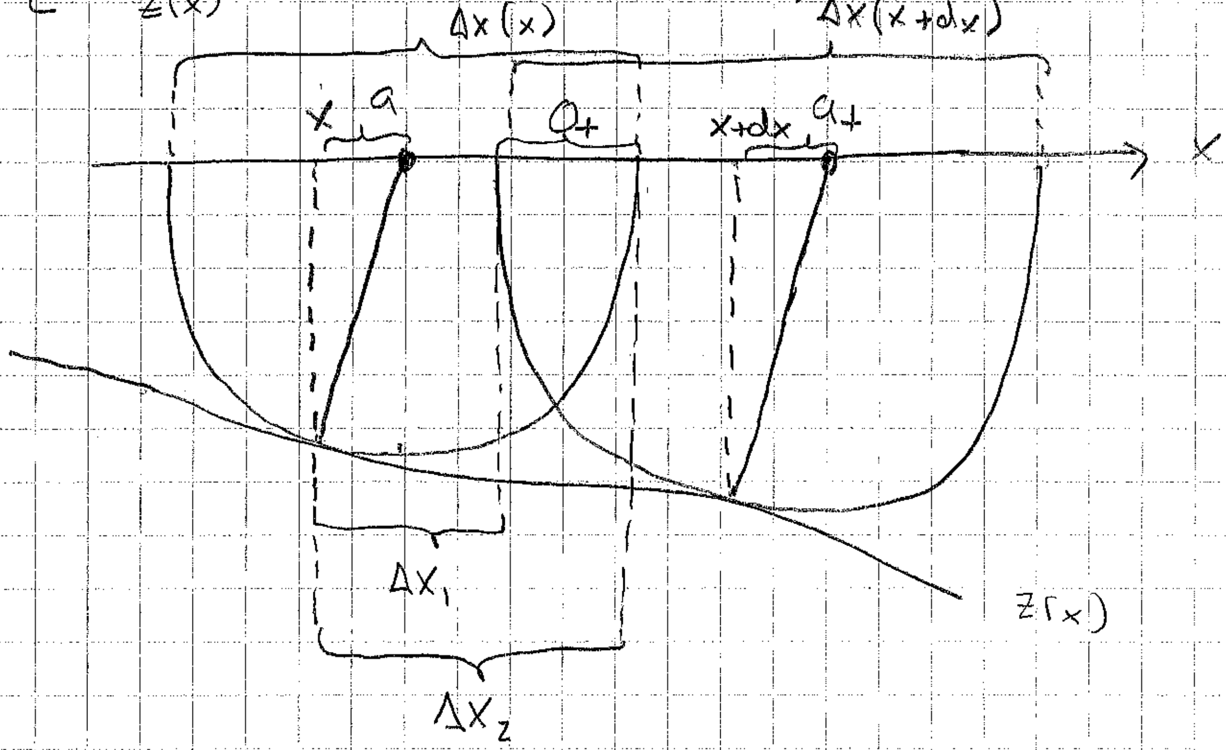
$$\Rightarrow \Delta x = 2\sqrt{2}\,|x|$$

If we want the length of the curve $z(x) = x$ we need the size of the circles to grow as $\Delta x = 2\sqrt{2}\,|x|$

Now let's continue on the general case $z = z(x)$, following ref. [2]

Still we have

$$\begin{cases} z(x)^2 + a^2 = \left(\frac{\Delta x}{2}\right)^2 & (1) \\ \frac{a}{z(x)} = z'(x) & (2) \end{cases}$$



$$\Delta x_1 = dx + a_+ - \frac{\Delta x(x+dx)}{2}$$

$$\Delta x_2 = a + \frac{\Delta x(x)}{2}$$

$$O_+ = \Delta x_2 - \Delta x_1 = a - a_+ + \frac{1}{2}\left(\Delta x(x) + \Delta x(x+dx)\right) - dx$$

Repeat for another circle at $x - dx$

$a_+ \to a$, $a \to a_-$

$$O_- = a_- - a + \frac{1}{2}\left(\Delta x(x-dx) + \Delta x(x)\right) - dx$$

$$O_\pm = \pm a \mp a_\pm + \frac{1}{2}\left(\Delta x(x) + \Delta x(x \pm dx)\right) - dx$$

$$O_+ = a(x) - a(x+dx) + \frac{1}{2}\left(\Delta x(x) + \Delta x(x+dx)\right) - dx = \Delta x(x) - a'(x)dx + \frac{1}{2}\Delta x'(x)dx - dx$$

