Journal Club 6/2/15

Introduction: AdS (in)stability

Literature:

- P. Bizoń, A. Rostworowski, "On weakly turbulent instability of anti-de Sitter Space", 1104.3702 [gr-qc]
- O. Dias, G. Horowitz, J. Santos, "Gravitational Turbulent Instability of Anti-de Sitter Space", 1109.1825 [hep-th]
- J. Jałmużna, A. Rostworowski, P. Bizoń, "A comment on AdS collapse of a scalar field in higher dimensions", 1108.4539 [gr-qc]
- A. Buchel, L. Lehner, S. Liebling, "Scalar Collapse in AdS", 1210.0890 [gr-qc]
- O. Dias, G. Horowitz, D. Marolf, J. Santos, "On the Nonlinear Stability of Asymptotically Anti-de Sitter Solutions, 1208.5772 [gr-qc]
- M. Maliborski, A. Rostworowski, "Time-periodic solutions in Einstein AdS - massless scalar field system", 1303.3186 [gr-qc]
- A. Buchel, S. Liebling, L. Lehner, "Boson Stars in AdS", 1304.4166 [gr-qc]
- M. Maliborski, A. Rostworowski, "What drives AdS unstable?", 1403.5434 [gr-qc]
- V. Balasubramanian, A. Buchel, S. Green, L. Lehner, S. Liebling, "Holographic Thermalization, stability of AdS and the FPU paradox", 1403.6471 [hep-th]
- P. Bizoń, A. Rostworowski, "Comment on 1403.6471" 1410.2631 [gr-qc]

- B. Craps, O. Evnin, J. Vanhoof, "Renormalization group, secular term resummation and AdS (in)stability", 1407.6273 [gr-qc]
- F. Dimitrakopolous, B. Freivogel, M. Lippert, I. Yang, "Instability corners in AdS space", 1410.5381 [hep-th]
- Ben. Craps, O. Evnin, J. Vanhoof, "Renormalization, averaging, conservation laws and ~~A~~ AdS (in)stability", 1412.3249 [gr-qc]

Review:

- P. Bizoń, "Is AdS stable?", 1312.5544 [gr-qc]

Motivation

Why does one want to study the stability of AdS (or asymptotically AdS) spacetime:

Via AdS CFT:

Instability of AdS leading to BH formation upon initial perturbation $\Longleftrightarrow$ equilibration / thermalization in CFT

Intuition:

AdS is a confining "box", the boundary prox + the effective attractive gravitational potential $\Rightarrow$ no dissipation by dispersion (unlike Minkowski + dS), boundary acts as mirror

Add finite excitation to this box

$\Rightarrow$ system is expected to explore all configurations consistent with conserved quantities

$\Rightarrow$ any finite excitation eventually finds itself inside its Schwarzschild radius

In contrast: Minkowski and dS are stable under small, finite perturbations [Christodoulou & Klainerman 1993, $\leftarrow$ Minkowski
Friedrich 1986] $\leftarrow$ dS

⚠ Positive energy theorem $\Rightarrow$ AdS is the unique ground state of Einstein gravity with $\Lambda < 0$, the unique solution with zero energy and therefore does not decay

Not in contradiction with the statement that small energy perturbations collapse to BH

Usually waves disperse at late times, this does not happen in AdS.

The topic nicely combines some of the topics we covered this term: e.g. turbulence, KAM theory, ..

The picture that emerged in previous years:
While there are initial perturbations of (asymptotically) AdS space that collapse no matter how small the initial perturbation is, there are also large classes of solutions that are stable when the initial perturbation is sufficiently small, e.g. standing waves or time-periodic solutions that can be arbitrarily close to AdS

Outline

- Recent historical development
- Rostworowski & Bizon 2011
- Anharmonic oscillator & secular terms
- Stable AdS solutions:
    - Boson Stars
    - Two time framework

Historical development:

2011 • Rostworowski & Bizon: 1104.3702 & 1108.4539

Perturbation of AdS_4 solution of gravity + massless scalar
→ evolution of spherical symmetric scalar always leads to gravitational collapse, even for arbitrarily small initial field amplitude (except for special non-resonant initial data)
→ resonances in the normal mode spectrum of the linearized problem lead to a direct turbulent cascade, i.e. energy transfer to high momentum modes leading to gravitational collapse

• Dias, Horowitz & Santos: 1109.1825
Support of non-linear instability claim by R & B in pure gravity

2012: • Dias, Horowitz, Marolf, Santos: 1208.5772
Many asymptotically anti-de Sitter solutions ~~are~~ seem to be nonlinearly stable: geons, boson stars, bhs
They are continuously connected to AdS and can therefore be seen as small perturbations of pure AdS

2013: • Maliborski & Rostworowski 1303.3186
numerical Construction of time-periodic solutions of gravity + scalar which are nonlinearly stable.

