Asymptotic Series & non-pert. effects.

References

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C.M. Bender, T.T. Wu, Phys Rev. 184, 1231 (1969)

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M. Mariño Lecture notes

① Motivation

Most of the series obtained in perturbation theory, in QM or QFT, are asymptotic rather than convergent. In fact, they typically have zero radius of convergence

Q: What causes this divergences and how can we extract a sensible answer from this series?

Let's start with the definition and properties of asymptotic series:

$S(g) = \sum_{n=0}^{\infty} a_n g^n$ is asymptotic to $f(g) \sim S(g)$.

when $g \to 0$ if $\frac{f(g)}{S(g)} \to 1$ as $g \to 0$.

Notice that due to it's definition, different functions can have the same asymptotic behavior.

$f(g) + Ce^{-A/g}$ will have exactly the same asymp. since the new term is non-perturb. in g (is invisible in pert theory.)

Every time you got $g \to 0$, you also have $e^{-A/g} \to 0$

This extra term that you can add to your series is called non-perturbative ambiguity

Asymptotic expansions are characterized by the fact that the a_n have a factorial divergence

$$\boxed{a_n \sim A^{-n} n! \quad \text{as } n \to \infty}$$

This makes $S(g)$ divergent since no power g^n can overcome $n!$ $\Rightarrow$ zero radius of convergence

This fact is the series telling you that you are missing some non-perturbative effects.

Later on, we will see that the large order behavior, encodes important information about what type of non-pert. eff. are you missing!

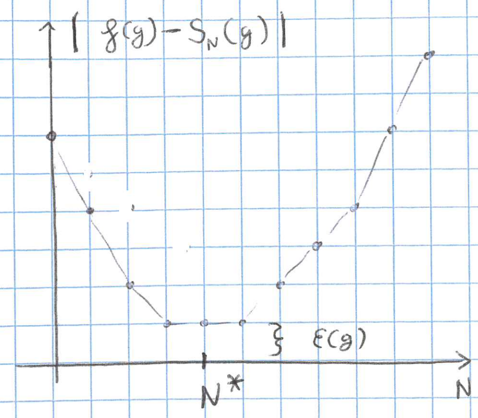
Normally, the series $S(g)$ will approach the exact value of $f(g)$ and then diverge factorially.

What is the optimal value for $N = N^*$ for which $S(g)$ is a good approximation?

What is the error or the resolution that $|f(g) - S_N(g)|$ will have?

Here $S_N(g) = \sum_{n=0}^{N} a_n g^n$ is the partial sum of $S(g)$

②. Optimal Truncation



To find the optimal value of N, we need to minimize

$$|a_n g^n| \sim n! \left(\frac{g}{A}\right)^n \quad \leftarrow \text{Stirling.}$$

$$\sim \exp\left\{ n \left(\log(n) - 1 - \log\left(\frac{A}{g}\right)\right)\right\}$$

$$\Rightarrow \boxed{N^* = \left|\frac{A}{g}\right|}$$

- The smaller is your expansion parameter, the more terms in your series you can use and the close to $f(g)$ you get.

Q: How close?

$$\boxed{\epsilon(g) \sim e^{-A/g}} \Rightarrow |f(g) - S_{N^*}(g)| \sim e^{-A/g}$$

The error is of the order of the non-pert ambiguity!

Example:

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{2} - g\frac{z^4}{4}}$$

← Non-pert definition! For $\mathrm{Re}(g) > 0$

We can expand for small g inside the integral

$$S(g) = \sum_{n=0}^{\infty} a_n g^n = \sum_{n=0}^{\infty} g^n \cdot \frac{(-4)^{-n}}{\sqrt{2\pi}\, n!} \int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{2}} z^{4n} \Rightarrow$$

$$a_n = (-4)^{-n} \frac{(4n-1)!!}{n!}$$

- To put this on the form $|A|^{-n} n!$ we use stirling.

