Non-Equilibrium QFT - Summary

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Literature

J.Berges, *Introduction to nonequilibrium quantum field theory*, [hep-ph/0409233]

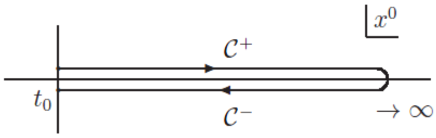


Figure 1. The close time contour $\mathcal{C}$ with components $\mathcal{C}^+$ going right from t_0 and $\mathcal{C}^-$ coming back the opposite way. Each point on $\mathcal{C}^-$ is considered to be after any point on $\mathcal{C}^+$. We have the usual time ordering on $\mathcal{C}^+$ but anti-time ordering on C^-.

1 Overview

The aim of nonequilibrium QFT is to understand the evolution of quantum fields which are initially **far from equilibrium**, in particular the process of **thermalization**. It will be interesting to see that a microscopically time-reversal invariant theory can explain macroscopic irreversable behaviour and the arisal of **universality**, i.e. the apparent loss of details about the initial state during the evolution. The approach presented employs very **few ingredients**:

- The specification of an initial state characterized by a density matrix ρ_0 (or equivalently by all correlation functions at initial time).
- The evolution of the quantum field under sole self-interactions of its constituents is completely determined by the Hamiltonian.

Since the QFT cannot be solved exactly, an approximation scheme needs to be employed. Traditional perturbative techniques fail for non-equilibrium situations due to the problem of **secularity**, since the coefficients in the perturbative expansion exhibit power-law-like growth with time invalidating the perturbative expansion after some finite time. Other approximation techniques on the level of the equations of motion fail to describe universality, since conservation laws are broken while conserved quantities are supposed to characterize the final state. These problems can be overcome by using non-perturbative approximation schemes for the so-called **n-particle irreducible (nPI) effective action.** Applying approximations on the level of the action preserves conservation laws. Moreover, we can employ schemes such that an infinitive series of contributions from the perturbative expansion are resummed solviong the problem of secularity and yielding uniform (valid for all times) approximations.

2 Nonequilibrium QFT

2.1 Nonequilibrium generating functional

Consider a scalar Hamiltonian field operator $\Phi(x) = \Phi(t, \boldsymbol{x})$. Consider the following generating functional of correlation functions:

$$Z[J, R; \rho_0] = \mathrm{tr}\left\{\rho_0 T_{\mathcal{C}} \exp i\left(\int_{x,\mathcal{C}} J(x)\Phi(x) + \frac{1}{2}\int_{xy,\mathcal{C}} R(x,y)\Phi(x)\Phi(y)\right)\right\}, \tag{2.1}$$

where $\int_{x,\mathcal{C}} = \int d^dx \int_{\mathcal{C}} dx^0$ and the closed contour $\mathcal{C}$ is shown in figure 1.

The closed time contour allows us to encode all possible time orderings of two operator intertions at times $> t_0$.A closed time path is also required due to the trace when we write the generating functional as a path integral. We write $\Phi^{\pm}(x) = \Phi^{\pm}(x^0, \boldsymbol{x})$ depending on the location of x^0 on either $\mathcal{C}^+$ or $\mathcal{C}^-$. The functional allows us to obtain all correlation functions via functional differentiation, i.e.

$$\left.\frac{\delta Z[J,R;\rho_0]}{i\delta J^\pm(x)}\right|_{J,R=0} = \mathrm{tr}\left\{\rho_0 \Phi(x)\right\} = \langle \Phi(x)\rangle \equiv \phi(x), \tag{2.2}$$

$$\begin{aligned}\left.\frac{\delta^2 Z[J,R;\rho_0]}{i\delta J^\pm(x) i\delta J^\pm(y)}\right|_{J,R=0} &= \mathrm{tr}\left\{\rho_0 T_{\mathcal{C}} \Phi^\pm(x)\Phi^\pm(y)\right\} = \begin{cases} \langle T\Phi(x)\Phi(y)\rangle & ++\ \text{component} \\ \langle \Phi(y)\Phi(x)\rangle & +-\ \text{component} \\ \langle \Phi(x)\Phi(y)\rangle & -+\ \text{component} \\ \langle \overline{T}\Phi(x)\Phi(y)\rangle & --\ \text{component} \end{cases} \\ &= \langle T_{\mathcal{C}}\Phi(x)\Phi(y)\rangle, \end{aligned} \tag{2.3}$$

where T and $\overline{T}$ denote time ordering and anti-time ordering. $\phi(x)$ is called the **macroscopic field**. The **propagator** $G(x,y)$ is defined by

$$\langle T_{\mathcal{C}}\Phi(x)\Phi(y)\rangle = G(x,y) + \phi(x)\phi(y). \tag{2.4}$$

$G(x,y)$ has four components $G^{\pm\pm}(x,y)$ and there is the algebraic identity

$$G^{+-} + G^{-+} = G^{++} + G^{--}. \tag{2.5}$$

2.2 Functional integral representation

The generating functional $Z[J,R;\rho_0]$ can be expressed as a functional integral by using complete sets of eigenstates of the Heisenberg operator at initial time,

$$\Phi^\pm(t_0,\boldsymbol{x})|\varphi^\pm\rangle = \varphi_0^\pm|\varphi^\pm\rangle. \tag{2.6}$$