2014: • Balasubramanian, Buchel, Green, Lehner & Liebling 1403.6471
Quasi-periodic solutions using two-time framework.
These solutions have finely tuned population of the initial state linear modes. (part of their numerics is wrong & → they found stable solutions with two modes initially populated)

- Craps, Evnin, Vanhoof: 1407.6273 & 1412.3249

  Resummation scheme for the secular terms that drive the nonlinear instability. The scheme closely resembles renormalization group treatment of UV divergences in perturbative QFT.

  → more about this tomorrow in Ben Craps' seminar

Rostworowski & Bizon 2011

Their paper 1104.3702 sparked the current interest into studying the stability of AdS under small perturbations

The Setup

Spherically symmetric, self-gravitating, massless scaler in 3+1 d + Einstein gravity with $\Lambda < 0$.

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \, \partial_\beta \phi - \tfrac{1}{2} g_{\alpha\beta} (\partial\phi)^2 \right)$$

$$\Box^2 \phi = 0$$

Metric ansatz: $ds^2 = \frac{\ell^2}{\cos^2 x} \left(-A e^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x \, d\Omega_2^2 \right)$

where $\ell^2 = -\frac{3}{\Lambda}$ and $A = A(t,x)$, $\delta = \delta(t,x)$, $-\infty < t < \infty$, $0 \leq x < \frac{\pi}{2}$

Writing $\Phi = \phi'$ and $\Pi = A^{-1} e^{\delta} \dot\phi$, the following set of e.o.m. emerges (in units where $4\pi G = 1$)

$$\dot\Phi = \left(A e^{-\delta} \Pi \right)', \qquad \dot\Pi = \frac{1}{\tan^2 x} \left(\tan^2 x \, A \, e^{-\delta} \Phi \right)'$$

$$A' = \frac{1 + 2\sin^2 x}{\sin x \cos x} (1 - A) - \sin x \cos x \, A \left(\Phi^2 + \Pi^2 \right) \qquad (*)$$

$$\delta' = -\sin x \cos x \left(\Phi^2 + \Pi^2 \right)$$

Pure AdS solution, $\phi = 0$, $A = 1$, $\delta = 0$

Boundary conditions:

- smoothness near origin: $\phi(t,x) = f_0(t) + O(x^2)$
  $A(t,x) = 1 + O(x^2)$
  $\delta(t,x) = O(x^2)$

- smoothness near boundary + finite total mass, normalizable mode, $\rho = \frac{\pi}{2} - x$
  $\phi(t,x) = f_\infty(t) \rho^3 + O(\rho^5)$
  $A(t,x) = 1 - 2M\rho^3 + O(\rho^6)$
  $\delta(t,x) = \delta_\infty(t) + O(\rho^6)$

where $M = \lim_{x \to \pi/2} m(t,x) = \int_0^{\pi/2} dx \left(A\Phi^2 + A\Pi^2 \right) (\tan x)^{d-1}$

→ problem is locally well-posed [Friedrich 1995, Holzegel & Smulevici 2011]

For smooth initial data ⇒ no freedom in imposing bndy. data

Numerical results :

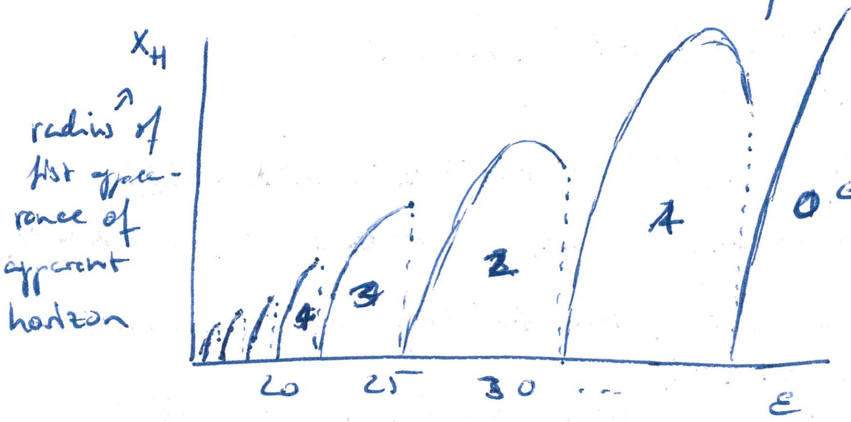
Gaussian type initial data $\rightarrow$ get initial A and δ from constraint eqs.

