SR Exam 2019 Solutions

1\. (a) For $x^\mu = x^\mu(\tau)$ in ref. frame S, with 4-coord. $x^\mu = (ct, x, y, z)$, we have $-c^2 d\tau^2 = -c^2 dt^2 + d\bar{x}^2 \Rightarrow$

$$d\tau = dt/\gamma \,, \quad \gamma = (1 - \bar{v}^2/c^2)^{-1/2},$$

$$U^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \vec{v}), \quad U^\mu U_\mu = -c^2.$$

$$A^\mu = \frac{dU^\mu}{d\tau} = (c\gamma\dot{\gamma}, \gamma\dot{\gamma}\bar{v} + \gamma^2 \dot{\bar{v}}) =$$

$$= \left(\gamma^4 \frac{\bar{v}\cdot\bar{a}}{c}, \gamma^4 \frac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2 \bar{a}\right), \text{ since}$$

$$\dot{\gamma} \equiv \frac{d\gamma}{dt} = \frac{\bar{v}\cdot\bar{a}}{c^2}\gamma^3.$$

In the inertial ref. frame comoving with the particle (i.e. having the same velocity as particle at each moment of time), $U_0^\mu = (c, \bar{0})$

$A_0^\mu = (0, \bar{a}_0) \Rightarrow A_0^\mu U_{\mu 0} = 0 \Rightarrow$

$\Rightarrow A^\mu U_\mu = 0$ in any other frame,

since $A^\mu U_\mu$ is Lor. invar. Also follows from $\frac{d}{d\tau}(U^\mu U_\mu) = 0$.

- $\frac{d\bar{p}}{dt} = \bar{f}$, where $\bar{p} = \gamma m \bar{v}$.

For $\bar{f} = (f_x, 0, 0)$, where $f_x = \text{const}$,

$$\frac{dp_x}{dt} = f_x \qquad \frac{dp_y}{dt} = 0 \qquad \frac{dp_z}{dt} = 0$$

$\Rightarrow p_x(t) = f_x t + p_{x0}$, $\quad p_y = p_{y0}$, $p_z = p_{z0}$.

$\Rightarrow \gamma m v_x = f_x t + p_{x0}$

Particle starts from rest $\Rightarrow p_{x0} = 0$, $p_{y0} = 0$, $p_{z0} = 0$. So, $\gamma^2 = (1 - v_x^2/c^2)^{-1}$.

$$\Rightarrow \gamma m v_x = f_x t \Rightarrow v_x = \frac{f_x t}{m\sqrt{1 + f_x^2 t^2/m^2c^2}}$$

$$\Rightarrow \boxed{\dot{x} = \frac{f_x t}{m\sqrt{1 + f_x^2 t^2/m^2c^2}}}$$

This can be integrated to find $x(t)$:

$$x(t) = \frac{f_x}{m} \int \frac{t\,dt}{\sqrt{1 + f_x^2 t^2 / m^2 c^2}}$$

With $\xi = \sqrt{1 + f_x^2 t^2 / m^2 c^2}$ we find

$$d\xi = \frac{1}{\sqrt{\ }} \frac{2t\,dt\,f_x^2}{m^2 c^2} \Rightarrow$$

$$x(t) = \frac{mc^2}{f_x} \left(\sqrt{1 + \frac{f_x^2 t^2}{m^2 c^2}} - 1 \right) + X_0 ,$$

where X_0 is integration constant (initial position).

<u>Note</u>: non-rel. limit corresponds to expanding the square root:

$$X(t) = x_0 + \frac{f_x t^2}{m \cdot 2} + \ldots$$

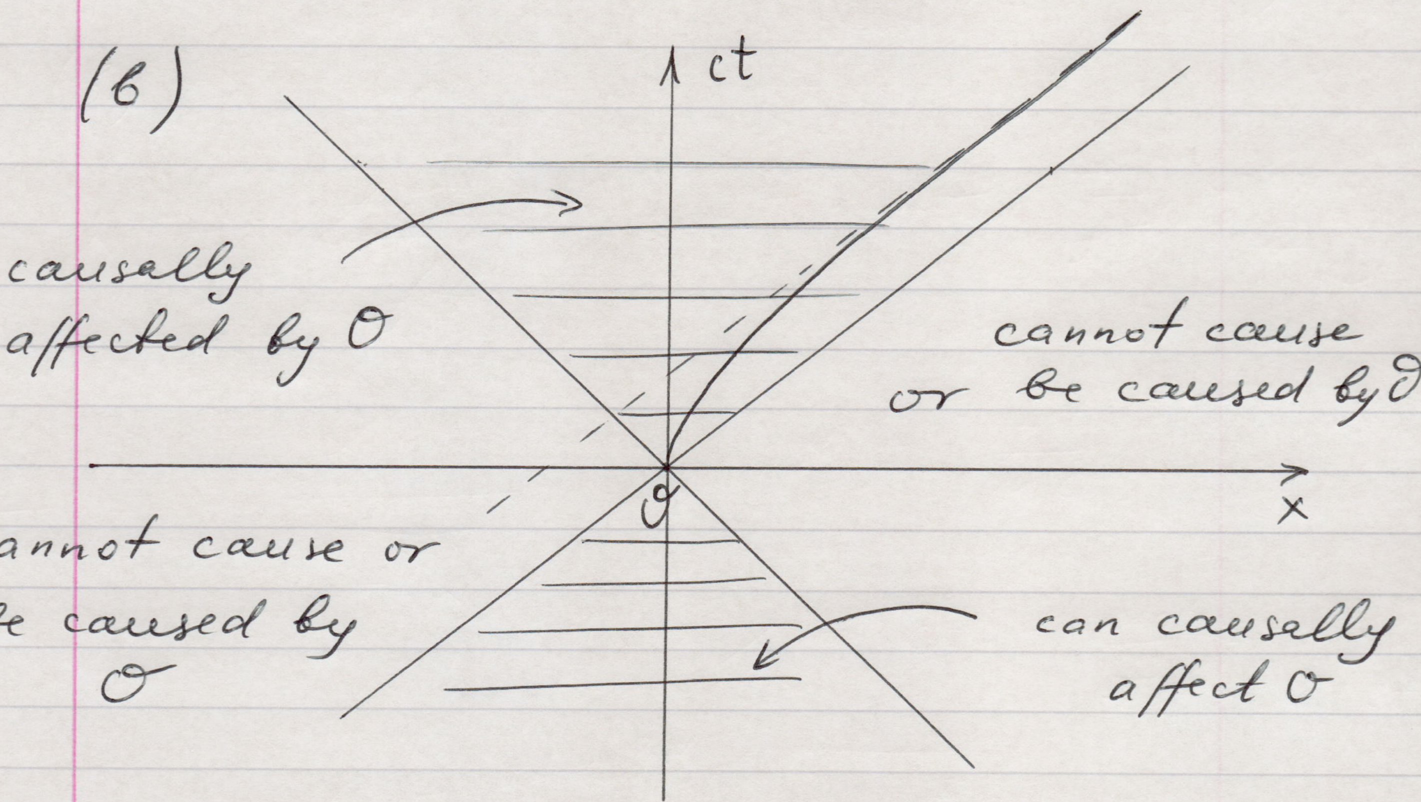
i.e. the usual $X(t) = X_0 + v_{0x} t + \frac{a_x t^2}{2}$,

where $v_{0x} = 0$, $a_x = f_x / m$.

Taking deriv. of $X(t)$, we find $\dot{X}$ as before, and also

$$\ddot{x}(t) = \frac{f/m}{\left(1 + f_x^2 t^2/m^2c^2\right)^{3/2}}$$

(b)



With $\bar{E} = (E_x, 0, 0)$, the force is $\bar{f} = (qE_x, 0, 0)$, so $f_x = qE_x$ and $x(t)$ is the same as found in part (a). In the limit $t \to \infty$, $x(t) \to ct - \frac{mc^2}{f} + \ldots$, so the trajectory approaches the line $ct = x + \frac{mc^2}{f}$ and $v_x \to c$ at late times as shown schematically in Fig. above.