Then, using the notation $[d\varphi_0^\pm] = \prod_{\boldsymbol{x}} d\varphi_0^\pm(\boldsymbol{x})$, $Z[J,R;\rho_0]$ is

$$\begin{aligned} Z[J,R;\rho_0] &= \mathrm{tr}\left\{\rho_0 T_{\mathcal{C}} \exp i\left(\int_{x,\mathcal{C}} J(x)\Phi(x) + \frac{1}{2}\int_{xy,\mathcal{C}} R(x,y)\Phi(x)\Phi(y)\right)\right\} \\ &= \int [d\varphi_0^+][d\varphi_0^-]\langle\varphi^+|\rho_0|\varphi^-\rangle \underbrace{\langle\varphi^-|T_{\mathcal{C}}\exp i\left(\int_{x,\mathcal{C}} J(x)\Phi(x) + \frac{1}{2}\int_{xy,\mathcal{C}} R(x,y)\Phi(x)\Phi(y)\right)|\varphi_0^+\rangle}_{\equiv(\varphi^-,t_0|\varphi^+,t_0)_{J,R}} \end{aligned} \tag{2.7}$$

where $(\varphi^-,t_0|\varphi^+,t_0)_{J,R}$ can be expressed as a path integral over a classical field $\varphi(x) = \varphi(x^0,\boldsymbol{x})$,

$$(\varphi^-,t_0|\varphi^+,t_0)_{J,R} = \int_{\varphi_0^+}^{\varphi_0^-} \mathcal{D}'\varphi \exp i\left(S[\phi] + \int_{x,\mathcal{C}} J(x)\varphi(x) + \frac{1}{2}\int_{xy,\mathcal{C}} R(x,y)\varphi(x)\varphi(y)\right), \tag{2.8}$$

where $S[\varphi] = \int d^dx \int_{\mathcal{C}} dx^0 \mathcal{L}(x)$ is the action along the closed contour and the prime on the path integral measure $\mathcal{D}'\varphi$ indicates that the integration of fields at initial time is exculded. The derivation of the path integral expression goes along the same lines as the corresponding derivation in standard field theory. One inserts N sets of eigenstates at intermediate times along the contour and uses the time evolution operator $\exp(iH\delta t)$ to compute the resulting N transition amplitudes in the limit $N\to\infty$.

The upshot of this section is the following: we can split the generating functional into two parts describing the statistical and quantum fluctuations, respectively:

$$\begin{aligned} &Z[J,R;\rho_0] \\ &= \underbrace{\int [d\varphi_0^+][d\varphi_0^-]\langle\varphi^+|\rho_0|\varphi^-\rangle}_{\text{initial conditions}} \underbrace{\int_{\varphi_0^+}^{\varphi_0^-} \mathcal{D}'\varphi \exp i\left(S[\varphi] + \int_{x,\mathcal{C}} J(x)\varphi(x) + \frac{1}{2}\int_{xy,\mathcal{C}} R(x,y)\varphi(x)\varphi(y)\right)}_{\text{quantum fluctuations}}. \end{aligned} \tag{2.9}$$

2.3 Initial conditions

If we choose Gaussian initial conditions, i.e. three- and higher point functions are zero at initial time, then we encode these initial conditions in the initial values of the linear and bilinear sources $J(x)$ and $R(x,y)$. This allows us to include the initial conditions in the functional integral. A Gaussian density matrix can be described by five parameters, ϕ_0, $\dot{\phi}_0$, ξ, η, σ,

$$\langle\phi^+|\rho_0|\phi^-\rangle = \frac{1}{\sqrt{2\pi\xi^2}}\exp\Big\{i\dot{\phi}_0(\varphi_0^+ - \varphi_0^-) - \frac{\sigma^2+1}{2}\left[(\varphi_0^+ - \phi_0)^2 + (\varphi_0^- - \phi_0)^2\right] \\ + i\frac{\eta}{2\xi}\left[(\varphi_0^+ - \phi_0)^2 - (\varphi_0^- - \phi_0)^2\right] + \frac{\sigma^2-1}{4\xi^2}(\varphi_0^+ - \phi_0)(\varphi_0^- - \phi_0)\Big\} \tag{2.10}$$

where we have surpressed the spatial $\boldsymbol{x}$-dependence of the fields and the five parameters. This initial density matrix is equivalent to the following initial correlation functions:

$$\phi_0 = \mathrm{tr}\ \{\rho_0\Phi(t)\}|_{t=t_0}\,, \tag{2.11}$$

$$\dot{\phi}_0 = \mathrm{tr}\ \{\rho_0\partial_t\Phi(t)\}|_{t=t_0}\,, \tag{2.12}$$

$$\xi^2 = \mathrm{tr}\ \{\rho_0\Phi(t)\Phi(t')\}|_{t=t'=t_0} - \phi_0^2, \tag{2.13}$$

$$\xi\eta = \mathrm{tr}\ \{\rho_0(\partial_t\Phi(t)\Phi(t') + \Phi(t)\partial_{t'}\Phi(t'))\}|_{t=t'=t_0} - \phi_0\dot{\phi}_0, \tag{2.14}$$

$$\eta^2 + \frac{\sigma^2}{4\xi^2} = \mathrm{tr}\ \{\rho_0\partial_t\Phi(t)\partial_{t'}\Phi(t')\}|_{t=t'=t_0} - \dot{\phi}_0^2 \tag{2.15}$$

with all higher correlators vanishing at initial time. We can write the Gaussian density matrix as

$$\langle\varphi^+|\rho_0|\varphi^-\rangle = \mathcal{N}\exp\left(if_{\mathcal{C}}[\phi]\right) \tag{2.16}$$

with some normalization $\mathcal{N}$ and

$$f_{\mathcal{C}}[\phi] = \alpha_0 + \int_{x,\mathcal{C}}\alpha_1(x)\varphi(x) + \frac{1}{2}\int_{xy,\mathcal{C}}\alpha_2(x,y)\varphi(x)\varphi(y), \tag{2.17}$$