Use

$$a = z'(x)\, z(x)$$

$$a_\pm = z'(x \pm dx)\, z(x \pm dx) = (z' \pm z''dx)(z \pm z'dx)$$
$$= z'z \pm (zz'' + z'^2)\, dx$$

$$\Delta x = 2\sqrt{z^2 + a^2} = 2z\sqrt{1 + z'^2}$$

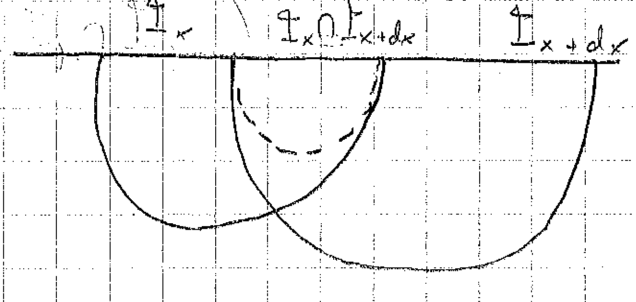
$$\Delta x(x \pm dx) = 2(z \pm z'dx)\sqrt{1 + (z' \pm z''dx)^2}$$
$$= 2z\sqrt{1 + z'^2 \pm 2z'z''dx} \pm 2z'\sqrt{1+z'^2}\, dx$$
$$= 2z\left(\sqrt{1+z'^2} \pm \frac{z'z''}{\sqrt{1+z'^2}}\, dx\right) \pm 2z'\sqrt{1+z'^2}\, dx$$
$$= 2z\sqrt{1+z'^2} \pm \left(2z'\sqrt{1+z'^2} + 2\frac{zz'z''}{\sqrt{1+z'^2}}\right) dx$$

$$O_\pm = \pm \cancel{zz'} \mp \left(\cancel{z'z} \pm (zz'' + z'^2)\right) dx$$
$$+ \frac{1}{2}\left(2z\sqrt{1+z'^2} + 2z\sqrt{1+z'^2} \pm \left(2z'\sqrt{1+z'^2} + 2\frac{zz'z''}{\sqrt{1+z'^2}}\right) dx\right) - dx$$
$$= 2z\sqrt{1+z'^2} - (1 + zz'' + z'^2)\, dx$$
$$\pm \left(2z'\sqrt{1+z'^2} + 2\frac{zz'z''}{\sqrt{1+z'^2}}\right) dx$$

$$E = \sum_x \left(S(I_x) - S(I_x \cap I_{x+dx})\right)$$
$$= \sum_x \left(S(I_x) - \frac{1}{2} S(I_x \cap I_{x+dx}) - \frac{1}{2} S(I_x \cap I_{x-dx})\right)$$

$$S(I_x) = \frac{1}{2G_N} \log \frac{\Delta x(x)}{\varepsilon}$$

$$S(I_x \cap I_{x \pm dx}) = \frac{1}{2G_N} \log \frac{O_\pm}{\varepsilon}$$



$$S(\mathcal{I}_x) - \frac{1}{2} S(\mathcal{I}_x \cap \mathcal{I}_{x+dx}) - \frac{1}{2} S(\mathcal{I}_x \cap \mathcal{I}_{x-dx})$$

$$= \frac{1}{2G_N}\left(\log \frac{\Delta x(x)}{\varepsilon} - \frac{1}{2}\log\frac{o_+}{\varepsilon} - \frac{1}{2}\log\frac{o_-}{\varepsilon}\right)$$

$$= \frac{1}{4G_N} \log \frac{\Delta x(x)^2}{o_+ o_-} = \frac{1}{4G_N} \log \left[\frac{\left(2z\sqrt{1+z'^2}\right)^2}{\left(2z\sqrt{1+z'^2} - (1+zz''+z'^2)dx\right)^2 + \mathcal{O}(dx^2)} \right]$$

$$= -\frac{1}{4G_N} \log \left[1 - \frac{1+zz''+z'^2}{2z\sqrt{1+z'^2}}\, dx \right]^2 + \mathcal{O}(dx^2)$$

$$= \frac{1}{2G_N} \frac{1+z'^2+zz''}{2z\sqrt{1+z'^2}}\, dx + \mathcal{O}(dx^2)$$

$$E = \lim_{dx\to 0} \frac{1}{4G_N} \sum_x \left[\frac{1+z'^2+zz''}{z\sqrt{1+z'^2}}\, dx + \mathcal{O}(dx^2) \right]$$

