Cargèse, 6 June 2014

Sean Hartnoll

DISORDER

① Disorder in QFT

② Holography

Consider a CFT in $(d+1)$ dim^{n}s.

(e.g. arising at QCP of some spin system)

$(T=0)$

QCP

$\langle\phi\rangle \neq 0 \quad g_c \quad \langle\phi\rangle = 0 \quad g$

↳ quantum fluctuations of ϕ

CFT: local operators $\mathcal{O}(x)$

$$\mathcal{O}(e^{\ell}x) = e^{-\Delta\ell}\,\mathcal{O}(x) \quad , \quad \Delta = \text{scaling dim}^n$$

Need to check CFT stable against perturbations:

$$L \to L + \lambda\,\mathcal{O}(x)$$

Example

$$L = (\partial\phi)^2 + \underbrace{m^2\phi^2}_{\mathcal{O}_1} + \underbrace{\lambda\phi^4}_{\mathcal{O}_2} + \dots$$

$\mathcal{O}$ (tuned in experiment at QCP)

Consider e.g. $L \to L + \lambda_0\phi^{20}$

$$\Rightarrow S = S_0 + \lambda_0 \int_{\epsilon} d^{d+1}x\ \mathcal{O}(x)$$

integrate out short distance modes:

$$\int D\phi \; e^{-S_0 + \left\{ \lambda_0 \int_{\epsilon e^{\ell}} dx\, \mathcal{O}(x) \; + \; \underbrace{\lambda_0 \int_{\epsilon}^{\epsilon e^{\ell}} d^{d+1}x\, \mathcal{O}}_{\text{no contribution to } 1^{st} \text{ order in } \lambda} \right\}}$$

rescale $x = e^{\ell} x'$ $(\ell > 0)$

$$\Rightarrow \lambda_0 \int_{\epsilon e^{\ell}} d^{d+1}x\, \mathcal{O}(x) = \lambda_0 e^{(d+1)\ell} \int_{\epsilon} d^{d+1}x' \underbrace{\mathcal{O}(e^{\ell}x')}_{e^{-\Delta \ell}\mathcal{O}(x')}$$

$$= \underbrace{\lambda_0 e^{[(d+1)-\Delta]\ell}}_{\lambda} \int_{\epsilon} d^{d+1}x'\, \mathcal{O}(x')$$

$$\Rightarrow \frac{d\lambda}{d\ell} = (d+1-\Delta)\lambda$$

if $\Delta < d+1$ $\to$ coupling grows as $\ell \to \infty$ (long wave lengths)

$\hookrightarrow$ relevant

In a cond. matter system $\to$ $\underbrace{\text{quenched}}_{\text{impurities not dynamical}}$ disorder

in CFT description: $S = \underset{CFT}{S_0} + \int d^dx\, dt\, V(x)\, \mathcal{O}(x,t)$

$V(x)$: space-dependent coupling

take most relevant oper. impurities couple to

e.g. charged disorder $V(x)\, J^t(x,t)$

Remark: quenched $\to$ $V(x,t) = V(x)$

Note: $V(x)$ irrelevant $\to$ mom. conserved in IR (translation inv. restored)

$V(x)$ relevant $\to$ mom. conservation broken!

Typically:

$V(x)$ very complicated

In fact: details should not matter!

$\rightarrow$ assume that we can draw $V(x)$ from some random distribution $P[V(x)]$

Note: • if $V(x)$ were e.g. $\delta(x)$ [point-source] or $\cos(\frac{x}{a})$ [periodic], it wouldn't be disorder

• it will important to distinguish between averaging over $V(x)$ (randomized potential) and $\phi(x)$ (path integral)

$$\langle Q \rangle_{dis} = \int DV \, P[V(x)] \left\{ \frac{\int D\phi \, e^{-S_d} Q}{Z_d} \right\}$$

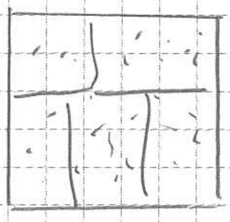
don't know how to calculate!

order crucial: calculate $\langle Q \rangle_{quantum}$ for each V and average them

note: Z_d depends on $V(x)$, too

$\hookrightarrow$ averages are taken with different weight / normalisation

Idea:



split vol with a given disorder structure into macroscopic blocks averaged over which $V(x)$ is randomized

Replica Trick

$$P[V(x)] = e^{-\int \frac{dx \, V(x)^2}{2\sigma^2}}$$

$\rightarrow$ Gaussian disorder: $\langle V(x) \rangle = 0$

$$\langle V(x) V(y) \rangle = \sigma^2 \delta(x-y)$$

- Consider n non-interacting copies of system
- put operator Q on one copy

$$F_n = \int D\phi_1 \dots D\phi_n \, e^{-\sum_n S_n} \, Q_1$$

For the time being: ignore $\frac{1}{Z_d}$ factors!

$$= Z^{n-1} \int D\phi_1 \, Q_1 \, e^{-S_1}$$

Find F_n → analytically continue $n \to 0$

$$\to \; F_0 = \frac{\int D\phi \, Q \, e^{-S}}{Z}$$ (a little formal...!)