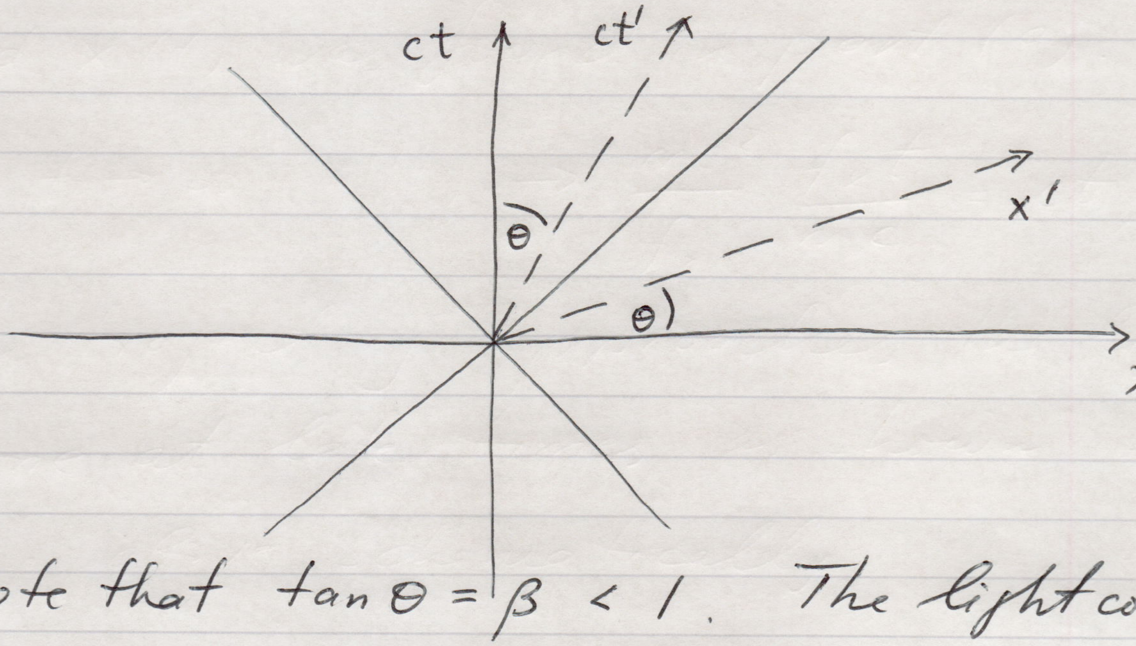
(c) The standard boost to S' is

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \end{cases}$$

This can be written as

$$\begin{cases} ct = \beta x + ct'/\gamma & (*) \\ ct = \dfrac{x}{\beta} - \dfrac{x'}{\gamma\beta} & (**) \end{cases}$$