where the α_i vanish except for the initial time. If we want to describe general (not necessarily Gaussian) initial conditions we just have to add terms with $\alpha_3(x,y,z)$, $\alpha_4(x,y,z,w),\ldots$. Shifting the sources in equation 2.9, $J(x)\to J(x)+\alpha_1(x)$ and $R(x,y)\to R(x,y)\to\alpha_2(x,y)$, we can write in the Gaussian case:

$$Z[J,R;\rho_0] = \int\mathcal{D}\varphi\exp i\left(S[\varphi] + \int_{x,\mathcal{C}}J(x)\varphi(x) + \frac{1}{2}\int_{xy,\mathcal{C}}R(x,y)\varphi(x)\varphi(y)\right). \tag{2.18}$$

We must recall in the future, that we can only set the sources to zero for times larger than the initial time.

2.4 Effective actions

Like in statistical mechanics, we can use the equivalent of the partition function, the generating functional, to get a free energy functional, which is the usual generating functional for connected correlation functions,

$$W[J,R] = -i\ln Z[J,R]. \tag{2.19}$$

Legendre transforms of the free energy functional lead to equivalent descriptions of physics (they are simply a change of variables - just like in thermodynamics). In the presence of approximations however, the different generating functionals can lead to different descriptions. However, the use of different effective actions is **no** approximation itself for the dynamics. In non-equilibrium physics it is particularly useful to Legendre transform w.r.t. the sources, since one obtains a functional of correlators whose initial values are better accessible.

For the rest of the text, we specialize to a $O(N)$ symmetric scalar model with action

$$S[\varphi] = \int_{x,\mathcal{C}} \left(\frac{1}{2} \partial_\mu \varphi_a(x) \partial^\mu \varphi_a(x) - \frac{m^2}{2} \varphi_a(x)\varphi_a(x) - \frac{\lambda}{4!N} (\varphi_a(x)\varphi_a(x))^2 \right) \tag{2.20}$$

We define the J and R dependent macroscopic field and propagator

$$\phi_a(x) = \frac{\delta W[J,R]}{\delta J_a(x)}, \qquad G_{ab}(x,y) = 2\frac{\delta W[J,R]}{\delta R_{ab}(x,y)} - \phi_a(x)\phi_b(y). \tag{2.21}$$

Legendre transforming once with respect to the leads to the 1PI effective action,

$$\Gamma^R[\phi] = W[J,R] - \int_{x,\mathcal{C}} J_a(x) \frac{\delta W[J,R]}{\delta J_a(x)} = W[J,R] - \int_{x,\mathcal{C}} J_a(x)\phi_a(x), \tag{2.22}$$

which explicitly still depends on the bilinear source R. The standard 1PI effective action is $\Gamma^{R=0}[\phi]$. The equations 2.21 translate into the stationarity conditions

$$\frac{\delta \Gamma^R[\phi]}{\delta \phi_a(x)} = -J_a(x), \qquad \frac{\delta \Gamma^R[\phi]}{\delta R_{ab}(x,y)} = \frac{W[J,R]}{R_{ab}(x,y)}. \tag{2.23}$$

Using the definition of $\Gamma^R[\phi]$ and subsequent shifting $\varphi_a(x) \to \varphi_a(x) + \phi_a(x)$ in the path integral we get

$$\begin{aligned} e^{i\Gamma^R[\phi]} &= \int \mathcal{D}\varphi \exp i \left\{ S[\phi] + \int_{x,\mathcal{C}} J_a(x)(\varphi_a(x) - \phi_a(x)) + \frac{1}{2}\int_{xy,\mathcal{C}} R_{ab}(x,y)\varphi_a(x)\varphi_b(y) \right\} \\ &= e^{iS^R[\phi]} \int \mathcal{D}\varphi \exp i \left\{ S^R[\phi+\varphi] - S^R[\phi] + \int_{x,\mathcal{C}} J_a(x)\varphi_a(x) \right\} \end{aligned} \tag{2.24}$$

where $S^R[\phi] = S[\phi] + \frac{1}{2}\int_{xy,\mathcal{C}} R_{ab}(x,y)\varphi_a(x)\varphi_b(y)$. Computing $S^R[\phi+\varphi] - S^R[\phi]$ explicitly using the action 2.20 on finds

$$S^R[\phi+\varphi] - S^R[\phi] = \frac{1}{2}\int_{xy,\mathcal{C}} \varphi_a(x)[iG^{-1}_{0,ab}(x,y;\phi) + R_{ab}(x,y)]\varphi_b(y) + S_{int}[\varphi,\phi] + \int_{x,\mathcal{C}} \varphi_a(x)\frac{\delta S^R[\phi]}{\delta\phi_a(x)} \tag{2.25}$$

where

$$S_{int}[\varphi,\phi] = -\frac{\lambda}{6N}\int_{x,\mathcal{C}} \phi_a(x)\varphi_a(x)\varphi_b(x)\varphi_b(x) - \frac{\lambda}{4!N}\int_{x,\mathcal{C}} (\varphi_a(x)\varphi_a(x))^2, \tag{2.26}$$

$$iG_{0,ab}(x,y;\phi)^{-1} = -\left(\Box_x + m^2 + \frac{\lambda}{6N}\phi_c(x)\phi_c(x) \right)\delta_{ab}\delta_{\mathcal{C}}(x-y) - \frac{\lambda}{3N}\phi_a(x)\phi_b(x)\delta_{\mathcal{C}}(x-y). \tag{2.27}$$