Idea: now average F_n over potential! (which comes without $\frac{1}{Z(V)}$) → the disorder will couple the n-copies

$$(\#) := \int DV \, \underbrace{P[V]}_{e^{-\int \frac{V(x)^2}{2\bar{V}^2}}} \int D\phi_1 \dots D\phi_n \, e^{-\sum_n S_n^{(0)} - \left\{ \int d^{d+1}x \, V(x) \, \mathcal{O}_1(x,t) + \dots + \int d^{d+1}x \, V(x) \mathcal{O}_n(x,t) \right\}}$$

Schematically:
$$\int dx \, e^{-x^2} \, e^{-xy_1 - xy_2} \sim e^{y_1^2 + y_2^2 + y_1 y_2}$$

coupling!

Do Integral over $V(x)$:

$$(\#) = \int D\phi_1 \dots D\phi_n \, e^{\underbrace{-\sum_n S_n^{(0)} - \frac{\bar{V}^2}{2} \int d^d x \int d\tau \, d\tau' \sum_{n,m} \mathcal{O}_n(x,\tau) \mathcal{O}_m(x,\tau')}_{=:(*)}}$$

$n \to 0$ limit of this the physical quantity

Scaling analysis:

$$\left.\begin{aligned} x &= e^{l} x' \\ t &= e^{l} t' \end{aligned}\right\} \quad (*) \sim \bar{V}^2 e^{(d+2)l - 2\Delta l}$$

Irrelevant if $d+2-2\Delta < 0 \Rightarrow \boxed{\frac{d+2}{2} < \Delta}$

Harris criterion (1970's)

Compare with homogeneous coupling: $d+1 < \Delta$

Note however: if e.g. homogeneous coupling $m^2 \to 0$ in exp., then inhomogeneous coupling may still be there!

e.g. Wilson-Fisher: $\mathcal{L} \sim (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4$

Is a disordered mass term $V(x)\phi^2$ relevant at QCP?

$d \geq 3$, ϕ^4 is irrelevant $\Rightarrow [\phi^2] = d-1 = \Delta$

$\Rightarrow$ Harris criterion satisfied for $d>4$

$\Rightarrow$ $d=3$ relevant

$\Rightarrow$ $d=2$ $[\phi^2] = 1.418\ldots$

↳ relevant!

$\Rightarrow$ Wilson-Fisher not a FP in presence of disorder

Sean Hartnoll - Lecture II

Cargèse, 7 June 2014

Yesterday: disorder $V(x)\ \mathcal{O}(x,t)$

relevant if $\frac{d+2}{2} > \Delta$ (violates Harris criterion)

Wilson - Fisher : $(2+1)d$ CFT

$$\int d^dx\, dt \left\{ \frac{1}{2}(\nabla\phi)^2 + \underbrace{m}_{\to 0}\ \phi^2 + \frac{\lambda}{4!}\phi^4 \right\}$$

relevant in $2+1$

$$\frac{d\lambda}{d\ell} = \underbrace{(3-d)}_{\text{set } d=3-\varepsilon}\lambda - \underbrace{\#}_{>0,\ O(1)\text{ number}}\lambda^2$$

controlled running

Feynman diagrammes

$\hookrightarrow$ fixed point $\frac{d\lambda}{d\ell} = 0$: $\lambda = \#\varepsilon$

$\hookrightarrow$ self-consistent sol^n for IR fixed point as $\lambda \sim O(\varepsilon)$

$$\int d^dx\, dt \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{\lambda}{4!}\phi^4 + V(x)\phi^2 \right\}$$

$$P[V] = e^{-\int \frac{V(x)^2}{2\bar{V}^2} d^dx}$$

Replicated Theory : $\bar{V}^2 \int dx\, dt\, dt' \sum_{a,b} \mathcal{O}_a \mathcal{O}_b$

$$\frac{d\bar{V}^2}{d\lambda} = (4-d)\bar{V}^2 + \#\bar{V}^4$$

$\downarrow$

"wrong" sign

$\hookrightarrow$ no perturbative fixed point

[chapter on disorder in Sachdev: "Quantum Phase Transitions"]

! It is not known what happens to the WF fixed point in 2+1 dim^n in the presence of disorder !