$$\phi' = \Phi(0,x) = 0 \qquad A^{-1} e^{\delta} \dot\phi = \Pi(0,x) = \frac{2\varepsilon}{\pi} \exp\left(-\frac{4 \tan x}{\pi^2 \sigma^2}\right)$$

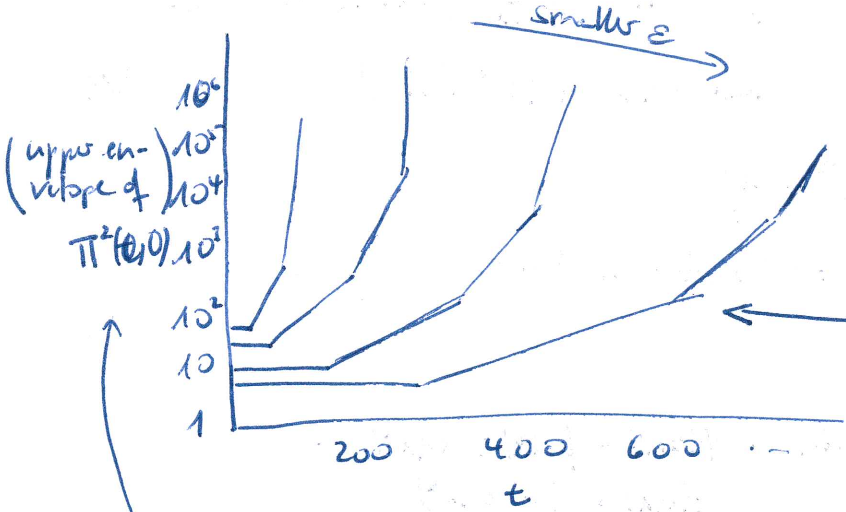
with σ fixed $\sigma = 1/16$. ε Amplitude which is varied.

$\rightarrow$ see animation

This was for one (small) ε, vary ε



$\leftarrow$ # of reflections



$\leftarrow$ clearly pronounced phases of exp. growth

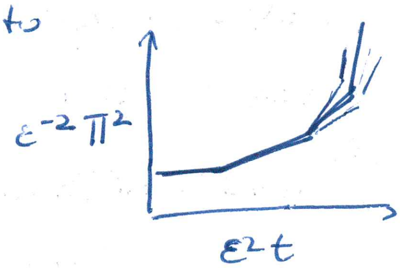
~~dt~~

kinks : come from different time scales $t, \varepsilon^2 t, \varepsilon^4 t, \ldots$ at which secular growth due to resonant mode mixing becomes important

connected to Ricci scalar at $x=0$

$$R(t,0) = -2\Pi^2(t,0)/\ell^2 - 12/\ell^2$$

Rescale to



and curves coincide

$\Rightarrow$ time of onset of exponential growth scales as $\boxed{\varepsilon^{2}}$

What is the <u>reason for the instability</u> (more mathematically than intuitive explanation earlier)?

Roughly: Resonances in the linearized spectrum $\overset{\text{for generic initial data}}{\Rightarrow}$ weakly turbulent energy transfer from low to high frequencies $\Rightarrow$ focusing of energy leads to BH formation.

<u>Perturbative ansatz</u>:

⌈Take initial data $(\phi, \dot\phi)|_{t=0} = (\varepsilon f(x), \varepsilon g(x))$ and expand

$$\phi = \sum_{j=0}^{\infty} \phi_{2j+1}\, \varepsilon^{2j+1}$$

, i.e. $(\phi_1, \dot\phi_1)|_{t=0} = (f(x), g(x))$ and other modes vanishing

$$A = 1 - \sum_{j=1}^{\infty} A_{2j}\, \varepsilon^{2j}$$

$$\delta = \sum_{j=1}^{\infty} \delta_{2j}\, \varepsilon^{2j}$$ ⌋

Expand ϕ, A, δ in powers of ε and solve the equations (*) perturbatively in ε with initial data $(\phi, \dot\phi)|_{t=0} = (\varepsilon f(x), \varepsilon g(x))$