$n! \sim n^n e^{-n}$

$(2n)!! = 2^n n! \sim 2^n n^n e^{-n}$ $\Rightarrow a_n \sim (-4)^{-n} \frac{(4n)!!}{n!} =$

$= (-4)^{-n} \frac{2^{2n}(2n)!}{n!} \sim (-4)^{-n} \cdot 4^n \cdot \frac{(2n)^{2n} e^{-2n}}{n^n e^{-n}} =$

$= (-4)^n n^n e^{-n} \sim \underline{\underline{(-4)^n n!}}$ $\Longrightarrow \boxed{|A| = \frac{1}{4}}$

Optimal truncation $N^* = \left|\frac{1}{4g}\right|$ and $\mathcal{E}(g) \sim e^{-1/4g}$

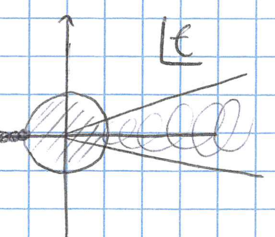
Can we extract the answer from all the a_n or even reconstruct $f(g)$??

. Borel Resummation & Padé-Borel

We define the Borel Transform of a series $S(g)$ as

$$\hat{S}(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$$

- Finite radius of convergence $\rho = |A|$ typically limitated by the nearest singularity or branch cut.



If we can analytically continue to $\mathbb{R}e(t)$ $(0,\infty)$ such that the integral

$$\boxed{BS(g) = \int_0^{\infty} dt\, e^{-t} \hat{S}(gt) = \frac{1}{g}\int_0^{\infty} dt\, e^{-t/g} \hat{S}(t)}$$ exists.

. $S(g)$ is Borel summable and $BS(g) \sim S(g)$

$\overset{\|}{??\ f(g)\ ??}$

Remark!

Even if $BS(g)\ \exists \not\Rightarrow BS(g) = f(g)$, they might be complex poles in the ℂ plane that dominate.

Q: What we do if we only have a finite # of a_n ?

. Padé-Borel extension

Padé approximant of $S(g)$ is given by a rational function $[L/M]$ /

$$\left[\frac{L}{M}\right] = \frac{p_0 + p_1 g + \dots + p_L g^L}{1 + q_1 g + \dots + q_M g^M} \quad \text{where} \quad S(g) - \left[\frac{L}{M}\right] = O(g^{L+M+1})$$

$\Rightarrow$ Padé approx $P_N^S(t) = \left[\frac{\left(\frac{N}{2}\right)}{\left(\frac{N+1}{2}\right)}\right]$ requires only $N+1$ coeff.

- Rational function with poles in the ℂ plane.

i) if $\hat{S}(t)$ has a pol, Padé will guess it's location and make the series converge much faster

ii) if $\hat{S}(t)$ has a branch cut, Padé, will mimic it by a series of poles as N grows.

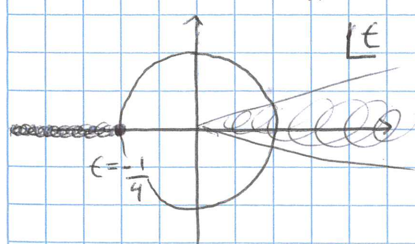
$$\Rightarrow B_N S(g) = \frac{1}{g}\int_0^{\infty} dt\, e^{-t/g} P_N^S(t)$$

Example $I(g)$.

$$\hat{S}(t) = \sum_{n=0}^{\infty} \frac{(-4)^{-n}(4n-1)!!}{n!\,n!} t^n = \frac{2K(\nu)}{\pi(1+4t)^{1/4}}$$

elliptic integral of first kind.

$\nu^2 = \frac{1}{2} - \frac{1}{2\sqrt{1+4t}}$



- Branc cut starting at $t = -\frac{1}{4} \to -\infty$
$= -|A|$

in this case $BS(g)\ \exists$ and $\boxed{BS(g) = I(g)}$ numerically in this case, we have $BS(g) = I(g)$ and we reconstruct the exact non-pert. function, without any ambiguity.