$\to$ IR description of disordered WF not known

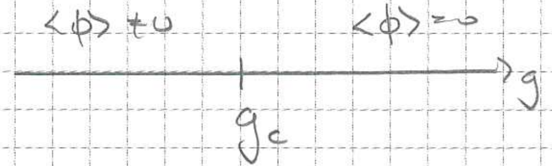
Griffiths - McCoy singularities

$$H = \sum_{\langle ij \rangle} S_i^z S_j^z J_{ij} + \sum_i g\, \sigma_i^x + \dots$$

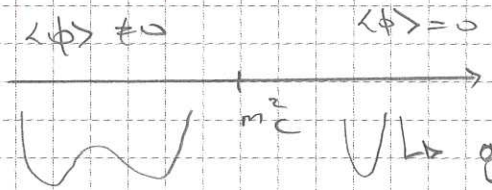
large J $\uparrow\uparrow\uparrow$ or $\downarrow\downarrow\downarrow$ (ferromagnetic)

Large g mixes $\uparrow$ and $\downarrow$ → tends to disorder the spins

Tune g to critical point

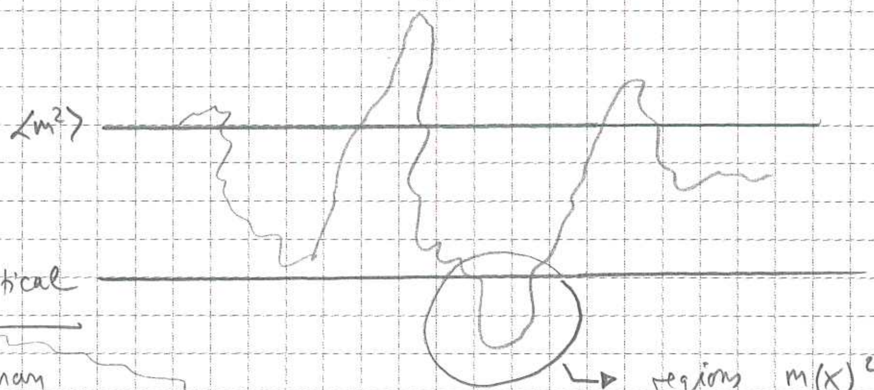


$$L = (\nabla\phi)^2 + m^2\phi^2 + \dots$$



IR: ↳ gapped: mass m^2

⤳ disorder



UV mass (IR mass = 0 at critical point)

↳ regions $m(x)^2 < m_c^2$

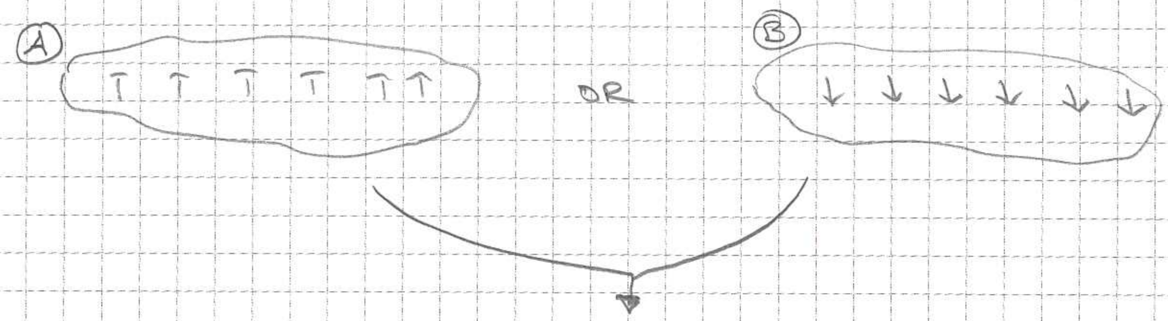
↳ $\exists$ rare but arbitrarily large regions with $m^2(x) < m^2_{crit}$

↳ locally order

- Spins independently disordered

$\hookrightarrow P_0 := P(m^2(x) < m_c^2) = \int_{-\infty}^{m_c^2} P(m_x^2)\, dm_x^2$

- region of size L : $(p_0^{L^d}) \Rightarrow P(L) \sim e^{-cL^d}$



what is ΔE ?

Will see : in the globally gapped phase, the "Goldstone" like flip of spins in a local region of $m^2(x) < 0$ will dominate the excitations!

$\hookrightarrow$ at $L = \infty$, exactly degenerate by $\mathbb{Z}_2$ symmetry $\phi \to -\phi$.

