Alejandra Castro   10 June 2014

HOLOGRAPHIC ENTANGLEMENT ENTROPY IN ADS_3/CFT_2

I) AdS_3 gravity

II) Wilson lines in EE

III) Gravitational anomalies, the cone, the anyon & the ribbon

1306.4338, 1405.2792 (see also 1306.4347)

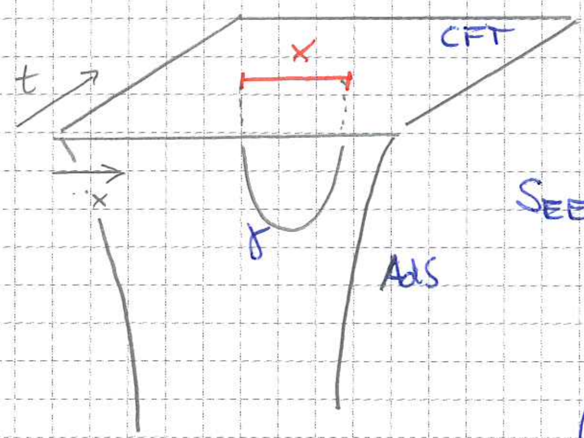
General relativity is wise:

i) it knows about thermodynamics; laws thermo = geom. theorems

$$S_{BH} = \frac{A_H}{4G} \qquad (\to S_{Wald} : \text{general for any covariant theory of gravity})$$

ii) it knows about quantum information

AdS/CFT   Ryu - Takayanagi



ρ_X : reduced density matrix

$$S_{EE} = -\mathrm{Tr}(\rho_X \log \rho_X)$$

$$= \frac{A_{min}(\gamma)}{4G}$$

$$\left(\underset{AdS_3}{=} \frac{L_\gamma}{4G} \right)$$

L_γ : geodesic distance

$\exists$ appropriate "S_{Wald}"

How general (universal) are these two properties for any theory of gravity?

- if you add higher derivative terms?
- if gravitational theory is non-local?
  - ↳ higher spin theories : graviton $s=2$
  - ↳ 3d gravity

$\boxed{\text{I} \quad AdS_3 \text{ Gravity}}$

Einstein-Hilbert + $\Lambda < 0$ (generalisations / modifications for $\Lambda = 0$, $\Lambda > 0$)

$$S_{EH} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \, (R - \Lambda) \, , \quad \Lambda = -2/\ell^2 \, , \ \ell = \text{radius of AdS}$$

Facts:

i) no local d.o.f. in $(2+1)d$ GR

indep. components of metric = # eom

↳ every sol^n is locally AdS_3

↳ no propagating graviton

ii) First ~~example~~ hint of AdS/CFT

Brown-Henneaux ('86)

↳ perturbative states : diffeos preserving AdS bdy

$x \to x + \xi$

$g \to g + \mathcal{L}_\xi g$

set of ξ's that leave bdy of AdS fixed and $Q[\xi] \neq 0$ (finite at ∞)

$$\{ Q[\xi], Q[\xi'] \} = Q[\xi''] + K \qquad c = \frac{3\ell}{2G}$$

$$\underbrace{\qquad\qquad}_{2\times \text{ Virasoro algebra}}$$

→ perturbative states of AdS_3 organise themselves in representation of Vir × Vir with $c = 3\ell/2G$

iii) $\underbrace{\text{Non-perturbative}}$ black hole (BTZ)

diffeos that are not globally defined

quotients AdS_3 / Γ

isommetry group: $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$L_0 + \bar{L}_0$: identify along this direction

iv) alternative representation of the theory in terms of a gauge theory

(Achucarro, Townsend ; Witten '89)

$(2+1)d$: $\exists$ top. gauge theory : Chern-Simons theory

$$S_{EH} = S_{CS}[A] - S_{CS}[\bar{A}]$$

$$S_{CS} = \frac{k}{4\pi} \int_M tr \left(A dA + \frac{2}{3} A \wedge A \wedge A \right)$$

gauge group

$A = \omega + e/\ell \quad \in \quad sl(2,\mathbb{R})$: $T^a \quad (L_0, L_1, L_{-1})$