* This also happens for the QM Quartic osilator.

example, general series

$$S(g) = \sum_{n=0}^{\infty} \frac{\Gamma(n+b)}{\Gamma(b)} A^{-n} g^{n}$$ its Borel transf is

$$\hat{S}(t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+b)}{n!\,\Gamma(b)} A^{-n} t^{n} = (1 - t/A)^{-b}$$

For $b \neq 1$ => we will have a branch cut at $t=A$.

For $b=1$ => we will have a pole at $t=A$.

in any case, we see that the singularities of the Borel transform are encoded in the large order behavior of $S(g)$!

Remark:
We can obtain information about non-pert. effects by looking at the Borel transform of a perturbation series, independently if $BS(g)$ exists or not!

How we relate them?

This is a general property of asymptotic series, no matter if they can be summed or not.

To see this, we will use $I(g)$ as an example.

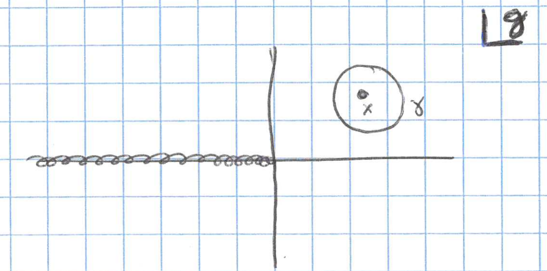
i) $I(g)$ has a branch cut $(-\infty, 0)$

ii) $\lim_{g \to 0} g\, I(g) \to 0$ (asymp series).

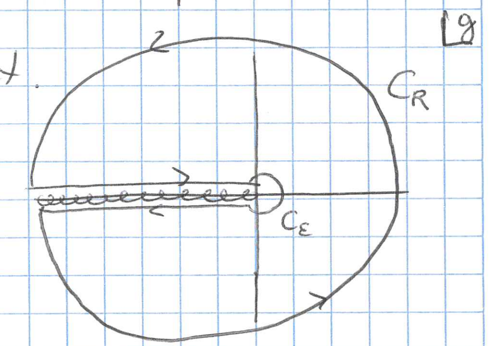
iii) $I(g) \sim g^{-1/4}$ as $g \to \infty$. (scaling $z = g^{-1/4} u$)

Using cauchy's thm.

$$I(g) = \frac{1}{2\pi i} \oint_{\gamma} dx \frac{I(x)}{x - g}$$



=> Deforming γ to encercle the cut.



$$I(g) = \frac{1}{2\pi i} \int_{-\infty}^{0} dx \frac{D(x)}{x - g}$$

where $D(x) = 2i \,\mathrm{Im}\, I(x + i0_+)$ Discontinuity of the cut.

$$I(g) = \frac{1}{\cancel{2\pi i}} \int_{-\infty}^{0} \sum_{n=0} \frac{1}{x} \frac{g^n}{x^n} \cancel{2i}\, \mathrm{Im} I(x)\, dx = \sum_{n=0} \frac{g^n}{\pi} \int_{-\infty}^{0} \frac{dx}{x^{n+1}} \mathrm{Im}\, I(x)$$

$$I(g) = \sum_{n=0} g^n \frac{1}{\pi} \int_{-\infty}^{0} \frac{dx}{x^{n+1}} \mathrm{Im}(I(x)) = \sum_{n=0} g^n \frac{1}{\pi} (-1)^{n+1} \int_{0}^{\infty} dx \frac{\mathrm{Im}(I(-x))}{x^{n+1}}$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi} (-1)^{n+1} \int_{0}^{\infty} dx \frac{\mathrm{Im}(I(-x))}{x^{n+1}}}$$

Dispersion relation

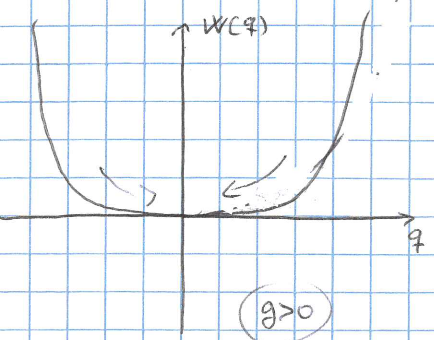
g strong $g \to \infty$

$\Updownarrow$

g weak (saddle point)

This relation is telling us something very profound.