At $O(\varepsilon)$: $\ddot\phi_1 + L\phi_1 = 0$, $L = -\frac{1}{\tan^2 x}\partial_x(\tan^2 x\, \partial_x)$

L has eigenvalues $\omega_j^2 = (3+2j)^2$ with eigenfunctions $e_j \propto {}_2F_1$

$\omega_j^2 > 0 \Rightarrow$ AdS linearly stable and

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\, e_j(x)$$

At $O(\varepsilon^2)$: <u>Backreaction</u> on the metric + ~~inhomogeneous eq.~~

$$A_2(t,x) = \frac{\cos^3 x}{\sin x}\int_0^x dy\, \tan^2 y\, \left(\dot\phi_1(t,y)^2 + \phi_1'(t,y)\right)$$

$$\delta_2(t,x) = -\int_0^x dy\, \sin y \cos y\, (-\,{''}\,-)$$

At $O(\varepsilon^3)$: <u>Inhomogeneous eq.</u>

$$\ddot\phi_3 + L\phi_3 = S(\phi_1, A_2, \delta_2)$$

$$= -2(A_2 + \delta_2)\ddot\phi_1 - (\dot A_2 + \dot\delta_2)\dot\phi_1 - (A_2' + \delta_2')\phi_1'$$

→ this inhom. diff. eq. has resonances where the eigenmode frequency on the left ~~side~~ coincides with the driving frequency on the right

Weakly anharmonic oscillator

Recap of how secular terms arise and how their growth invalidates perturbation theory. Consider anharmonic oscillator

$$\ddot{x} + \omega^2 x + \varepsilon x^3 = 0 \quad , \quad x(0)=1,\ \dot{x}(0)=0$$

Expand $x = x_0 + \varepsilon x_1 + \dots$

$$\Rightarrow \quad \ddot{x}_0 + \omega^2 x_0 = 0 \quad \Rightarrow \quad x_0(t) = \cos(\omega t)$$

$$\Rightarrow \quad \ddot{x}_1 + \omega^2 x_1 + x_0^3 = 0$$

i.e. $$\ddot{x}_1 + \omega^2 x_1 = -\frac{1}{4}\cos(3\omega t) - \frac{3}{4}\cos\omega t$$

this term gives rise to resonance

1209.2369

$$\Rightarrow \quad x_1 = \frac{1}{32\omega^2}(\cos 3\omega t - \cos\omega t) - \frac{3}{8\omega} t \sin\omega t$$

secular term: invalidates expansion for $t \gtrsim \frac{1}{\varepsilon}$

We can absorb the secular term by a frequency shift

$$x(t) = \cos\left[\left(1 + \frac{3\varepsilon}{8\omega^2} + \dots\right)\omega t\right] + \frac{\varepsilon}{32}\cos\left[3\left(1+\frac{3\varepsilon}{8\omega^2}+\dots\right)\omega t\right]$$

→ more accurate description for longer time intervals

so called Poincaré-Lindstedt method

One needs more elaborate methods when dealing with multiple resonances.

In the case of the perturbative treatment of the gravity + scalar systems, we get equations:

$$O(\varepsilon^0): \quad \ddot{\phi}_1 + L\phi_1 = 0 \quad \Rightarrow \quad \phi_1 = \sum_{j=0}^{\infty} \underbrace{a_j \cos(\omega_j t + \beta_j)\, e_j(x)}_{\equiv \phi_1^j}$$

$$\xrightarrow[e_j]{\text{project on}} \quad \ddot{\phi}_1^j + \omega_j^2 \phi_1^j = 0$$

Call $\phi_n^j \equiv$ projection of ϕ_n on e_j

$$\Rightarrow \quad \ddot{\phi}_n^j + \omega_j^2 \phi_n^j = S_n^j(\phi_1, \dots, \phi_{n-1})$$

The RHS will have terms like $\propto \cos\left[(\omega_{i_1} \pm \omega_{i_2} \pm \dots)t + b_{i_1} \pm b_{i_2} \pm \dots\right]$

where i_1, i_2 can be any kind of mode numbers

If $\omega_{i_1} \pm \dots \neq \omega_j$ → No secular term

$\omega_{i_1} \pm \dots = \omega_j$ → term $\propto t\sin\left[(\omega_{i_1} \pm \dots)t + b_{i_1} \pm \dots\right]$

Two cases:

