

Relativistic Hydrodynamics: Second Order Constituent Relations

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1 Hydrodynamics Assumptions

Microscopic Theory Consider a Poincaré-invariant field theory in $(d + 1)$ space-time dimensions with conserved current densities $T^{\mu\nu}$ and J^μ satisfying $(d + 1 + 1)$ conservation equations. Let P^μ and N be the $(d + 1 + 1)$ associated conserved charges.

Thermal Equilibrium Global thermal equilibrium is described by a density operator given by the exponential of conserved charges P^μ , N ; the manifold of equilibrium states can be parameterised by a unit timelike vector u^μ (4-velocity) and scalars μ (chemical potential) and T (temperature).

Hydrodynamics Assumptions

1. Close to global thermal equilibrium, the system is still fully determined by the conservation of current densities. That implies that $T^{\mu\nu}$ and J^μ must be functions of $(d + 2)$ parameters which can be interpreted as local 4-velocity $u^\mu(x)$, local chemical potential $\mu(x)$ and local temperature $T(x)$. $u^\mu(x)$, $\mu(x)$ and $T(x)$ are the values that these thermodynamic quantities assume in regions of local thermal equilibrium which are large compared to microscopic scales (such as mean free time and mean free path) but approximately local on macroscopic, thermodynamic scales. That means that when the conserved charge densities $T^{\mu\nu}$ and J^μ are expressed in terms of $u^\mu(x)$, $\mu(x)$ and $T(x)$ (constituent relations), we are not dealing with the microscopically defined current densities but rather with their expectation values over regions of local thermodynamic equilibrium. These expectation values are still to obey the conservation equations.
2. The assumed proximity to global thermal equilibrium at $t \rightarrow \infty$ is manifest in two consequences: First, as we consider late times (compared to microscopic interaction scales) only long-lived fluctuations of small frequencies and thus (assuming a normal dispersion relation) small wave-vectors are relevant. This is a statement about the

wavelength of hydrodynamic fluctuations. Second, the local charge densities assume values close to the ones in global equilibrium. This is a statement about the amplitude of hydrodynamic fluctuations. These assumptions allow for two expansions, one in gradients ∂ (small wave-vectors) of our local thermodynamic quantities and one in fluctuations δ of local charge densities around their values in global thermal equilibrium.

2 Constituent Relations

We want to express $T^{\mu\nu}$ and J^μ in terms of (u^μ, μ, T) . For that purpose, decompose the current densities in components transverse and parallel to u^μ :

$$\begin{aligned} T^{\mu\nu} &= \mathcal{E} u^\mu u^\nu + \mathcal{P} \Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu} \\ J^\mu &= \mathcal{N} u^\mu + j^\mu \end{aligned} \quad (2.1)$$

with

$$\Delta^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \quad (2.2)$$

where \mathcal{E} , \mathcal{P} and \mathcal{N} are scalars, q^μ and j^μ are vectors transverse to u^μ and $t^{\mu\nu}$ is a transverse, traceless, symmetric tensor; all expansion parameters are functions of the $(d+2)$ parameters $u^\mu(x)$, $\mu(x)$ and $T(x)$. The explicit decomposition can for instance be found in [1].

Let us now expand the constituent parameters in long-wave lengths, that is, in gradients of our local thermodynamic variables (u^μ, μ, T) . Lorentz tensor structure and the orthogonality condition $u^\mu q_\mu = u^\mu j_\mu = 0$ imply that q^μ and j^μ cannot depend on (u^μ, μ, T) but only on their derivatives. Thus we can write