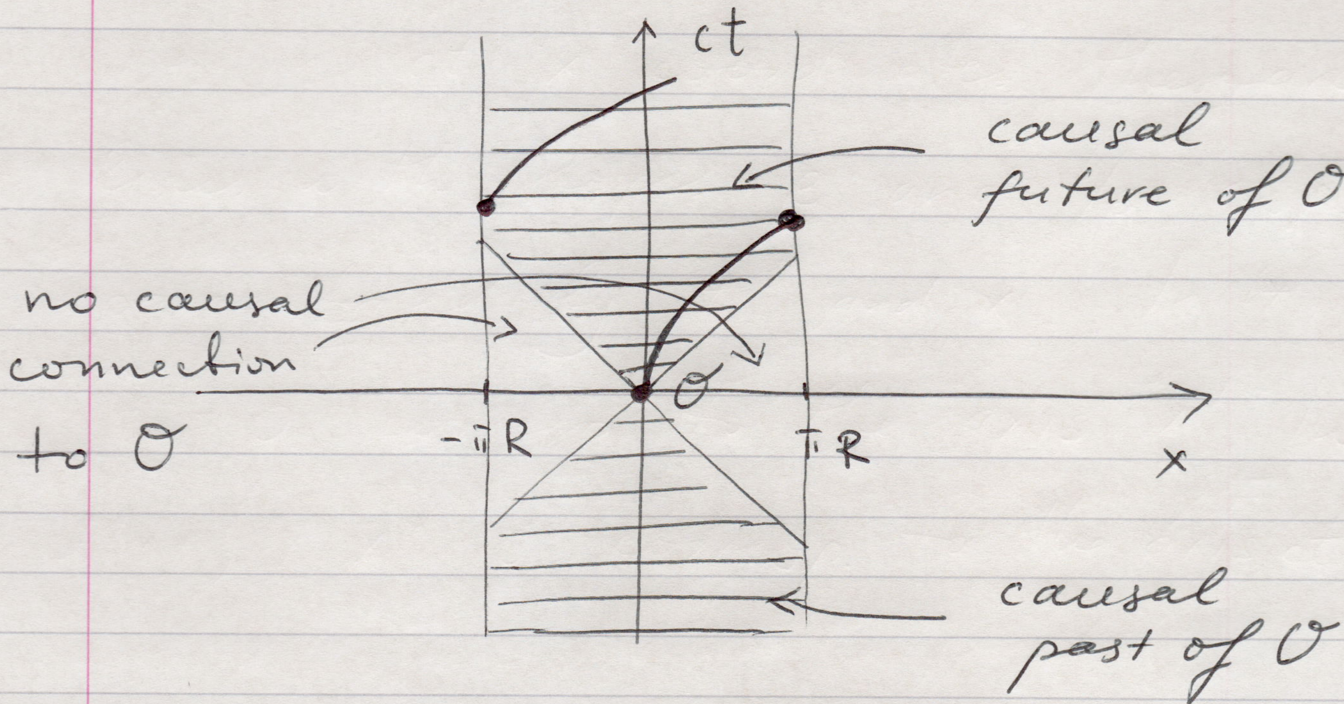
i.e. in ct-x coordinates, these are lines with slopes β and $1/\beta$, resp., parametrised by t' and x'. In $(*)$, $t'=0$ corresp. to the axis x'. In $(**)$, $x'=0$ corresp. to the axis ct'.



Note that $\tan\theta = \beta < 1$. The light cone

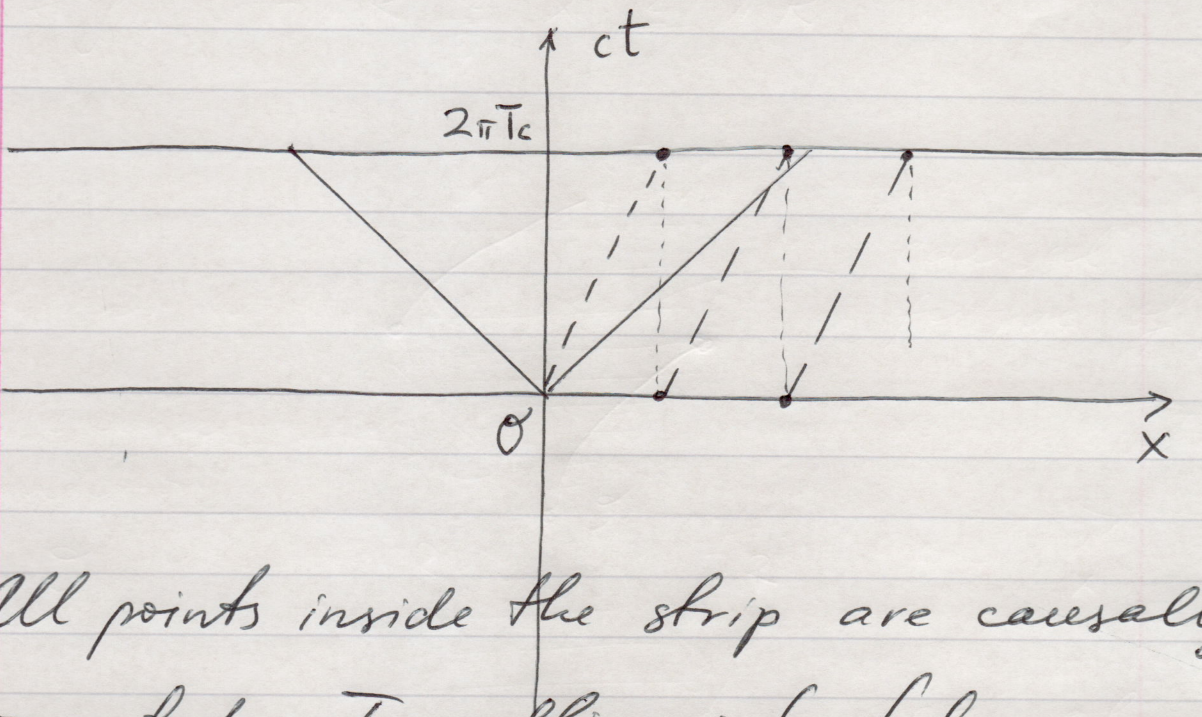
$|ct| = |x|$ is transformed into itself: $ct' = x'$ is the line $ct = x$. The causal regions are therefore the same as before.