1. Non-resonant spectrum: $\sum_j m_j \omega_j = 0$ only has the solution $m_j = 0$

   $\Rightarrow$ $\omega_j = \omega_{i_1} \pm \omega_{i_2} \pm \ldots$ can only be satisfied if ω_j appears on the RHS once time more with a "+" than with a "−"

   $\Rightarrow$ $\cos[(\omega_{i_1} \pm \ldots)t + b_{i_1} \pm \ldots] = \cos(\omega_j t + b_j)$

   $\Rightarrow$ secular term $\varepsilon^n A_n^j(a)\, t \sin(\omega_j t + b_j)$

   $\Rightarrow$ can be absorbed into frequency shift

   $$\cos((\omega_j + \alpha)t + b_j) = \cos(\omega_j t + b_j) - \alpha t \sin(\omega_j t + b_j)$$

2. Resonant spectrum: There are more general solutions to $\sum_j m_j \omega_j = 0$

   $\Rightarrow$ resonant terms $\cos(\omega_j t + \underbrace{b_{i_1} \pm b_{i_2} \pm \ldots}_{\text{generally complicated combination of phases}})$

   ~~cannot be absorbed~~

   $\Rightarrow$ secular term cannot be absorbed just in a shift of the frequency $\varepsilon^n t A_n^j(a)\, t \sin(\omega_j t + b_{i_1} \pm \ldots)$

   One can however shift both a_j and ω_j in $a_j \cos(\omega_j t + b_j)$ to get secular term, but shift in a_j has to be time dependent, i.e. not better or worse than original secular term

Physically: Terms that look like amplitude drifts (i.e. time dependent amplitudes)

$\longrightarrow$ significant energy transfer between modes.

⚠ The failure of the Poincaré Lindstedt method does not necessarily lead to instability → other resummation techniques might produce long-period oscillations of energy between modes etc...

Back to Rostworowski & Bizoń

The eigenmode spectrum of the unperturbed problem is highly resonant $\omega_j = \cancel{\ldots} (3+2j)^2$

One finds: For one-mode initial data → absorption by shift of frequency possible

For multiple modes in initial state → cannot eliminate secular terms ⇒ non-linear mode coupling and drift of energy to higher frequencies

See the shift of Energy between modes in animation

This already happens for two-modes initial data, i.e.

$\phi(0,x) = \varepsilon(\gamma e_0(x) + \delta e_1(x))$ ~~As~~ ~~see~~

Animation for some initial mode occupation. → turbulence

$E_j \propto j^{-\alpha}$

$\alpha \approx 1.2$

For Einstein's equations, the transfer of energy to high frequency modes cannot proceed forever because concentration of energy on smaller and smaller scales leads to BH formation.

Result generalized: higher dim [R&B 2011]
complex scalar [Buchel, Lehner & Liebling 2012]
pure gravity [Dias, Horowitz, Santos 2011]

Stable Solutions

More recently, nonlinearly stable solutions ~~of~~ in AdS where found.

Maliborski & Rostworowski 2013

arXiv: 1303.3186

Same model as in [R&B 2011]: gravity + scalar

Perturbative constr.:

Seek time-periodic solutions of the e.o.m. of the form

$$\phi = \varepsilon \cos(\omega_\gamma \tau) e_\gamma(x) + \sum_{\text{odd } \lambda} \varepsilon^\lambda \phi_\lambda(\tau, x)$$

where $\tau = \Omega_\gamma t$; $\Omega_\gamma = \omega_\gamma + \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda \omega_{\gamma,\lambda}$

i.e. one dominant mode in the limit $\varepsilon \to 0$ (and for one-mode initial data stability was shown in [R&B 2011]

Similar expansion for A and δ

$$A = 1 + \sum_{\text{even } \lambda \geq 2} \varepsilon^\lambda A_\lambda(\tau, x)$$

$$\delta = \sum_{\substack{\text{even} \\ \lambda \geq 2}} \varepsilon^\lambda \delta_\lambda(\tau, x)$$

Numerical constr.

Ansatz

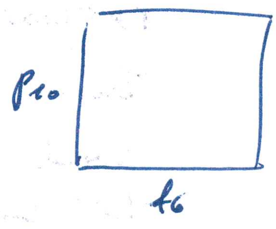
$$\phi = \sum_{0 \leq j \leq K} f_j(\tau) e_j(x) = \sum_{0 \leq i \leq N} \sum_{0 \leq j \leq K} f_{ij} \cos((2i+1)\tau) e_j(x)$$

$$\Pi = \sum_{0 \leq j \leq K} p_j(\tau) e_j(x) = \sum_{0 \leq i \leq N} \sum_{0 \leq j \leq K} p_{ij} \sin((2i+1)\tau) e_j(x)$$

Sim. for δ and A

⇒ solve e.o.m. on grid.