$$\begin{aligned} \mathcal{E} &= \epsilon(T, \mu) + f_{\mathcal{E}}^{(1)}(\partial T, \partial \mu, \partial u^\mu) + \mathcal{O}(\partial^2) \\ \mathcal{P} &= p(T, \mu) + f_{\mathcal{P}}^{(1)}(\partial T, \partial \mu, \partial u^\mu) + \mathcal{O}(\partial^2) \\ \mathcal{N} &= n(T, \mu) + f_{\mathcal{N}}^{(1)}(\partial T, \partial \mu, \partial u^\mu) + \mathcal{O}(\partial^2) \\ q_\mu &= q_\mu^{(1)}(\partial T, \partial \mu, \partial u^\mu) + \mathcal{O}(\partial^2) \\ t_{\mu\nu} &= q_{\mu\nu}^{(1)}(\partial T, \partial \mu, \partial u^\mu) + \mathcal{O}(\partial^2) \\ j_\mu &= j_\mu^{(1)}(\partial T, \partial \mu, \partial u^\mu) + \mathcal{O}(\partial^2) \end{aligned} \quad (2.3)$$

where (1) indicates that the respective function depends to first order on gradients of (u^μ, μ, T) .

Correspondingly, we can expand the charge densities in gradients:

$$\begin{aligned} T_{\mu\nu} &= T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + \mathcal{O}(\partial^2) \\ J_\mu &= J_\mu^{(0)} + J_\mu^{(1)} + \mathcal{O}(\partial^2) \end{aligned} \quad (2.4)$$

Ordinary dissipative hydrodynamics takes into account first order gradient terms.

2.1 Zeroth Order Gradient Expansion

An expansion of the current densities to zeroth order in gradients means that we ignore their spatial variation or in other words that we neglect the interplay between different patches of local equilibrium: At zeroth order we look at individual regions of local thermal equilibrium isolated from each other. The current densities take the form

$$\begin{aligned} T_{\mu\nu}^{(0)}(x) &= \epsilon(T(x), \mu(x)) u_\mu(x) u_\nu(x) + p(T(x), \mu(x)) \Delta_{\mu\nu}(x) \\ J_\mu^{(0)} &= n(T, \mu) u_\mu. \end{aligned} \quad (2.5)$$

As we have just argued that zeroth order describes the local thermal equilibrium within each region (labeled by x), we can readily interpret $\epsilon(x)$, $p(x)$ and $n(x)$ as local equilibrium energy density, pressure and charge density of an ideal fluid with 4-velocity u^μ in this region. The zeroth order quantities $\epsilon(x)$, $p(x)$ and $n(x)$ thus have to satisfy equilibrium thermodynamic relations and are related to the values of the thermodynamic quantities $(\mu(x), T(x))$ within the same region x of local equilibrium by an applicable equation of state. We can use this relation to obtain $T(n, \epsilon)$, $\mu(n, \epsilon)$ and trade (u^μ, μ, T) for their conjugate variables to use the local charge densities (u^μ, n, ϵ) as thermodynamic variables in terms of which we express the conserved current densities.

2.2 Frame Choice

For higher orders in the gradient expansion, we encounter a gauge freedom: (u^μ, μ, T) do not have a microscopic definition out of global thermal equilibrium, they are merely auxiliary fields used to parameterise the microscopically defined conserved current densities $T^{\mu\nu}$ and J^μ . We are thus free to reparameterise (u^μ, μ, T) as long as $T^{\mu\nu}$ and J^μ do not change. As, however, the zeroth order values do have a microscopic definition as local fluid 4-velocity and local equilibrium charge densities, we will only be able to shift (u^μ, μ, T) by functions that vanish for vanishing gradients (this can also be seen by observing that any zeroth order change would necessarily change $T_{\mu\nu}^{(0)}$ and $J_\mu^{(0)}$).

Let us reparameterise the thermodynamic variables

$$(u^\mu, \mu, T) \rightarrow (u'^\mu, \mu', T') = (u^\mu, \mu, T) + (\delta u^\mu, \delta \mu, \delta T). \quad (2.6)$$

where $(\delta u^\mu, \delta \mu, \delta T)$ are functions that depend on first order gradients of (u^μ, μ, T) and vanish for vanishing gradients. Using the decomposition (2.1) and requiring invariance of $T^{\mu\nu}$ and J^μ one finds

$$\begin{aligned} \delta \mathcal{E} &= 0 \\ \delta \mathcal{P} &= 0 \\ \delta \mathcal{N} &= 0 \\ \delta q^\mu &= -(\mathcal{E} + \mathcal{P}) \delta u^\mu \\ \delta j^\mu &= -\mathcal{N} \delta u^\mu \\ \delta t_{\mu\nu} &= 0. \end{aligned} \quad (2.7)$$