$\hookrightarrow$ at finite L : • spin picture : $g\sigma_x^i$ flips a spin

• $\sum_{\langle ij \rangle} J_{ij}\, \sigma_i^z \sigma_j^z + g\sigma_x^i$

to see mixing between (A) and (B), need to go to order g^{L^d}

to get overlap between states (have to flip L^d spins)

splitting $\Delta E \sim g^{L^d} \sim e^{-\bar{c}L}$

- contribute to susceptibilities : spectral representation

$$\mathrm{Im}\, \chi(\omega) = \sum_\alpha |\langle \alpha | \hat{O} | 0 \rangle|^2\, \delta(\omega - (E_\alpha - E_0))$$

"amount of generated heat" = $\sum_{\text{energy states } \alpha}$ (overlap with electr field)^2 · (frequency $\doteq E_\alpha - E_0$)

$\omega^2 < \langle m^2 \rangle$:

$$\mathrm{Im}\,\chi(\omega) \sim \int dL \; P(L) \; \delta(\omega - \Delta E_L)$$

$$\sim \int dL \; e^{-cL^d} \; \delta\left(\omega - e^{-\bar{c}L^d}\right)$$

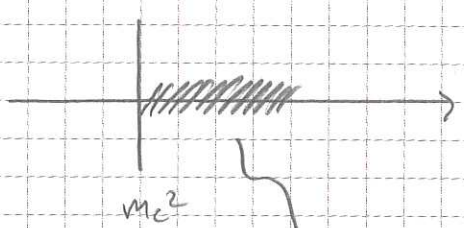
$$\sim \omega^{\frac{c}{\bar{c}} - 1} \qquad \text{(up to logs)}$$

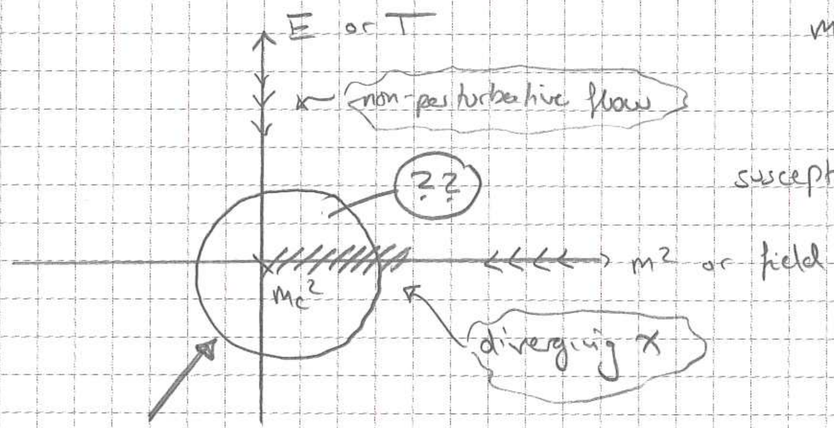
$y := e^{-\bar{c}L^d}$, $e^{-cL^d} = y^{c/\bar{c}}$

$\frac{dy}{dL} = -\bar{c}\, d\, L^{d-1}\, y$

- power law spectral weight $\rightarrow$ system gapless
- $c, \bar{c}$ non-universal numbers:

$\frac{c}{\bar{c}} < 1 \Rightarrow \lim_{\omega \to 0} \chi(\omega) = \infty$





in 1+1 solvable

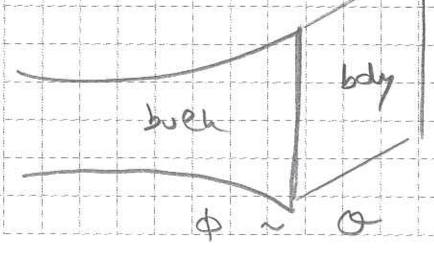
(Dan Fisher)

$\hookrightarrow$ infinite disorder fixed point

Disorder in Holography

(1402.0872 with Santos)