Remark: $sl(2,\mathbb{R})$ non-compact ↔ AdS
⇕
$su(2)$ compact ↔ dS

$\bar{A} = \omega - e/\ell \quad \in \quad sl(2,\mathbb{R})$

$A = T_a A^a_\mu dx^\mu$, $\omega = \omega^a T_a$, $\omega^a = \omega^{bc}_\mu \epsilon_{bc}{}^a dx^\mu$

$e^a = e^a_\mu dx^\mu$

$k = \frac{\ell}{4G}$ Chern-Simons level

$$g_{\mu\nu} = \# \, \mathrm{Tr}\left((A-\bar{A})_\mu (A-\bar{A})_\nu \right)$$

eom:

$$\underbrace{\begin{aligned} F &= dA + A^2 = 0 \\ \bar{F} &= d\bar{A} + \bar{A}^2 = 0 \end{aligned}}_{\text{vanishing field strength} \; \hookrightarrow \; \text{topological theory}} \quad \left. \begin{aligned} &\text{Cartan eq}^n \quad (\omega(e)) \\ &\text{Einstein eq}^n \quad (d\omega + \omega^2 + e^2 = 0) \end{aligned} \right.$$

Warning: why CS might not equal EH at quantum level.

- $A = 0 = \bar{A}$
- $Z_{CS}[M]$ does not include a sum over topologies (e.g. one would exclude BH's)

BTZ in CS Theory

$$M\ell = \frac{r_+^2 + r_-^2}{8G\ell}, \quad J = \frac{r_+ r_-}{4G\ell}, \quad r_\pm : \begin{matrix} \text{outer} \\ \text{inner} \end{matrix} \text{ horizon}$$

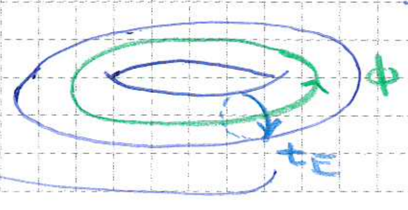
$$ds^2_{BTZ} = \frac{\ell^2}{2} \mathrm{Tr}\left(g^{-1} dg \, g^{-1} dg\right), \quad g \in SL(2,\mathbb{C}) \quad (\text{or } SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$$

$$\text{quotient} \quad g \sim \gamma g \gamma^+, \quad \gamma = \begin{pmatrix} e^{i\pi\tau} & 0 \\ 0 & e^{i\pi\bar{\tau}} \end{pmatrix}, \quad \tau = r_- + i r_+$$

Euclidean BTZ BH:

solid torus

contractable, shrinks to zero at $r = a$ (horizon)



ϕ ~ non-contractable cycle, size = horizon area

$$A = b^{-1} a b + b^{-1} db$$

$$\bar{A} = b \bar{a} b^{-1} + b db^{-1} \quad , \quad b = e^{\phi(\rho)} \to \text{radial direction}$$

$$a = (L_1 - \mathcal{L} L_{-1}) dx^{+} \quad , \quad x^{\pm} = t \pm \phi$$

$$\bar{a} = -(L_{-1} - \bar{\mathcal{L}} L_1) dx^{-}$$

$$\frac{c}{6} \mathcal{L} = \ell M + J$$

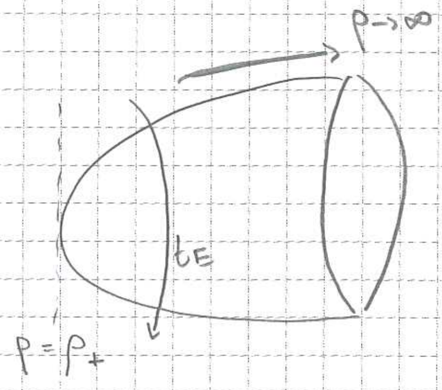
$$\frac{c}{6} \bar{\mathcal{L}} = \ell M - J$$

Topology : solid torus

↳ smooth solutions $\quad \mathrm{Hol}_{\gamma_{t_E}} \begin{pmatrix} A \\ \bar{A} \end{pmatrix} = \pm 1\!\!1$

constraint $t_E \sim t_E + \beta$

$\beta \sim M \Rightarrow$ Hawking temperature!



$$\mathcal{L} = \frac{1}{4\beta^2} \qquad (J = 0 , \mathcal{L} = \bar{\mathcal{L}})$$