Imagine that you want to calculate the energy for Quartic harmonic oscilator $g>0$, as a series

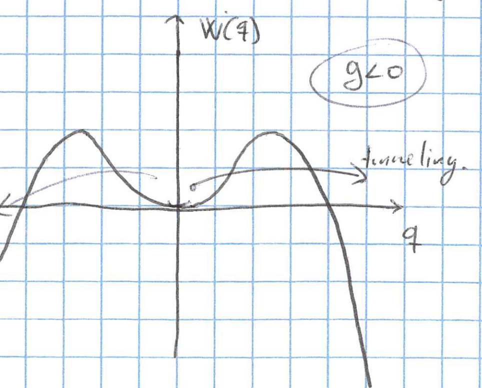


$$W(q) = \frac{q^2}{2} + g\frac{q^4}{4}.$$

$$E(g) = \sum_{n=0}^{\infty} a_n g^n = \frac{1}{2} + \frac{3}{14}\left(\frac{g}{4}\right) - \frac{21}{8}\left(\frac{g}{4}\right)^2 + \frac{333}{16}\left(\frac{g}{4}\right)^3 + \dots$$

This is a divergent series, and we would like to know, what is the asymptotic behavior of $\underline{\underline{a_n}}$ $n \to \infty$.

Looking at the previus equation, we se that the asymptotic a_n is encoden in the imaginary part of the energy, when we change the sign of the coupling!



$$W(q) \to \frac{q^2}{2} + g\frac{q^4}{4} \; !$$

But $\mathrm{Im}(E)$ is related to the semiperiod of decay!

$$\boxed{|\mathrm{Im}(E(-g))| = +\frac{\Gamma}{2}}$$

...uation:

> The asymptotic behavior of a_n for $g>0$ of the energy levels, is related to the decay rate through tuneling in a ptheory where $g \to -g$! i.e. instanton solutions that is non-perturbative in $|g|$.

Example. General Discontinuity.

We want to calculate the asymp a_n. Using the formula for a_n, we have thas asymptotically.

$$a_n = \frac{(-1)^{n+1}}{2\pi i}\int_0^\infty dx \frac{\mathrm{Disc}(S(-x))}{x^{n+1}} \quad \text{where} \quad \mathrm{Disc}(S(-g)) = C_0 e^{-A/g} g^{-b} i$$
$$= 2i\,\mathrm{Im}(S(-g))$$

$$a_n = \frac{(-1)^{n+1}}{2\pi}\int_0^\infty C_0 \frac{x^{-b} e^{-A/x}}{x^{n+1}} = \frac{(-1)^{n+1}}{2\pi} A^{-b-n} C_0 \int_0^\infty dx\, x^{n+b+1} e^{-x}$$

$$= C_0\frac{(-1)^{n+1}}{2\pi} \cdot A^{-b-n}\,\Gamma(n+b) \implies \boxed{a_n \sim C_0\frac{(-1)^{n+1}}{2\pi} A^{-b-n}\,\Gamma(n+b)}$$

4. Lateral Borel & Instantons

Q: What can we doo when Borel Summability Fails?

There are 2 Main reasons why $\mathcal{B}S(g) \nexists$

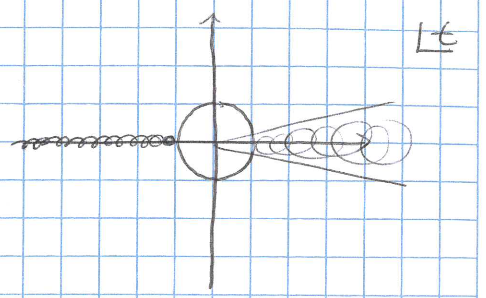
i) Perturbation series arround unstable minimum
(Quartic harmonic $g<0$, Cubic oscilator)

ii) Degenerate ground state
(Double-well potential, ~~γ potential $\gamma=1$~~)