- large N $\rightarrow$ classical gravity

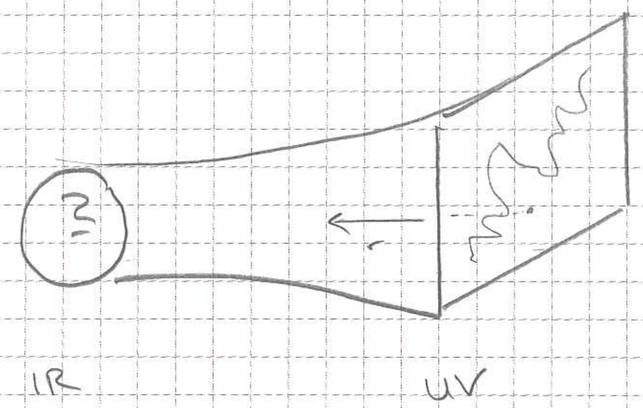


$$Z[\phi \to \phi_0] = \langle e^{\int O \phi_0 dx} \rangle_{QFT}$$

adding source $\phi_0(x)$ for O in QFT

↳ solve bulk eom subject to bdy cdt that $\phi \to \phi_0(x)$

Need to solve bulk eq^{ns} with disordered bdy cdt $\phi(r, x)$



irrelevant case: disorder will get smaller as you go in

Note: $\phi(r,x) \to$ classical system, but PDE

Bulk

$$L = R + \frac{2}{L^2} - 2(\nabla\phi)^2 + V(\phi)$$

(1+1 CFT, $d=1$)

How to make a random function:

$$\phi_0(x) = \bar{V} \sum_{n=1}^{N-1} A_n \cos(k_n x + \gamma_n)$$

$k_n = n\Delta k$, $\Delta k = \frac{k_0}{N}$

short distance cut-off $\frac{2\pi}{k_0}$,

long distance cut-off $\frac{2\pi N}{k_0}$

$A_n = 2\sqrt{\Delta k}$

γ_n: draw randomly from a unitary distribution $(0, 2\pi)$

Theorem

define disorder average $\langle f \rangle_R = \lim_{N\to\infty} \int_0^{2\pi} \prod \frac{d\gamma_i}{2\pi} f$

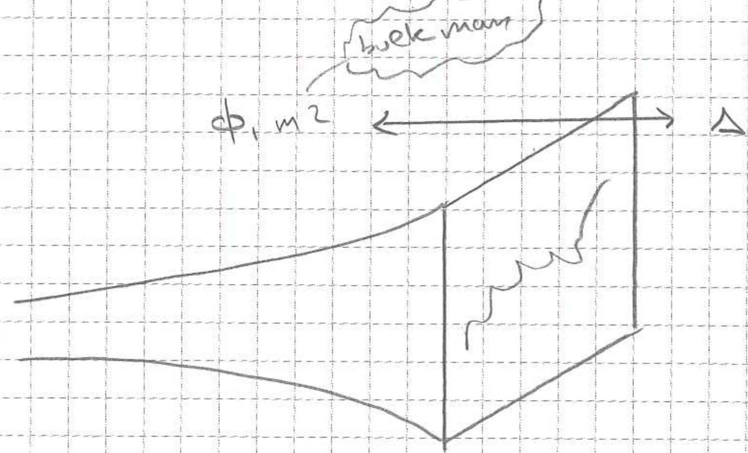
then $\langle \phi_0(x) \rangle = 0$

$$\langle \phi_0(x)\, \phi_0(y) \rangle_R = \bar{V}^{-2} \delta(x-y)$$

and higher point functions factorise

i.e. $\int \mathcal{D}V\, e^{-\int \frac{V(x)^2}{\bar{V}^2}} \longleftrightarrow \int \prod_i d\gamma_i$

Note: making amplitudes A_n n-dependent
↳ will change $\delta(x-y)$ to other short range correlation (e.g. power law decay)



choose m^2 s.t. Δ saturates Harris criterion → marginal

⟶ marginally relevant

expand near boundary → logarithmic growth
{ (ϕ_0 1st order, backreaction on metric 2nd order, backreaction on ϕ (→ marginally relevant) 3rd order)

Analytically:

$$ds^2(r,x) = -A(x,r)\frac{dt^2}{r^2} + \underbrace{\frac{dr^2}{r^2}}_{\text{gauge-fix}} + B(x,r)\frac{dx^2}{r^2}$$

$$\langle ds^2 \rangle = \int ds^2 \, d\gamma \qquad \text{average metric}$$

$$= \frac{1}{r^2}\left\{ -(1 + \#\log r)\,dt^2 + dr^2 + \#\,dx^2 \right\} + \#\log^2 r$$

$$\rightsquigarrow \quad r^{1+\varepsilon} = r\,e^{\varepsilon \log r} = r(1 + \log r + \dots)$$

Resum logs: Lifshitz metric

analytically not controlled, but can check with numerics

$$\langle ds^2 \rangle = -\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx^2}{r^2}$$

↳ scale invariant!

UV   CFT

IR   disordered "horizons"

↳ on average scale invariant!

→ disordered fixed point

Remark: • $z \neq 1$: disorder has $O(1)$ effect in IR!

• z is given as power series in $\bar{V}$

• sol^n has pp-singularity