→ find closed curves on slices of phase space, e.g.



→ evidence for stability. → animation

Boson stars

arXiv: 1304.4166, Buchel, Lehner & Liebling

Model: Complex scalar + gravity in 4d ~~AdS~~ with $\Lambda = -\frac{3}{L^2}$ $\rightarrow L=1$

$$S = \frac{1}{16\pi G}\int_{M_4} d^4\xi \,(R + 6 - 2\partial\phi\partial\phi^*)$$

Conserved $U(1)$ charge Q

The resulting e.o.m. have stationary solutions called boson stars where the scalar field ϕ varies harmonically

$$\phi_1(x,t) + i\phi_2(x,t) = \frac{\phi(x)}{\cos^2 x}\, e^{i\omega t}, \quad A(t,x) = a(x),\ \delta(t,x) = d(x)$$

$\rightarrow$ standing wave!

For small scalar amplitude, ~~these~~ solutions are known analytically, for larger amplitudes they can be found numerically

[They are superpositions of the linear modes of pure AdS!]

These solutions are linearly stable and numerical results indicate that they are also non-linearly stable for perturbations with amplitudes smaller than some ε.

Boson stars are unstable in asymptotically flat backgrounds as they radiate energy ~~and finally settle~~ to infinity.

Heuristic argument: Widely distributed mass-energy distorts space sufficiently to introduce dispersion and therefore oppose the concentrating effect of the instability

They also claim that the model of R & B of 2011 is non linearly stable ~~that ZR~~ for $\sigma \geq 0.4$ ~~that~~ but Rostworowski disagrees (Maliborski, Rostworowski, "A Comment on "Boson stars in AdS"", 1307.2875) $\rightarrow$ they say that BLL found a stability island and the turbulent cascade is restored for larger σ.

which is consistent with this argument

Two Timescales Formalism

arXiv: 1403.6471, Balasubramanian, Buchel, Green, Lehner & Liebling

Idea: two "times": $t \rightarrow$ characterizes original normal modes

$\tau = \varepsilon^2 t \rightarrow$ characterizes timescale at which energy transfer between modes occurs

$\rightarrow$ determine large class of quasi-periodic solutions that have tuned initial energy spectra such that the flow of energy in and out of each mode vanishes on timescale τ

Setup: gravity + massless scalar, but $\phi = \phi(t,\tau,x)$, $A = A(t,\tau,x)$, $\delta = \delta(t,\tau,x)$ $\rightarrow$ same for their expansion coefficients

Find $\partial_t^2 \phi_{(1)} + L\phi_{(1)} = 0$ just as previously

$\Rightarrow \phi_{(1)}(t,\tau,x) = \sum_{j=0}^{\infty} \left(A_j(\tau)\, e^{-i\omega_j t} + \bar{A}_j(\tau) e^{i\omega_j t} \right) e_j(x)$

$\uparrow$ τ dependent amplitude, both abs. value & phase

At $O(\varepsilon^3)$: $\partial_t^2 \phi_{(3)} + L\phi_{(3)} + 2\partial_t\partial_\tau \phi_{(1)} = S_{(3)}(t,\tau,x)$

$\rightarrow$ handle resonances by using add. term $\partial_t \partial_\tau \phi_{(1)}$ and freedom in choosing $A_j(\tau)$

$\rightarrow$ set of equations for $A_j = \alpha_j \exp(-i\beta_j \tau)$ if we want periodic solutions (have to truncate to $j \leq j_{max}$)

$\rightarrow$ leads to solutions where all modes are initially populated. In this case, also the cascade picture breaks and one can again absorb all secular terms into frequency shifts.

The paper apparently contains some wrong results when they claim that also two-mode initial data can lead to quasi-periodic evolution. Numerics by Bizoń. lead to BH formation for the same model + same initial conditions

1410.2831 [gr-qc]

The Two-Time Framework only covers the first non-trivial scale, one could introduce many more

$\varepsilon t, \varepsilon^2 t, \varepsilon^3 t, \dots$

There might be more complicated cases where dependencies like $\varepsilon^{3/2} t$ arise

Problem: one has to guess in advance what the relevant scales are

$\rightarrow$ cured by renormalization group method which we will hear about tomorrow.