In the one loop approximation (only consider terms up to quadratic order in the fluctuating field φ), we get

$$\begin{aligned} \Gamma^{R\,(1\text{ loop})}[\phi] =& S^R[\phi] - i\ln\int \mathcal{D}\varphi \exp\left\{ -\frac{1}{2}\int_{xy,\mathcal{C}} \varphi_a(x)[G^{-1}_{0,ab}(x,y;\phi) - iR_{ab}(x,y)]\varphi_b(y) \right\} \\ =& -i\ln(\det{}_{\mathcal{C}}[G_0^{-1}(\phi) - iR])^{-\frac{1}{2}} + \text{const} \\ =& \frac{i}{2}\mathrm{tr}_{\mathcal{C}}\ln[G_0^{-1}(\phi) - iR] + \text{const}. \end{aligned} \tag{2.28}$$

The second of the stationarity conditions 2.23 up to one loop reads

$$\frac{\delta\Gamma^{R\,(1\text{ loop})}}{\delta R} = \frac{1}{2}(\phi\phi + [G_0^{-1}(\phi) - iR]^{-1}) \quad \Rightarrow \quad \left(G^{(1\text{ loop})}\right)^{-1} = G_0^{-1} - iR. \tag{2.29}$$

Legendre transforming the free energy w.r.t. both the linear and the bilinear source term yields the 2PI effective action,

$$\begin{aligned}\Gamma[\phi, G] &= W[J, R] - \int_{x,\mathcal{C}} J_a(x)\frac{\delta W[J,R]}{\delta J_a(x)} - \int_{xy,\mathcal{C}} R_{ab}(x,y)\frac{\delta W[J,R]}{\delta R_{ab}(x,y)} \\ &= W[J,R] - \int_{x,\mathcal{C}} J_a(x)\phi_a(x) - \frac{1}{2}\int_{xy,\mathcal{C}} R_{ab}(x,y)(\phi_a(x)\phi_b(y) + G_{ab}(x,y)),\end{aligned} \tag{2.30}$$

and the stationarity conditions 2.21 become

$$\frac{\delta\Gamma[\phi,G]}{\delta\phi_a(x)} = -J_a(x) - \int_{y,\mathcal{C}} R_{ab}(x,y)\phi_b(y), \qquad \frac{\delta\Gamma[\phi,G]}{\delta R_{ab}(x,y)} = -\frac{1}{2}R_{ab}(x,y). \tag{2.31}$$

A calculation analogous to 2.24 shows

$$\begin{aligned}\Gamma[\phi, G] = S[\phi] - i\ln\int\mathcal{D}\varphi\exp i\Bigg\{&\frac{i}{2}\int_{xy,\mathcal{C}}\varphi_a(x)\left(G^{-1}_{0,ab}(x,y;\phi) - iR_{ab}(x,y)\right)\varphi_b(y) \\ &-\int_{x,\mathcal{C}}\varphi_a(x)\frac{\delta(\Gamma[\phi,G] - S[\phi])}{\delta\phi_a(x)} - \frac{1}{2}\int_{xy,\mathcal{C}} G_{ab}(x,y)R_{ab}(x,y)\Bigg\}.\end{aligned} \tag{2.32}$$

The linear term in the fluctuating field φ represents tadpoles which guarantee a vanishing of the expectation value of the fluctuating field. It is taken into account by taking ϕ to describe the macroscopic field. In the 1 loop approximation we then get

$$\begin{aligned}\Gamma^{(1\text{ loop})}[\phi, G] &= S[\phi] - i\ln\int\mathcal{D}\varphi\exp i\Bigg\{\frac{i}{2}\int_{xy,\mathcal{C}}\varphi_a(x)\left(G^{-1}_{0,ab}(x,y;\phi) - iR_{ab}(x,y)\right)\varphi_b(y) \\ &\qquad -\frac{1}{2}\int_{xy,\mathcal{C}} G^{(1\text{ loop})}_{ab}(x,y)R_{ab}(x,y)\Bigg\} \\ &= S[\phi] + \frac{i}{2}\mathrm{tr}_{\mathcal{C}}\,\ln\left(G^{(1\text{ loop})}\right)^{-1} + \frac{i}{2}\mathrm{tr}_{\mathcal{C}}\,G_0^{-1}(\phi)G^{(1\text{ loop})} + \text{const},\end{aligned} \tag{2.33}$$

where we have used 2.29 to get to the last line. Crucially, we can therefore write the **full 2PI effective action** as the following where Γ_2 captures thr "rest" which we have not considered in the one loop approximation:

$$\boxed{\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{tr}_{\mathcal{C}}\,\ln G^{-1} + \frac{i}{2}\,\mathrm{tr}_{\,\mathcal{C}}\,G_0^{-1}(\phi)G + \Gamma_2[\phi,G] + \text{const}.} \tag{2.34}$$

The second of the stationarity equations 2.23 then gives

$$G^{-1}_{ab}(x,y) = G^{-1}_{0,ab}(x,y;\phi) - iR_{ab}(x,y) - \Sigma_{ab}(x,y;\phi,G), \tag{2.35}$$

where

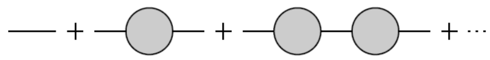
$$\Sigma_{ab}(x,y;\phi,G) \equiv 2i\frac{\delta\Gamma_2[\phi,G]}{\delta G_{ab}(x,y)} \tag{2.36}$$

is called **self-energy**. Equation 2.35 can be inverted using the geometric series:

$$\begin{aligned}G =&(G_0^{-1} - iR)^{-1} + (G_0^{-1} - iR)^{-1}\Sigma(G_0^{-1} - iR)^{-1} \\ &+ (G_0^{-1} - iR)^{-1}\Sigma(G_0^{-1} - iR)^{-1}\Sigma(G_0^{-1} - iR)^{-1} + \dots .\end{aligned} \tag{2.37}$$

We can also write the propagator as

$$G = $$



(2.38)

where the line indicates the classical propagator and the "blob" is the sum of all 1PI diagrams. We conclude that Σ is the sum of all 1PI diagrams. From this we can conclude that Γ_2 only contains 2PI contributions, since it is obtained by opening one propagator line in diagrams contributing to the self-energy.