Entanglement Entropy

$$S_{EE} = \frac{L_\gamma}{4G} = \lim_{n \to 1} \frac{1}{1-n} m \int_\gamma ds \quad , \quad m \to \frac{1}{1-n} \frac{c}{6} = \frac{1}{4G(1-n)}$$

$$= -\mathrm{Tr}(\rho_x \log \rho_x)$$

$$= \lim_{n \to 1} \underbrace{\frac{1}{1-n} \underbrace{\mathrm{Tr}\, \rho_x^n}_{\text{partition fnt replicated manifold}}}_{\text{Renyi entropy}}$$

$$= \log \mathrm{Tr}\, \rho_x^n$$

Alejandra Castro - lecture II 13 June 2014

last 2 things I told you not to forget for:

- non-linear / non-local : higher spin theories

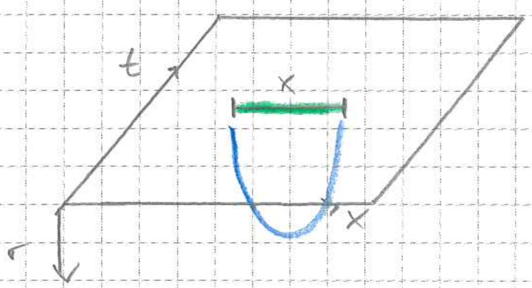
toymodel : (2+1)d gravity

- higher derivatives modification & new data

gravitational anomalies : (2+1)d gravity TMG

(top massive gravity)

Revisit EE in AdS_3 / CFT_2 :



$$S_{EE} = -\mathrm{Tr}\,(\rho_x \log \rho_x)$$

$$= \frac{L_\gamma}{4G_3} \quad \text{Ryu-Takayanagi}$$

Einstein gravity $\longleftrightarrow$ Chern-Simons

$(g_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$ $\qquad SO(2,2) = SL(2) \times SL(2)$

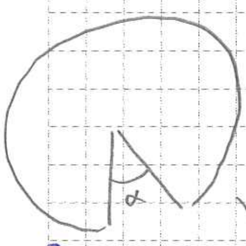
$(A, \bar{A})$

- Brown-Henneaux : perturbative states
- BHs thermodyn.

(?) How to formulate Ryu-Takayanagi in terms of CS ?

$$S_{EE} = \lim_{n\to 1} \underbrace{S_n}_{\text{Renyi entropy}}$$

$$= \lim_{n\to 1} \frac{1}{1-n} \underbrace{\log \mathrm{tr}\, \rho_x^n}_{\log \frac{Z_n}{(Z_1)^n}} \sim \log Z_{grav} \underset{n\to 1}{\sim} S_{grav} + O(n-1)^2 \qquad Z_{grav} \sim e^{S_{grav}}$$



$R \sim \alpha\,\delta(..)$

δ-curvature singul.

$\sim m\,\delta()$

massive particle ≙ δ source

$$= \lim_{n\to 1} \frac{1}{1-n} S_{cone} \longrightarrow \text{massive particle}$$

$$\overset{GR}{=} \boxed{\lim_{n\to 1} \frac{1}{1-n}\, m \underbrace{\int ds}_{x}}$$

$$m = (1-n)\frac{c}{\ell 6} = (1-n)\frac{1}{4G} \quad , \quad c = \frac{3\ell}{2G_3}$$

$$S_{EE} = \frac{1}{4G_3} L_\gamma$$

only in the limit $n \to 1$ this can be treated as a probe, otherwise $\frac{1}{G}$ is very happy and will backreact

Goal: design massive particle in Chern-Simons formulation

Wilson line:

$$W_R(C) = \mathrm{Tr}_R \left(P \exp \int_\gamma A \right)$$

R: rep of gauge group G

C: curve on M

If C is closed (loop) $\rightarrow$ $W_R(C)$ gauge invariant, topologically invariant

C is open $\rightarrow$ $W_R(C)$ only depends on two end points

Natural place to locate m is in R

$R \rightarrow$ Hilbert space $\mathcal{H}$

$$\mathrm{Tr}_R \rightarrow \mathrm{Tr}_{\mathcal{H}} \sim \underbrace{\int \mathcal{D}U \, e^{-S(U)}}_{\text{path integral}}$$

U: probe for aux. system

↳ design system such that upon quantisation $\mathcal{H} = R$ is the Hilbert space of the auxiliary system

$S_{free}(U)$: invariant under G

$$\underbrace{\mathrm{Tr}_R \left(P \exp \int A \right)}_{\substack{\text{evolution op.} \\ \text{along curve}}} = \int \mathcal{D}U \exp\Big(- S_{free}(U) + \underbrace{S_{int}(U, A)}_{\text{interaction}} \Big)$$

↓

G a local symm.

which R makes contact with massive particle ?