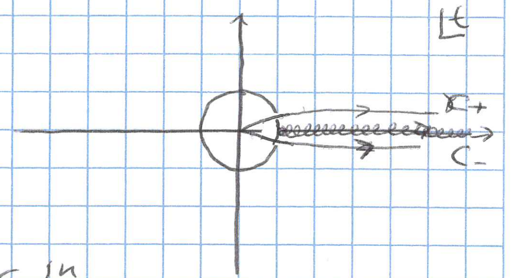
This potencials lack of a trivial Borel Summation because they have singularities in the real positive $\mathbb{C}$ plane or $\mathrm{Re}(t)>0$

This singularities tell you that there are non-pert. effects (Instantons, ~~Renormalons~~) that you have to take into account.

>0 $\Longrightarrow$ ∃ Borel transf, ∃ BS (Bender, Wu)



<0 $\Longrightarrow$ ∃ Borel transf, ∄ BS



We need to deform the contour in order to avoid the sing. or branchcuts.

Lateral Borel Transforms

$$\boxed{B_\pm S(g) = \frac{1}{g}\int_{C_\pm} dt\; e^{-t/g}\,\hat{S}(t)}$$

The transform pick an imaginary part, which is encoded in the discontinuity function

$$\mathcal{E}(g) = \frac{1}{2\pi i}\left(B_+S(g) - B_-S(g)\right) = \frac{1}{\pi}\,\mathrm{Im}\left(BS(g)\right) \sim \frac{1}{\pi}\,\mathrm{Im}\left(S(g)\right)$$

This non-pert imaginary contribution suggests that instead of considering $S(g)$, we should consider a generalized Trans Series, to make $S(g) \in$ Real

$$\boxed{S(g) = \sum_{e=0}^{\infty} C^e\, S^{(e)}(g)}$$

Here $S^{(e)} \sim e^{-eA/g}\sum_{n=0}^{\infty} C_n^e\, g^n$ with

$C^e \in \mathbb{C}$, and $S^{(0)} = \sum_{n=0}^{\infty} a_n g^n$ initial guess.

This is the type of series that you obtain when you look at e-instanton corrections to your solution!

~~Example: Instanton in the Quartic harmonic g<0.~~

Instantons in Q.M

Ground state energy of QM system with potential $W(q)$ can be extracted from small T of the partition function.

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) \quad \text{as} \quad E = -\lim_{\beta\to\infty}\frac{1}{\beta}\log[Z(\beta)]$$

In euclidean path integral

$$Z(\beta) = \int \mathcal{D}q\; e^{-S[q]} \qquad S[q] \text{ euclidean action}$$

$$S[q] = \int_{-\beta/2}^{\beta/2} dt\left[\frac{1}{2}\dot{q}^2 + W(q)\right]$$ and we integrate over periodic trajectories

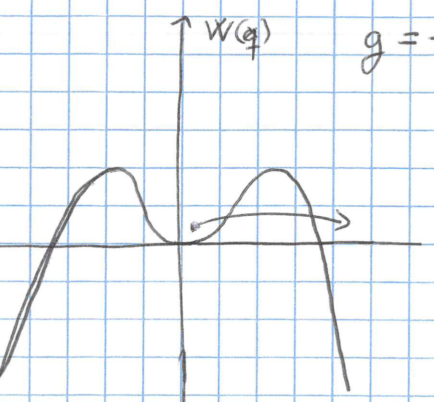
$$q(-\beta/2) = q(\beta/2)$$

Note: Euclidean action can be though of as the action in Lagrang mechanics with

$$\boxed{V(q) = -W(q)}$$

Example: Instanton in Quartic harmonic $g<0$.

- We want to calculate the instanton contribution to the Quartic potential with $g<0$ because it will give us the imaginary part of the energy which contains valuable information about the asymptotic series an. of the Quartic potential with $g>0$.