(d) $x = x + 2\pi R$: space is compact, e.g. $-\pi R \le x \le \pi R$



The trajectory is shown above, $x = \pi R$ is identified with $x = -\pi R$. The particle passes through the same point (e.g. O) infinitely many times. The source of energy is the external field, so there is no contradiction with the conservation of energy. Also, the accelerated particle will radiate losing energy in the process.

(e) $t = t + 2\pi T$: time is compact



All points inside the strip are causally connected. Travelling into future, one may come to a given point in the past -> this is problematic and can lead to various paradoxes such as preventing your own birth etc.

2. (a) In S', the 4-coordinates of the endpoints of the rod are

$$A' : (ct', 0, ut')$$
$$B' : (ct', L, ut')$$

Transforming to S:

$$ct = \gamma_v (ct' + \beta_v x')$$
$$x = \gamma_v (x' + \beta_v ct')$$
$$y = y'$$

Applying this to A':

$$ct = \gamma_v ct'$$
$$x = \gamma_v \beta_v ct'$$
$$y = ut'$$

$$\Rightarrow A : (ct, \beta_v ct, ut/\gamma_v)$$

Applying Lor. transf. to B':

$$ct = \gamma_v (ct' + \beta_v L)$$
$$x = \gamma_v (L + \beta_v ct')$$

$$y = ut'$$

$$\Rightarrow B: \left(ct,\ \gamma_v\left(L + \beta_v \frac{ct}{\gamma_v} - \beta_v^2 L\right),\ \frac{ut}{\gamma_v} - \beta_v \frac{uL}{c} \right).$$

So, $\Delta x = \gamma_v L + \cancel{\beta_v ct} - \gamma_v \beta_v^2 L - \cancel{\beta_v ct}$

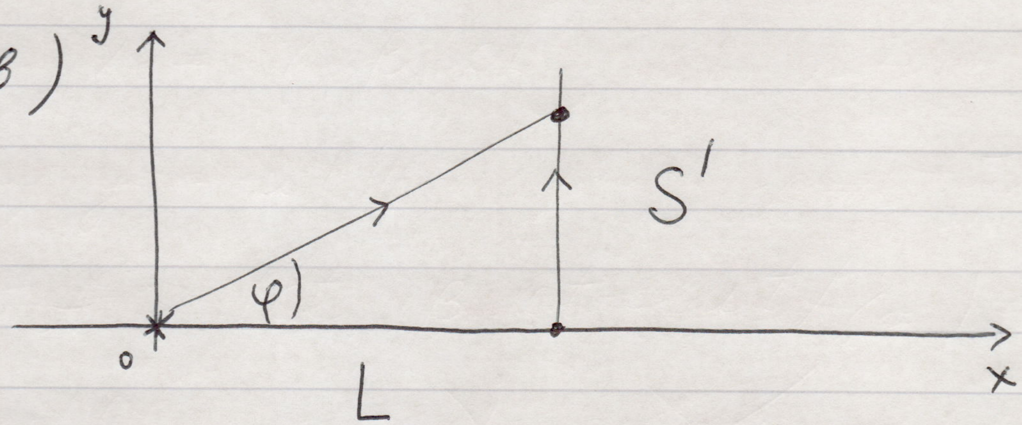
$$= \gamma_v L (1 - \beta_v^2) = L/\gamma_v.$$

$$\Delta y = \frac{ut}{\gamma_v} - \frac{\beta_v uL}{c} - \frac{ut}{\gamma_v} = -\frac{\beta_v uL}{c}.$$