$$= \frac{1}{4G_N} \int_{x_i}^{x_f} dx\, \frac{1+z'^2+zz''}{z\sqrt{1+z'^2}}$$

$$= \frac{1}{4G_N} \int_{x_i}^{x_f} dx \left(\frac{\sqrt{1+z'^2}}{z} + \frac{z''}{\sqrt{1+z'^2}} \right)$$

$$= \frac{1}{4G_N} \int_{x_i}^{x_f} dx\, \frac{\sqrt{1+z'^2}}{z} + \frac{1}{4G_N} \underbrace{\int_{x_i}^{x_f} dz'\, \frac{1}{z''} \frac{z''}{\sqrt{1+z'^2}}}_{= \int dz' \frac{1}{\sqrt{1+z'^2}} = \Big|_{x_i}^{x_f} \operatorname{arcsinh}(z')}$$

This is exactly the length of the curve $z = z(x)$, if the total derivative term vanishes.

$x = \sinh y$;

$ch^2 - sh^2 = 1$ $\qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$

Comments:

- The continuum limit of E is given as follows.

$$O_+ = \Delta x(x) - \left(1 + a'(x) + \tfrac{1}{2}\Delta'(x)\right) dx$$
$$O_- = \Delta x(x) - \left(1 + a'(x) - \tfrac{1}{2}\Delta'(x)\right) dx$$

$$E = \lim_{dx \to 0} \sum_x \left(S(\Delta x(x)) - \tfrac{1}{2} S(O_+) - \tfrac{1}{2} S(O_-) \right)$$
$$= \lim_{dx \to 0} \sum_x \left[-\frac{1}{2} \frac{dS(\Delta x)}{d\Delta x}\left(-1 - a' - \tfrac{1}{2}\Delta'\right) - \frac{1}{2}\frac{dS(\Delta x)}{d\Delta x}\left(-1 - a' + \tfrac{1}{2}\Delta'\right)\right]$$
$$= \int dx \left(1 + a'(x)\right) \cdot \frac{dS(\Delta x)}{d\Delta x} \quad \text{, where}$$

$$a(x) = z'(x)\, z(x)$$
$$\Delta x = 2 z(x) \sqrt{1 + z'(x)^2}$$

- Denote the midpoint of the intervals as

$$x_c = x + a(x) \quad \Rightarrow \quad \frac{dx_c}{dx} = 1 + a'(x)$$

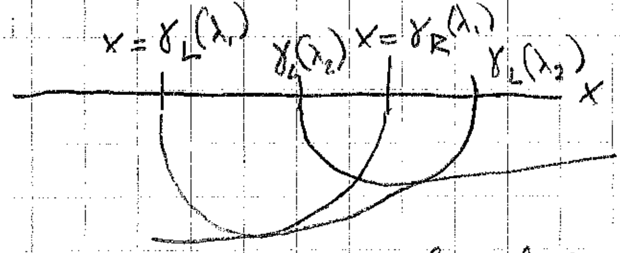
$$E = \int dx \frac{dx_c}{dx} \frac{dS(\Delta x)}{d\Delta x} = \int dx_c \frac{dS(\Delta x)}{d\Delta x}$$

- The curve $z(x)$ determines $\Delta x(x)$ and $x_c(x)$ (or $a(x)$)

- Need to know bulk metric to know which curve E computes! The construction is thus state dependent!

Proof for an arbitrary spacetime

Here we will follow ref. [3].
Rather than parametrizing the set of of boundary surfaces with $\Delta x(x)$, we will introduce a parameter $\lambda \in (0,1)$, and specify the left and right endpoints of the intervals as $\gamma_L(\lambda)$ and $\gamma_R(\lambda)$.



$$\text{Length} = S(\gamma_L(\lambda), \gamma_R(\lambda))$$

Assume x is compactified and $\gamma_L(\lambda+1) = \gamma_L(\lambda)$
$\gamma_R(\lambda+1) = \gamma_R(\lambda)$.

$$E = \sum_\lambda \Big(S(\gamma_L(\lambda), \gamma_R(\lambda)) - S(\gamma_L(\lambda+d\lambda), \gamma_R(\lambda)) \Big)$$

$$= \oint d\lambda \, \frac{dS(\gamma_L, \gamma_R)}{d\gamma_L^\mu} \frac{d\gamma_L^\mu}{d\lambda}$$

$$= \oint d\lambda \, \frac{dS(\gamma_L(\lambda'), \gamma_R(\lambda))}{d\lambda'} \Big|_{\lambda'=\lambda}$$

Preliminary mathematical results:

$$S = \int_{s_i}^{s_f} ds \, \mathcal{L}(\gamma, \dot\gamma) \quad , \quad \mathcal{L} = |\dot\gamma| = \sqrt{g_{\mu\nu}(\gamma)\dot\gamma^\mu \dot\gamma^\nu}$$