2.5 Evolution equations

The non-equilibrium evolution equations are obtained from the stationarity conditions 2.23. In the symmetric regime ($\phi = 0$) we only need to consider the evolution equation for the propagator which we have obtained in 2.35. This can be rewritten as

$$\int_{z,\mathcal{C}} G^{-1}_{0,ac}(x,z)G_{cb}(z,y) - \int_{z,\mathcal{C}} \left[\Sigma_{ac}(x,z) + iR_{ac}(x,z)\right] G_{cb}(z,y) = \delta_{ab}\delta_{\mathcal{C}}(x-y) \tag{2.39}$$

and in the symmetric regime

$$iG^{-1}_{0,ab}(x,y) = -(\Box_x - m^2)\delta_{ab}\delta_{\mathcal{C}}(x-y). \tag{2.40}$$

Consequently, the evolution equation for the propagator is

$$\boxed{(\Box_x + m^2)G_{ab}(x,y) + i\int_{z,\mathcal{C}} \left[\Sigma_{ac}(x,z) + iR_{ac}(x,z)\right] G_{cb}(z,y) = \delta_{ab}\delta_{\mathcal{C}}(x-y).} \tag{2.41}$$

Defining the **spectral function** $\rho_{ab}(x,y)$ and the **statistical function** $F_{ab}(x,y)$ as

$$\rho_{ab}(x,y) = i\langle[\Phi_a(x), \Phi_b(y)]\rangle, \qquad F_{ab}(x,y) = \frac{1}{2}\langle\{\Phi_a(x), \Phi_b(y)\}\rangle \tag{2.42}$$

we can write

$$G_{ab}(x,y) = F_{ab}(x,y) - \frac{i}{2}\rho_{ab}(x,y)\mathrm{sgn}_{\mathcal{C}}\,(x^0 - y^0). \tag{2.43}$$

Splitting up the self-energy in a local and a non-local part

$$\Sigma_{ab}(x,y) = -i\Sigma^{(0)}_{ab}(x)\delta(x-y) + \overline{\Sigma}_{ab}(x,y) \tag{2.44}$$

and then splitting up the non-local part in analogy to 2.43,

$$\overline{\Sigma}_{ab}(x,y) = \Sigma^F_{ab}(x,y) - \frac{i}{2}\Sigma^\rho_{ab}(x,y)\mathrm{sgn}_{\mathcal{C}}\,(x^0 - y^0), \tag{2.45}$$

it is a purely algebraic task to derive the **coupled evolution equations for the spectral and statistical function** from 2.41:

$$\boxed{\begin{aligned}
(\Box_x + M^2_{ac}(x))F_{cb}(x,y) &= -\int_{t_0}^{x_0} dz\, \Sigma^\rho_{ac}(x,z)F_{cb}(z,y) + \int_{t_0}^{y_0} dz\, \Sigma^F_{ac}(x,z)\rho_{cb}(z,y) && (2.46)\\
(\Box_x + M^2_{ac}(x))\rho_{cb}(x,y) &= -\int_{y_0}^{x_0} dz\, \Sigma^\rho_{ac}(x,z)\rho_{cb}(z,y) && (2.47)
\end{aligned}}$$

where we have used the definition $M^2_{ab}(x) = m^2\delta_{ab} + \Sigma^{(0)}_{ab}(x)$, we have assumed $x^0 > 0$, which makes $R_{ab}(x,z)$ vanish, and we have to keep in mind that $M^2 = M^2(F)$, $\Sigma^F = \Sigma^F(\rho, F)$ and $\Sigma^\rho(\rho, F)$. Crucially, we see so-called **memory integrals** emerging that integrate over the time history of the configuration and which capture causality. The equations describe a unitary time evolution if no further approximations are made. They can be solved numerically after approximations given initial conditions for F, while the initial conditions for ρ are given by the equal time commutation relations.

In the **non-symmetric regime** the statistical function is

$$F_{ab}(x,y) = \frac{1}{2}\langle\{\Phi_a(x), \Phi_b(y)\}\rangle - \phi_a(x)\phi_b(y) \tag{2.48}$$

and the general form of the equations 2.46 and 2.47 remains unchanged, except that the mass squared matrix $M^2_{ab}(x)$ receives ϕ-dependent contributions:

$$M^2_{ab}(x) = \left(m^2 + \frac{\lambda}{6N}\phi^2(x)\right)\delta_{ab} + \frac{\lambda}{3N}\phi_a(x)\phi_b(x) + \Sigma^{(0)}_{ab}(x). \tag{2.49}$$

We obtain the **field evolution equation** from the first stationarity condition in 2.23 using 2.34:

$$\boxed{\left[\left(\Box_x + m^2 + \frac{\lambda}{6N}\left[\phi^2(x) + F_{cc}(x,x)\right]\right)\delta_{ab} + \frac{\lambda}{3N}F_{ab}(x,x)\right]\phi_b(x) = \frac{\delta\Gamma_2}{\delta\phi_a(x)}} \tag{2.50}$$