We can therefore choose δu^μ such that $q_\mu^{(1)} = 0$. From (2.3) we find furthermore

$$0 = \delta \mathcal{E} = \frac{\partial \epsilon}{\partial T} \delta T + \frac{\partial \epsilon}{\partial \mu} \delta \mu + \delta f_{\mathcal{E}}^{(1)} + \mathcal{O}(\partial^2) \quad (2.8)$$

$$0 = \delta \mathcal{P} = \dots$$

$$0 = \delta \mathcal{N} = \dots$$

We can therefore choose δT and $\delta \mu$ such that

$$\delta f_{\mathcal{E}}^{(1)} = -f_{\mathcal{E}}^{(1)} \quad (2.9)$$

$$\delta f_{\mathcal{P}}^{(1)} = -f_{\mathcal{P}}^{(1)} \quad (2.10)$$

which means that

$$\mathcal{E} = \epsilon + \mathcal{O}(\partial^2) \quad (2.11)$$

$$\mathcal{N} = n + \mathcal{O}(\partial^2)$$

2.3 First Order Gradient Expansion

In the Landau frame we have the following first order expansion:

$$T_{\mu\nu}^{(1)} = (\mathcal{P} - p) \Delta_{\mu\nu} + t_{\mu\nu}^{(1)} \quad (2.12)$$

$$J_\mu^{(1)} = j_\mu^{(1)} \quad (2.13)$$

We need to find the constituent relations for the scalar $\mathcal{P} - p$, the transverse, traceless symmetric tensor $t^{\mu\nu}$ and the transverse vector j^μ . For our purposes it will be most convenient to switch to the local thermodynamic variables (u^μ, ϵ, n) .

From these variables we can build the following first order quantities with well-defined Lorentz tensor structure:

1. Three scalars: $u^\lambda \partial_\lambda \epsilon$, $u^\lambda \partial_\lambda n$, $\partial^\lambda u_\lambda$
2. One traceless, transverse symmetric tensor: $\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} (2\partial_{(\alpha} u_{\beta)} - (2/d)\eta_{\alpha\beta} \partial^\lambda u_\lambda)$
3. Three transverse vectors: $\Delta^{\mu\nu} \partial_\nu \epsilon$, $\Delta^{\mu\nu} \partial_\nu \mu$, $\Delta^{\mu\nu} u^\lambda \partial_\lambda u_\nu$

However, we can make use of the zeroth order conservation equations which are first order in derivatives and can be decomposed in the following tensor structure:

1. Two scalar equations: $u_\nu \partial_\mu T^{(0)\mu\nu} = 0$ and $\partial_\mu J^{(0)\mu} = 0$
2. One transverse vector equation: $\Delta_{\sigma\nu} \partial_\mu T^{(0)\mu\nu} = 0$

We can use them to express for instance all first order scalars in terms of $\partial^\lambda u_\lambda$ and all first order transverse vectors in terms of $\Delta^{\mu\nu} \partial_\nu \mu$ and $\Delta^{\mu\nu} u^\lambda \partial_\lambda u_\nu$. We hence end up with the following constituent relations:

$$T^{(1)\mu\nu} = -\zeta \Delta^{\mu\nu} \partial_\lambda u^\lambda - \eta \sigma^{\mu\nu} \quad (2.14)$$

$$J^{(1)\mu\nu} = -D \Delta^{\mu\nu} \partial_\nu n + \alpha \Delta^{\mu\nu} u^\lambda \partial_\lambda u_\nu$$

where ζ is the shear viscosity, η is the bulk viscosity, D is the diffusion constant and α is another parameter.