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot\gamma^\mu} = \frac{g_{\mu\nu}\dot\gamma^\nu}{|\dot\gamma|}.$$

$$\mathcal{L} = \dot\gamma^\mu P_\mu$$

Let $\Gamma(s,\lambda)$ be a continuous set of geodesics parametrized by continuous label $\lambda \in (0,1)$ s.t. $\Gamma(s,1) = \Gamma(s,0)$

Define a convenient quantity

$$R(s) = \oint d\lambda \, \Gamma'^{\mu} p_{\mu} \quad , \quad \Gamma'^{\mu} = \partial_x \Gamma^{\mu}$$

This quantity has some convenient properties:

$$S_{12} = \int_{s_1}^{s_2} ds \, \mathcal{L}(\Gamma, \dot{\Gamma}) \quad , \text{where } s_1 \,\&\, s_2 \text{ arbitrary.}$$

Change λ and let $s_2 = s_2(\lambda)$, $s_1 = s_1(\lambda)$

$$dS_{12} = (s_2' - s_1') \mathcal{L} \, d\lambda + \int_{s_1}^{s_2} ds \left(\frac{\partial \mathcal{L}}{\partial \Gamma^{\mu}} \Gamma'^{\mu} d\lambda + \frac{\partial \mathcal{L}}{\partial \dot{\Gamma}^{\mu}} \dot{\Gamma}'^{\mu} d\lambda \right)$$

$$= (s_2' - s_1') \mathcal{L} \, d\lambda + \int_{s_1}^{s_2} ds \left[\left(\frac{\partial \mathcal{L}}{\partial \Gamma^{\mu}} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Gamma}^{\mu}} \right) \right) \Gamma'^{\mu} d\lambda \right.$$

$$\left. + \frac{\partial \mathcal{L}}{\partial \dot{\Gamma}^{\mu}} \Gamma'^{\mu} d\lambda \right|_{s_2} - \left. \frac{\partial \mathcal{L}}{\partial \dot{\Gamma}^{\mu}} \Gamma'^{\mu} d\lambda \right|_{s_1}$$

Integrate over λ from 0 to 1 and use periodicity of S_{12}

$$0 = \oint_0^1 d\lambda \, \frac{dS_{12}}{d\lambda} = \oint d\lambda \, \mathcal{L} \, (s_2' - s_1') + R(s_2) - R(s_1)$$

$$\left[\; R(s_2) = R(s_1) - \oint d\lambda \, \mathcal{L} \, (s_2' - s_1') \; \right]$$

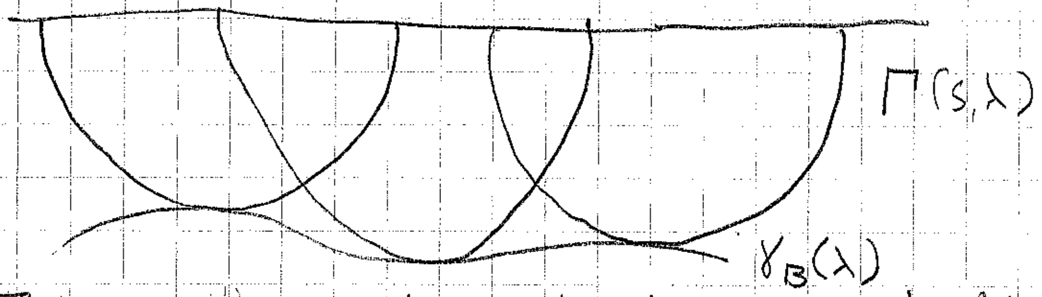
Now we are ready for the proof.

$$E = \oint d\lambda \, \frac{dS(\gamma_L, \gamma_R)}{d\gamma_L^{\mu}} \frac{d\gamma_L^{\mu}}{d\lambda} \quad , \quad \begin{aligned} \Gamma^{\mu}(s_L, \lambda) &= \gamma_L^{\mu}(\lambda) \\ \Gamma^{\mu}(s_R, \lambda) &= \gamma_R^{\mu}(\lambda) \end{aligned}$$

$$= \oint d\lambda \left. \frac{d\Gamma^{\mu}}{d\lambda} p_{\mu} \right|_{s = s_L} = R(s_L)$$

Use $\dfrac{dS(\gamma_L, \gamma_R)}{d\gamma_L^{\mu}} = \dfrac{\partial \mathcal{L}}{\partial \dot{\Gamma}^{\mu}} = p_{\mu}$