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ fit m, m, R

L_0, L_1, L_{-1} $\quad \bar{L}_0, \bar{L}_1, \bar{L}_{-1}$

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$\rightsquigarrow$ look at infinite dim rep $\Rightarrow$ highest weight rep

highest weight state $|h\rangle$

$$L_0|h\rangle = h|h\rangle$$

$$L_1|h\rangle = 0$$

create states $(L_{-1})^n|h\rangle \Rightarrow h+n$

h large

Casimir $C_2 = 2L_0^2 - (L_{-1}L_1 + L_1L_{-1}) \longrightarrow C_2 = 2h(h-1) \simeq \frac{m^2}{2}$

$h \leftrightarrow m$

$\bar{C}_2 = C_2 \Rightarrow m \neq 0,\ s = 0$

(Ref. $\Lambda = 0$: Witten '89, Carlip '89)

$$S(U, P, A) = S_{free} + S_{int} = \int_\gamma ds \left\{ \mathrm{Tr}(PU^{-1}DU) + \lambda(s)\left(\mathrm{Tr}\,P^2 - C_2\right) \right\}$$

P: conj mom.

$\lambda(s)$: Lagrange multiplier

C_2: Casimir (constant)

$U \to L(s)\, U\, R(s)$, $L, R \in SL(2)$

$P \to R^{+} P R$ (right)

$P_L := U P U^{-1}$ (left)

$\hookrightarrow$ implies $C_2 = \bar{C}_2$

$$DU = \frac{dU}{ds} + AU - U\bar{A}$$

ext. derivative along the curve

gauge trafo

$$A \to L(A + d)L^{-1}$$

$$\bar{A} \to R^{-1}(\bar{A} + d)R$$

$$A = A_3 = A_\mu \frac{dx^\mu}{ds}$$

Geodesic eq^n :

integrate out p and then $x(s)$:

$$S(U, A, \bar{A}) = \sqrt{C_2} \int ds \sqrt{\mathrm{Tr}\,(U^{-1} DU)^2}$$

→ action indep. of x^μ, depends only on topology

Eom

$$\frac{d}{ds}\left((A^u - \bar{A})_\mu \frac{dx^\mu}{ds} \right) + [\bar{A}_\mu, A_\nu^u] \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (*)$$

$$A^u := U^{-1}\frac{dU}{ds} + U^{-1}AU$$

Set $U = \mathbb{1}$ ⇝ eom for $x^\mu(s)$

$$A = \omega + e/\ell$$
$$\bar{A} = \omega - \frac{e}{\ell}\,,$$

$(*)$ = geodesic eq^n

$$S(U = \mathbb{1}, A, \bar{A}) = \sqrt{C_2} \int ds \sqrt{\mathrm{Tr}(A - \bar{A})^2}$$
$$= \sqrt{2C_2} \int ds \sqrt{g_{\mu\nu} \dot{x}^\mu(s) \dot{x}^\nu(s)}$$

→ minimal area!

$$\leadsto \quad W_R(C) \sim \exp\left(-\overbrace{\sqrt{2C_2}}^{\text{massive probe}}\, L_\gamma\right)$$

for $G = SO(2,2)$ valid

saddle point approx

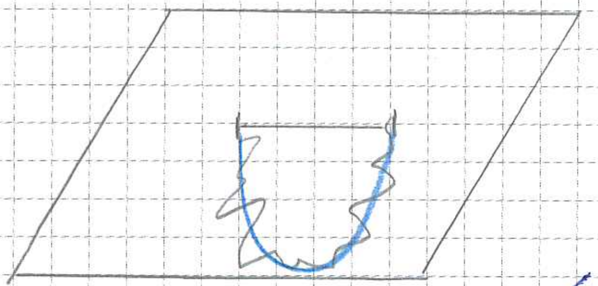
⇒ Proposal :

$$S_{EE} = -\log W_2(C) \qquad \text{with} \quad \sqrt{2C_2} = \frac{c}{6}$$

↳ formulation potential to go beyond classical limit

formulation in terms of Wilson loops

↳ emphasises non-local nature of entanglement



$W_R(C)$ captures hol EE in a non-local way

↳ set it up for higher spin theory

bdy : $U(s_i) = \mathbb{1} = U(s_f)$

↳ respect Lorentz invariance $L = R^{-1}$

(thus: answer the same for all spacelike curves with same endpoints)