Then $\tan\theta = \dfrac{\Delta y}{\Delta x} = -\dfrac{\beta_v u \cancel{L}}{c \cancel{L}} \gamma_v$

$$\Rightarrow \tan\theta = -\beta_v \beta_u \gamma_v.$$

(b)



A photon emitted from the origin in S has $k^\mu = \left(\frac{\omega_0}{c}, \bar{k}_0\right) = \left(\frac{\omega_0}{c}, k_{0x}, k_{0y}\right)$

where $k_{0x} = |\bar{k}| \cos\varphi$, $k_{0y} = |\bar{k}| \sin\varphi$,

$|\bar{k}| = \omega_0 / c$, $\cos\varphi = L / \sqrt{L^2 + v^2 t^2}$,

$\sin\varphi = vt / \sqrt{L^2 + v^2 t^2}$ – assuming it is received at the origin of S'.

The 4-velocity of S' in S is

$$u^\mu = (\gamma c,\ 0,\ \gamma v).$$

In S', $k'^\mu = \left(\frac{\omega'}{c},\ \bar{k}'\right)$ and

$$u'^\mu = (c,\ 0,\ 0).$$

Since $k^\mu u_\mu = k'^\mu u'_\mu$, we have

$$\boxed{-\omega' = -\omega_0 \gamma + k_{0y} \gamma v}$$

$$\Rightarrow \omega' = \omega_0 \gamma \left(1 - \frac{v}{c} \frac{vt}{\sqrt{L^2 + v^2 t^2}}\right).$$

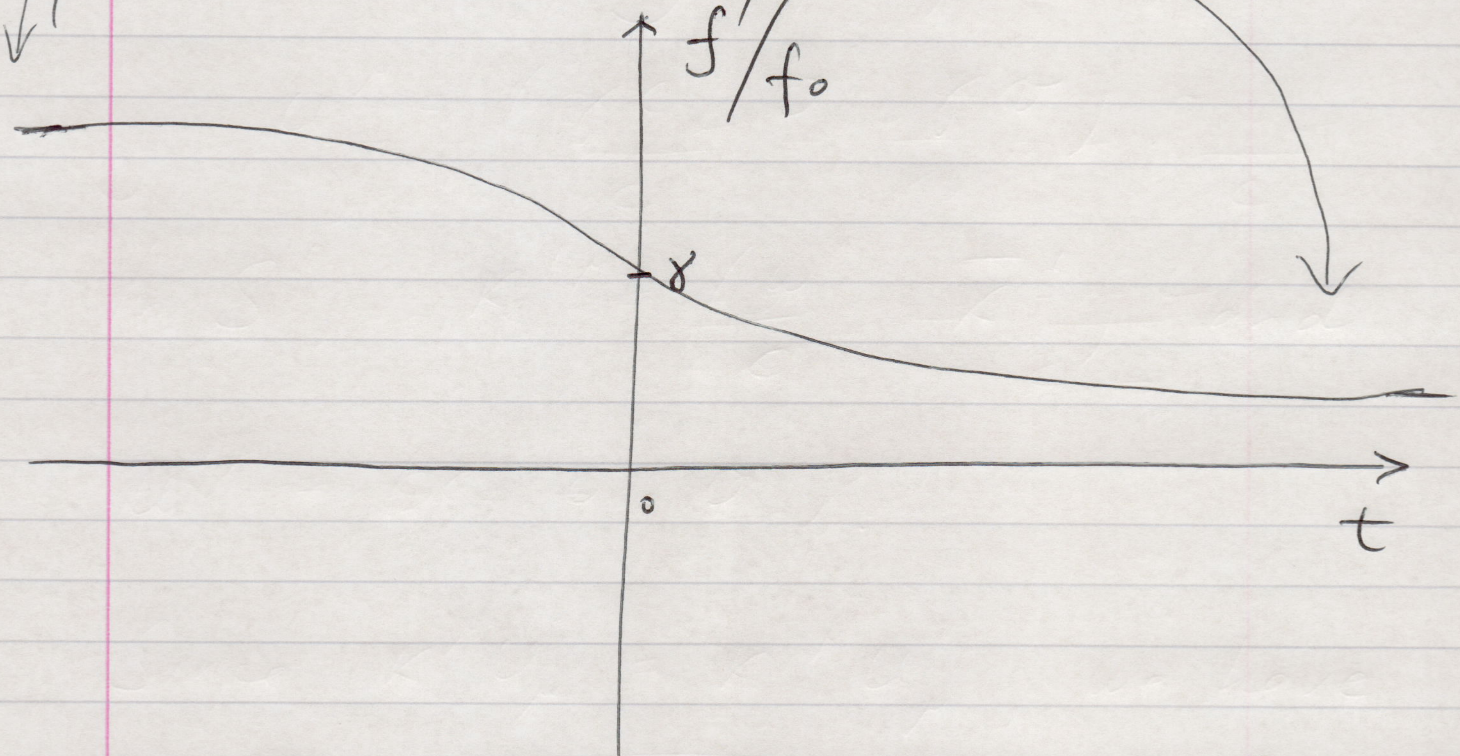
or $$f' = f_0 \gamma \left(1 - \frac{v}{c} \frac{vt}{\sqrt{L^2 + v^2 t^2}}\right).$$