Call the curve whose lenth we are computing $\gamma_B^\mu(\lambda)$



This curve is tangent to Γ at the point where γ_B and Γ meet. This happens at some value $s = s_B(\lambda)$ of the affine parameter.

$$\begin{cases} \gamma_B^{\mu}(\lambda) = \Gamma^\mu(s_B(\lambda), \lambda) & (*) \\ \dot{\Gamma}^\mu(s,\lambda)\Big|_{s=s_B(\lambda)} = \gamma_B'^{\mu}(\lambda)\,\alpha(\lambda) & (**) \end{cases}$$

Arbitrary proportionality constant.

$$E = R(s_L) = R(s_B(\lambda)) + \oint d\lambda\, \mathcal{E}\, (s_B'(\lambda) - s_B'(\lambda))$$

We can choose to fix $s_L = c$ for all λ
$\Rightarrow s_L'(\lambda) = 0$

$$E = \oint d\lambda\, \Gamma'^\mu p_\mu \Big|_{s=s_B(\lambda)} + \oint d\lambda\, \mathcal{E}\, s_B'(\lambda)$$

Use $(*)$ to compute Γ'^μ

$$\gamma_B'^{\mu}(\lambda) = \dot{\Gamma}^\mu(s_B(\lambda),\lambda)\, s_B'(\lambda) + \Gamma'^\mu(s,\lambda)\Big|_{s=s_B(\lambda)}$$

Then use $(**)$ to rewrite p_μ

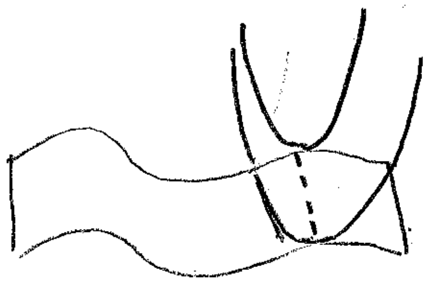
$$p_\mu = \frac{g_{\mu\nu}\dot{\Gamma}^\nu}{|\dot{\Gamma}|} = \frac{g_{\mu\nu}\,\alpha\,\gamma_B'^\nu}{|\alpha\gamma_B'|} = \frac{g_{\mu\nu}\gamma_B'^\nu}{|\gamma_B'|}$$

$$E = \oint d\lambda\, \gamma_B'^\mu \frac{g_{\mu\nu}\gamma_B'^\nu}{|\gamma_B'|} - \oint d\lambda \left(\dot{\Gamma}^\mu s_B' p_\mu - \mathcal{E}\, s_B'\right)$$

$$= \oint d\lambda \sqrt{g_{\mu\nu}\gamma_B'^\mu \gamma_B'^\nu} + \int d\lambda\, s_B' \underbrace{\left(\mathcal{E} - \dot{\Gamma}^\mu p_\mu\right)}_{=0} \qquad \square$$

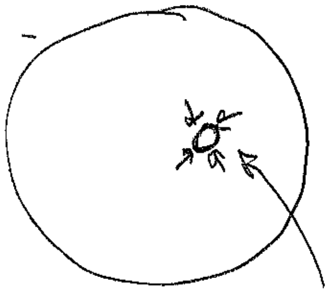
Subsequent developments

The formalism can be generalized to higher dimensions. Surfaces with planar symmetry were done in [2]



Time dependent (but spacelike) surfaces were done in [3].

[4] used the construction to define bulk points as limits of curves. Suggested a background independent definition of points and distances between them.



Rough idea: $L_\gamma = 0 \Rightarrow$ Some specific $\Delta x(x)$ and $a(x)$ for which $E = \int dx\,(1+a')\,\frac{dS}{d\Delta x} = 0$.

These $\Delta x(x)$ actually extremize the following quantity:

$$\text{“}\; I = \int d\theta \sqrt{-\frac{d^2 S}{d\Delta x^2}\left(1-(\Delta x'(x))^2\right)} \;\text{”}$$

Only CFT data. $\delta I = 0 \Rightarrow$ Diff. eq. for $\Delta x(x)$

$\Rightarrow$ Sol'n is some particular point A described by $\Delta x_A(x)$

[4] also defines a distance between points

$$d(A,B) = \frac{1}{2} E[\gamma(x)] \ , \ \gamma(x) = \min\{\Delta x_A(x) \ , \Delta x_B(x)\}.$$

[3], [4]

↪ New problem appears as not all points can be reached with minimal length geodesics.

[5] gives an information theoretic interpretation for E.