$g=-\lambda<0$

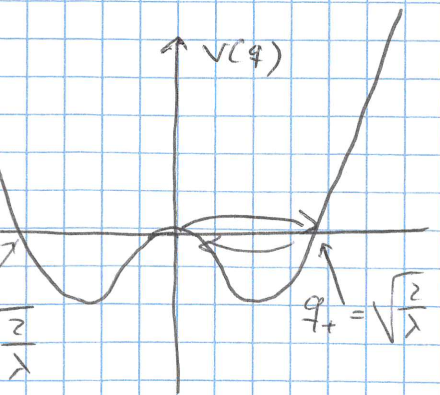
$$W(q) = \frac{1}{2}q^2 - \lambda\frac{q^4}{4}$$

We know that the contribution will be of the order of

$\sim e^{-S_c}$ where S_c is the classical action of the instanton.

- how we calculate it??

- We use the tric that the tunneling can be thought of as a normal trajectory in $-W(q)=V(q)$ potential.



$$\boxed{V(q) = -\frac{1}{2}q^2 + \lambda\frac{q^4}{4}}$$

$$\Big\downarrow \text{ e.o.m}$$

$$\boxed{\ddot{q} - q + \lambda q^3 = 0}$$

In the limit in which $\beta\to\infty$, we want to consider solutions that have $E=0$, and are solutions of $V(q)=0$.

Classically, we have $q_\pm = \pm\sqrt{\frac{2}{\lambda}}$, however, we also have special configurations where we consider <u>non-trivial</u> <u>saddle points</u> ~~of~~ the classical action.

These correspond to solutions that interpolate between the 2 solutions. In our case.

$$\boxed{q_c(t) = q_\pm \frac{1}{\cosh(t)}} \qquad \begin{cases} q_c \to 0 & t \to \pm\infty = \pm\beta/2 \\ q_c \to q_\pm & t\to 0 \end{cases}$$

#The action of this solution can be calculated as:

$$S_c = \int_{-\beta/2}^{\beta/2} dt\left[\frac{1}{2}\dot{q}_c^2 - V(q_c)\right] = 2\int_{q=0}^{q_\pm}\sqrt{2W(q)}\,dq$$

where energy conservation $E = 0 = \frac{1}{2}\dot{q}_c^2 + V(q_c)$

$$S_c = 2\int_0^{\sqrt{\frac{2}{\lambda}}} dq\, q\sqrt{1-\frac{\lambda q^2}{2}} = \frac{4}{3\lambda}$$

$$\Longrightarrow \quad \mathrm{Im}(E_0) \sim q_\pm e^{-\frac{4}{3\lambda}} \sim \frac{1}{\sqrt{\lambda}}e^{-\frac{4}{3\lambda}}$$

$$\boxed{\mathrm{Im}(E_0) = \frac{4}{\sqrt{2\pi\lambda}}e^{-\frac{4}{3\lambda}}}$$ exact result.

If we now compare this with the solution for the general Discontinuity.

$$\frac{\text{Disc}}{2\pi i} = \frac{C_0}{2\pi} e^{-A/g} g^{-b} = \text{Im}(E_0) = \frac{4}{\sqrt{2\pi\lambda}} e^{-\frac{4}{3\lambda}}$$

$$\Longrightarrow \quad C_0 = 2\sqrt{\frac{2}{\pi}}, \quad b = \frac{1}{2}, \quad g = \lambda, \quad A = 4/3$$

$$\boxed{a_n \sim (-1)^{n+1} \frac{\sqrt{6}}{\pi^{3/2}} \left(\frac{3}{4}\right)^n \Gamma\left(n+\frac{1}{2}\right)}$$

The famous result in Bender, Wu !

Large order behavior in asymp. series

$\Updownarrow$

(nearest) Singularities in the complex LE plane

$\Updownarrow$

Generated by non-perturbative instanton effects. in Q.M.

- Complex instantons
- Sing. in $\mathbb{R}e(t) < 0$

$\Downarrow$

- Borel summable

$\not\Downarrow$

- BS is non perturb solution !

(Complex instantons might be not subleading!)

- Real instantons
- Sing. in $\mathbb{R}e(t) > 0$

$\Downarrow$

- Lateral Borel + Trans series extension

$\Downarrow$

- Unstable minima or degeneration

(in Degenerate cases, if you consider all instanton contrib you can obtain the exact solution fixing c_e !!)