2.6 Thermal equilibrium

A brief look at thermal equilibrium is useful to get an intuition for the spectral and the statistical function. The simplification of being in thermal equilibrium will bring about a relation between the two functions. The canonical density matrix in equilibrium is

$$\rho_\beta = \frac{1}{Z_\beta}e^{-\beta H} \tag{2.51}$$

where $Z_\beta = \mathrm{tr}\ e^{-\beta H}$. As always, we want to write the generating functional of correlation functions 2.1 as a path integral, but notice beforehand, that we can write

$$e^{-\beta H} = e^{-i(-i\beta)H} \tag{2.52}$$

the initial density matrix as translation operator in imaginary time, so that we can obtain the path integral by adding the imaginary time piece from t_0 to $t_0 - i\beta$ to the end of the time contour, see figure 2. For notational convenience we add source terms on the part of the contour as well, so that we get

$$\begin{aligned} Z[J,R;\rho_\beta] &= \mathrm{tr}\left\{\rho_\beta T_{\mathcal{C}_\beta}\exp i\left(\int_{x,\mathcal{C}_\beta} J(x)\Phi(x) + \frac{1}{2}\int_{xy,\mathcal{C}\beta} R(x,y)\Phi(x)\Phi(y)\right)\right\} \\ &= \int_{\text{periodic}} \mathcal{D}\varphi \exp i\left\{S_\beta[\varphi] + \int_{x,\mathcal{C}_\beta} J(x)\varphi(x) + \frac{1}{2}\int_{xy,\mathcal{C}_\beta} R(x,y)\varphi(x)\varphi(y)\right\} \end{aligned} \tag{2.53}$$

where periodicity means $\varphi(t_0 - i\beta, \boldsymbol{x}) = \varphi(t_0, \boldsymbol{x})$.

We may again split the propagator:

$$G^{(eq)}(x-y) = F^{(eq)}(x-y) - \frac{i}{2}\rho^{(eq)}(x-y)\,\mathrm{sgn}_{\mathcal{C}_\beta}(x^0 - y^0). \tag{2.54}$$

The periodicity conditions yields

$$\langle\Phi(y)\Phi(x)\rangle|_{x^0=t_0} = \langle\Phi(x)\Phi(y)\rangle|_{x^0=t_0-i\beta} \tag{2.55}$$

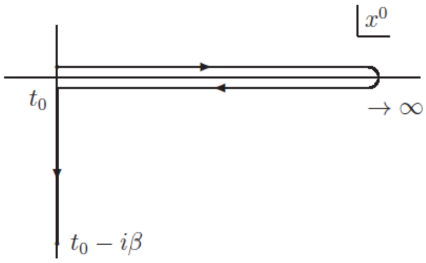


Figure 2. The time contour $\mathcal{C}$ is modified by an additional piece in imaginary time to obtain $\mathcal{C}_\beta$. All points on the imaginary piece are considered to be later that the points on $\mathcal{C}$.

After plugging in the decomposition 2.54 and Fourier trnasforming (equilibrium is translation invariant) we obtain the **fluctuation-dissipation relation**,

$$F^{(eq)}(\omega, \boldsymbol{p}) = -i \left(\frac{1}{2} + n_\beta(\omega) \right) \rho^{(eq)}(\omega, \boldsymbol{p}), \tag{2.56}$$

where $n_\beta(\omega) = (e^{\beta\omega} - 1)^{-1}$.

3 Approximate non-equilibrium dynamics

3.1 Perturbative loop expansion

The perturbative expansion of the 2PI action for small coupling goes along the same lines as the standard perturbative expansion, but

- Instead of the classical propagator we use the dressed propagator $G^{-1} = G_0^{-1} - iR - \Sigma$
- Only 2PI contributions are kept.

We can use usual Feynman diagrams, but only consider diagrams which remain connected after cutting any two lines. The vertices are determined by the interaction terms in 2.26, so there is a 3- and a 4-point vertex each coming with one power of λ. We get the expansion

$$\Gamma_2[\phi, G] = \Gamma_2^{(2\text{ loop})}[\phi, G] + \Gamma_3^{(3\text{ loop})}[\phi, G] \ldots \tag{3.1}$$

Where the power counting is in general difficult and can even be time dependent. In the case of $N = 1$ and $m^2 < 0$ we get $\phi = \pm\sqrt{-\frac{6m^2}{\lambda}}$, i.e. $\phi \sim \frac{1}{\sqrt{\lambda}}$. At two loop order there are two contributions:

$$\Gamma_2^{(2a)}[G] = -i \cdot 3 \left(-i \frac{\lambda}{4!} \right) \int_{x,\mathcal{C}} G^2(x, x) = -\frac{\lambda}{8} \tag{3.2}$$

$$\Gamma_2^{(2b)}[\phi, G] = -i \cdot 6 \cdot \frac{1}{2} \left(-i \frac{\lambda}{6} \right)^2 \int_{xy,\mathcal{C}} \phi(x)\phi(y) G^3(x, y) = i \frac{\lambda^2}{16} \tag{3.3}$$

The two diagrams would contribute ar the same order λ because the two three-vertices in $(2b)$ contribute $1/\lambda$. Of course, the restriction of only considering 2PI diagrams reduces the number of diagrams significantly, e.g. in the symmetric regime there is only one diagram at each order up to fifth order, where there are two.