3 Expansion in Fluctuations around Global Equilibrium

We have expanded the conserved current densities to first order $\mathcal{O}(\partial)$ in gradients of local charge densities (u^μ, n, ϵ) . By taking first order gradient effects into account we take care of small dissipative effects. We would now like to perform the second hydrodynamic expansion, namely in fluctuations δ of the charge densities around their values in global thermal equilibrium. In order to be sensitive to charge current fluctuations which require both a fluctuation in the velocity (net movement of particles) and a fluctuation in the charge density of roughly the same wavelength (net charge of moving particles), we need to expand up to second order in fluctuations $\mathcal{O}(\delta^2)$. We shall, however, ignore effects of order $\mathcal{O}(\delta^2\partial)$ which are more strongly suppressed.

Let us go into the frame in which the fluid will be at rest in the limit of global equilibrium ($t \rightarrow \infty$) and let us further restrict ourselves to a charge conjugation invariant system with zero chemical potential in the equilibrium limit. We can then write

$$\begin{aligned}\epsilon(x) &= \bar{\epsilon} + \delta\epsilon(x) + \mathcal{O}(\delta^2) \\ u^\mu &= (1, \vec{v}) + \mathcal{O}(\delta^2)\end{aligned}\tag{3.1}$$

where $\bar{\epsilon}$ denotes the constant value of the energy density in global equilibrium and use the first order fluctuations $(\delta\epsilon(x), n(x), v^i(x))$ as local variables. Using the equation of state we can expand the local equilibrium pressure around its global equilibrium value as

$$\begin{aligned}p(x) &= \bar{p} + \frac{\partial \bar{p}}{\partial \bar{\epsilon}} \delta\epsilon(x) + \frac{\partial \bar{p}}{\partial \bar{n}} n(x) \\ &= \bar{p} + v_s^2 \delta\epsilon(x) .\end{aligned}\tag{3.2}$$

Note that the invariance under charge conjugation requires the absence of any linear terms in n from our constituent relations.

Observe the following relation between the variables $(\delta\epsilon(x), n(x), v^i(x))$ and the conserved charge densities in our frame using (2.1), (2.11) and (3.1):

$$\begin{aligned}T^{00}(x) &= \bar{\epsilon} + \delta\epsilon(x) + \mathcal{O}(\delta^2, \partial^2) \\ \pi^i(x) &= T^{0i}(x) = (\bar{\epsilon} + \bar{p}) v^i(x) + \mathcal{O}(\delta^2, \partial^2) \\ J^0(x) &= n(x) + \mathcal{O}(\delta^2, \partial^2)\end{aligned}\tag{3.3}$$

Using this together with (2.5), (2.14) and (3.2) we finally end up with the following constituent relation for the current densities T^{ij} and J^i in terms of the charge densities T^{00} , π^i and J^0 :

$$\begin{aligned}T_{ij} &= \delta_{ij} \left(\bar{p} + v_s^2 \delta\epsilon - \frac{\zeta}{\bar{\epsilon} + \bar{p}} \partial_k \pi^k \right) + \frac{1}{\bar{\epsilon} + \bar{p}} \pi_i \pi_j \\ &\quad - \frac{\eta}{\bar{\epsilon} + \bar{p}} \left(2\partial_{(i} \pi_{j)} - \frac{2}{d} \delta_{ij} \partial_k \pi^k \right) + \mathcal{O}(\partial^2, \delta^3, \delta^2 \partial) \\ J_i &= J^0 v_i - D \partial_i J^0\end{aligned}\tag{3.4}$$

References

- [1] P. Kovtun, J. Phys. A **45** (2012) 473001 [arXiv:1205.5040 [hep-th]].
- [2] P. B. Arnold and L. G. Yaffe, Phys. Rev. D **57** (1998) 1178 [hep-ph/9709449].
- [3] P. Kovtun and L. G. Yaffe, Phys. Rev. D **68** (2003) 025007 [hep-th/0303010].
- [4] L. P. Kadanoff and P. C. Martin, Annals of Physics **24** (1963) 419-469 [DOI: [http://10.1016/0003-4916\(63\)90078-2](http://10.1016/0003-4916(63)90078-2)].
- [5] R. Balescu, *Equilibrium and Nonequilibrium Statistical Physics*, John Wiley and Sons, New York (1975).