Gravitational Anomalies

CFT : $c_L, c_R; c_L + c_R$

$$T(z)\,T(0) \sim \frac{c_L/2}{z^4} + \dots$$

$$\bar{T}(\bar{z})\,\bar{T}(0) \sim \frac{c_R/2}{\bar{z}^4} + \dots$$

diff. anomaly $\quad \nabla^i T_{ij} = \frac{c_L - c_R}{16\pi}\, \varepsilon^{kl}\, \partial_k \partial_m \Gamma^m{}_{jl}\,, \quad T_{ij} = T_{ji}$

loc. anomaly $\quad \hat{T}_{ab} \neq \hat{T}_{ba}$

AdS$_3$: Einstein-Hilbert $+ \frac{1}{32\pi G\mu} \int \underbrace{\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^2\right)}_{\text{higher der. diff inv.}} \rightsquigarrow$ TMG

Brown-Henneaux: $\quad c_L = \frac{3\ell}{2G}\left(1 - \frac{1}{\mu\ell}\right)$

$$c_R = \frac{3\ell}{2G}\left(1 + \frac{1}{\mu\ell}\right)$$

$$T_{ij} = T_{ji}$$

$\leadsto$ all sol$^{\text{ns}}$ of EH are sol$^{\text{ns}}$ of TMG

BTZ : modification to the entropy

Wald: $$S_{BTZ} = \underbrace{\frac{2\pi r_+}{4G_3}}_{\text{area}} + \underbrace{\frac{2\pi r_-}{4G_3\mu}}_{\text{grav. CS}}$$

Goal:

modification of R-T in the presence of grav. anom

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_x^n$$

twist field

$$\underset{\text{1 interval}}{\mathrm{Tr}\, \rho_x^n} = \langle \Phi_+(x)\, \bar\Phi(0) \rangle_{C^n/\mathbb{Z}}$$

fixed conf dim

twist fields have $$h_L = \frac{c_L}{24}\left(n - \frac{1}{n}\right) \qquad c = \frac{c_L + c_R}{2}$$

$$h_R = \frac{c_R}{24}\left(n - \frac{1}{n}\right)$$

$$\Delta = h_L + h_R = \frac{c}{12}\left(n - \frac{1}{n}\right) \simeq \frac{c}{6}(n-1) + O(n-1)^2$$

$$s = h_L - h_R = \frac{\bar c}{12}\left(n - \frac{1}{n}\right) \simeq \frac{\bar c}{6}(n-1) + \cdots$$

Conjecture (can be proved)

correct object for generalisation of RT is massive spinning particle in the bulk

anyon

Mathisson '37

Papapetriou '51 — study spinning massive particle in GR

Dixon '70

$$\rightarrow \nabla (m V^\mu + V_\alpha \nabla S^{\mu\alpha}) = \frac{1}{2} V^\nu S^{\rho\sigma} R^\mu{}_{\nu\rho\sigma}$$

$$\nabla S + [v, p] = 0 \quad \longleftrightarrow \quad \nabla S^{\alpha\beta} + V^\alpha V_\rho \nabla S^{\beta\rho} - V^\alpha V_\rho \nabla S^{\alpha\rho} = 0$$

↳ in 3d: $S^{\alpha\beta} = s\,\varepsilon^{\alpha\beta\nu} v_\nu = s(n^\alpha \tilde{n}^\beta - \tilde{n}^\beta n^\alpha)$

$\vee$

2 normal vectors

$$S_{EE} = \underbrace{\frac{1}{4G_3}\int ds}_{\text{length of the curve}} + \underbrace{\frac{1}{4G_3\mu}\int \tilde{n}\cdot\nabla n}_{\text{twist of the curve}}$$

⇝ line → ribbon
(which admits twist)

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