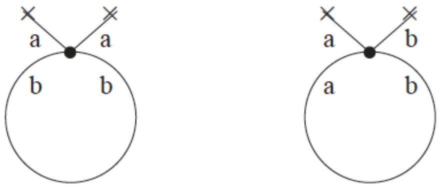


Figure 3. A leading and next to leading order diagram only differing in their index contraction.

We can use these two diagrams to get a consistent approximation up to two loops. From the two expressions $\Gamma_2^{(2a)}$ and $\Gamma_2^{(2b)}$ one can now obtain the self-energy via the definition 2.36:

$$\begin{aligned}\Sigma(x,y) &= -i\Sigma^{(0)}(x)\delta_{\mathcal{C}}(x-y) + \Sigma^F(x,y) - \frac{i}{2}\mathrm{sgn}_{\mathcal{C}}(x^0-y^0)\Sigma^\rho(x,y)\\ &= 2i\frac{\delta\Gamma_2[\phi,G]}{\delta G(x,y)}.\end{aligned} \tag{3.4}$$

We find, that to second order

$$\Sigma^{(0)}(x) = \frac{\lambda}{2}G(x,x), \tag{3.5}$$

$$\Sigma^F(x,y) = -\frac{\lambda^2}{2}\left(F^2(x,y) - \frac{1}{4}\rho^2(x,y)\right)\phi(x)\phi(y), \tag{3.6}$$

$$\Sigma^\rho(x,y) = -\lambda^2 F(x,y)\rho(x,y)\phi(x)\phi(y), \tag{3.7}$$

$$\frac{\Gamma_2^{(2\text{ loop})}}{\delta\phi(x)} = \frac{\lambda^2}{2}\int_{t_0}^{x_0} dy\,\rho(x,y)\left(F^2(x,y) - \frac{1}{12}\rho^2(x,y)\right)\phi(y). \tag{3.8}$$

Plugging these results in the equations 2.46 and 2.47 we obtain the evolution equations in this approximation.

3.2 Non-perturbative 1/N expansion

In this section we discuss a non-perturbative approximation scheme of the 2PI effective action. Since the loop expansion is restricted to weakly coupled systems, we identify $1/N$, where N is the number of field components, as a possible small parameter in the theory:

$$\Gamma_2[\phi,G] = \underbrace{\Gamma_2^{\mathrm{LO}}[\phi,G]}_{\sim N^1} + \underbrace{\Gamma_2^{\mathrm{NLO}}[\phi,G]}_{\sim N^0} + \underbrace{\Gamma_2^{\mathrm{NNLO}}[\phi,G]}_{\sim N^{-1}} + \ldots \tag{3.9}$$

As a non-perturbative scheme the $1/N$ expansion can be used to describe systems with non-perturbatively large fluctuations like extreme non-equilibrium phenomena.

We consider again the $O(N)$-symmetric scalar model with action 2.20. Since Γ_2 is a $O(N)$ singlet each term in Γ_2 is. We can thus classify the contributions to Γ_2 by how many irreducible $O(N)$ singlets they contain, of which each contributes a power on N. The irreducible singlets are

$$\phi^2, \qquad \mathrm{tr}\,(G^n), \qquad \mathrm{tr}\,(\phi\phi G^n), \tag{3.10}$$

where only the invariants with $n < N$ are irreducible. On the other hand, vertices contribute a factor of $1/N$, as can be seen in the action. As an example we consider the term $\mathrm{tr}\,(G_0^{-1}G)$ from Γ:

$$\begin{aligned} i\,\mathrm{tr}\,\left(G_0^{-1}G\right) &= -\left(\left(\Box_x + m^2 = \frac{\lambda}{6N}\phi_c\phi_c\right)\delta_{ab}\delta_{\mathcal{C}} + \frac{\lambda}{3N}\phi_a\phi_b\delta_{\mathcal{C}}\right)G_{ab}\\ &= -\Big(\underbrace{(\Box_x + m^2)\,\mathrm{tr}\,G}_{\sim N^1\to\mathrm{LO}} + \underbrace{\frac{\lambda}{6N}\phi^2\,\mathrm{tr}\,G}_{\sim N^1\to\mathrm{LO}} + \underbrace{\frac{\lambda}{3N}\,\mathrm{tr}\,(\phi\phi G)}_{\sim N^0\to\mathrm{NLO}}\Big)\end{aligned} \tag{3.11}$$

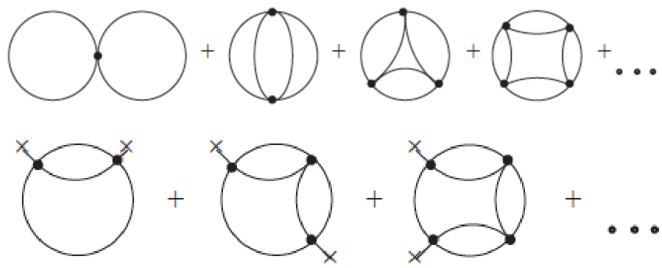


Figure 4. The two infinite series of diagrams contributing to Γ_2 at NLO. The second of the series is only present in the non-symmetric regime.

The diagrams corresponding to the second and third term can be seen in figure 3. There is only one diagram contributing to Γ_2 to leading order, the diagrams contributing in the NLO can be seen in 4. We note, that the NLO contribution resums an **infinite series** of diagrams which **contribute to all orders in the perturbative expansion**. This resummation heals the problem of secularity. Once we have obtained Γ_2 to a certain order in the approximation we can again find the evolution equation of the spectral and statistical function as well as the macroscopic field.

3.3 LO fixed points and NLO thermalization

We look at the case where $\phi = 0$, i.e. the symmetric regime, where there is one leading order contribution to Γ_2 and a series of diagrams at NLO, which can be seen in 4. We find

$$\Gamma_2^{\mathrm{LO}}[G] = -\frac{\lambda}{4!N}\int_x G_{aa}(x,x)G_{bb}(x,x), \tag{3.12}$$

$$\Gamma_2^{\mathrm{NLO}}[G] = \frac{i}{2}\,\mathrm{tr}\,\ln[B(G)], \tag{3.13}$$

$$B(x,y;G) = \delta(x-y) + i\frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y), \tag{3.14}$$

where 3.13 analytically sums the infinite series from figure 4, which we can see from

$$\begin{aligned}
&\mathrm{tr}\,\ln[B(G)] = \\
&\quad \int_x \left(i\frac{\lambda}{6N}G_{ab}(x,x)G_{ab}(x,x)\right) \\
&\quad -\frac{1}{2}\int_{xy}\left(i\frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y)\right)\left(i\frac{\lambda}{6N}G_{a'b'}(y,x)G_{a'b'}(y,x)\right) \\
&\quad +\frac{1}{3}\int_{xyz}\left(i\frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y)\right)\left(i\frac{\lambda}{6N}G_{a'b'}(y,z)G_{a'b'}(y,z)\right)\left(i\frac{\lambda}{6N}G_{a''b''}(z,x)G_{a''b''}(z,x)\right) \\
&\quad -\ldots
\end{aligned} \tag{3.15}$$

Each of the terms scales as $(\mathrm{tr}\ G^2/N)^n \sim N^0$, so they all contribute at the same order. In the symmetric regime we have $G_{ab}(x,y) = G(x,y)\delta_{ab}$. In order to find the evolution equations 2.47 we need to compute the self energies 2.36:

$$\Sigma^{\mathrm{LO}}(x,y;G) = 2i\frac{\delta\Gamma_2^{\mathrm{LO}}[\phi,G]}{\delta G(x,y)} = -\frac{i\lambda}{6}G(x,x)\delta(x-y) = -\frac{i\lambda}{6}F(x,x)\delta(x-y). \tag{3.16}$$

where in the last step we have used that ρ vanishes at equal times. So at leading order the self-energy is local and only contributes to the time dependent mass term 2.49. The same calculation needs to be performed at NLO, but it is too lengthy for this summary.

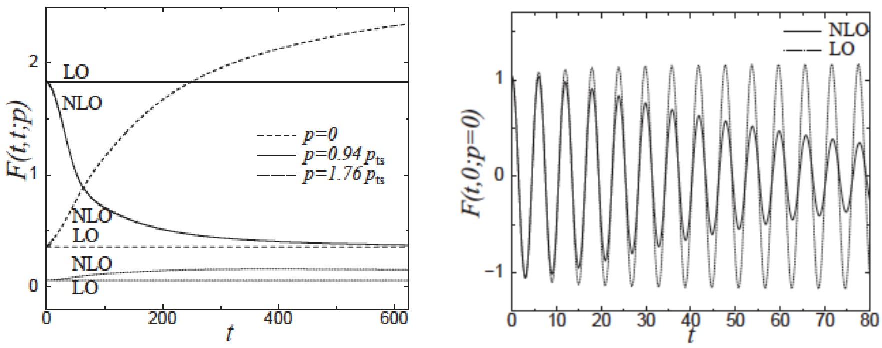


Figure 5. **Left**: The equal time statistical function for different momentum modes in the LO and NLO approximation. We see that the systems remains fixed in the LO approximation indicating that it was initialized in a LO fixed point, while in the NLO approximation the low-energetic mode becomes populated and the momentum peak at p_i decays. **Right**: The effective loss of details about the initial conditions is indicated by a decay of correlations with the initial time at NLO. At LO, there is no decay, thus there cannot be thermalization.

If we choose spatially homogeneous initial conditions for F and ρ we can Fourier transform to $F(t, t', \boldsymbol{x})$ and $\rho(t, t', \boldsymbol{x})$ since the system will stay homogeneous in the symmetric regime. The LO equations of motion become:

$$\left[\partial_t^2 + \boldsymbol{p}^2 + M^2(t : F)\right] F(t, t'; \boldsymbol{p}) = 0, \tag{3.17}$$

$$\left[\partial_t^2 + \boldsymbol{p}^2 + M^2(t : F)\right] \rho(t, t'; \boldsymbol{p}) = 0, \tag{3.18}$$

where

$$M^2(t; F) = m^2 + \frac{\lambda}{6N} \int_{\boldsymbol{p}} F_{cc}(t, t', \boldsymbol{p}). \tag{3.19}$$

So the spectrum just consists of quasi-particle modes of energy $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + M(t)^2}$ with infinite lifetime. Each mode particle number is conserved, so there is an infinite number of conserved charges, that prohibit dynamics and prevent an effective loss of details about the initial conditions. This leads to non-thermal fixed points in the LO approximation. Only at NLO order these fixed points vanish. For a distribution that is initially peaked around p_i in 1+1d, we get the time evolution in figure 5

Of potential interest:

- Jelena Smolic, Milena Smolic, *2PI Effective Action and Evolution Equations of $N = 4$ super Yang-Mills*, arXiv:1111.0893 [hep-th]
- Juergen Berges, Kirill Boguslavski, Soeren Schlichting, Raju Venugopalan, *Universal attractor in a highly occupied non-Abelian plasma*, arXiv:1311.3005 [hep-ph]
- Juergen Berges, Kirill Boguslavski, Soeren Schlichting, Raju Venugopalan, *Turbulent thermalization of the Quark Gluon Plasma*, arXiv:1303.5650 [hep-ph]