The limits $t \to \pm\infty$ are, corresp:

$$f'/f_0 = \sqrt{\frac{1+\beta}{1-\beta}}, \quad t \to -\infty,$$

$$f'/f_0 = \sqrt{\frac{1-\beta}{1+\beta}}, \quad t \to +\infty,$$

$\beta = v/c$.



(c) This part is not very clearly formulated. One may note the following:

- $f_0 < f$, i.e. motion with $\bar{v} \neq 0$ is used to effectively increase f in the ref frame of the atoms. In the limit

$\beta = v/c \to 0$ we expect no absorption/emission, and, moreover, β should be larger than some threshold value: e.g. at $t \to -\infty$, it is determined by the condition $f/f_0 = \sqrt{\frac{1+\beta}{1-\beta}}$, i.e.

$$\beta > \beta_* = \frac{(f/f_0)^2 - 1}{(f/f_0)^2 + 1}.$$

- For $\beta > \beta_*$, at some point along the trajectory the condition

$$f/f_0 = \gamma\left(1 - \beta \frac{vt}{\sqrt{L^2 + v^2t^2}}\right)$$

is satisfied and the absorption occurs.

- Absorption/emission occur within a frequency interval $\Gamma \sim 1/\tau$ in S' and $1/\gamma\tau$ in S.

- the frequency will decrease with time as the eq. shows. The amplitude follows the standard $\sim e^{-\Gamma t}$ fall-off.

3. (a) An invariant in the context of relativity is the quantity which is the same in all inertial frames (e.g. a length of a 4-vector, $A^\mu A_\mu$, or any other scalar). A conserved quantity Q is the one with $\dot{Q} = 0$ as a result of dynamics (on the eq. of motion) – one can write a covariant form of this conservation law by introducing the appropriate 4-current J^μ:
$\partial_\mu J^\mu = 0$.

. Tensors such as $T^{\mu\dots}_{\nu\dots}$ transform as $T'^{\mu\dots}_{\nu\dots}(x') = \frac{\partial x'^\mu}{\partial x^\rho}\dots\frac{\partial x^\lambda}{\partial x'^\nu}\dots T^{\rho\dots}_{\lambda\dots}(x)$,
where in special rel. $\partial x'^\mu / \partial x^\rho =$
$= \Lambda^\mu_{\ \rho} =$ const is the matrix of Lor. transf., e.g. $\Lambda^\mu_{\ \rho} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) $Y'^{\mu} = \dfrac{\partial x'^{\mu}}{\partial x^{\rho}} Y^{\rho}$

E.g. if $Y^0 = 0$ and $Y'^0 = 0$ in <u>all</u> frames: $0 = \dfrac{\partial x'^0}{\partial x^k} Y^k$, i.e. a linear combination $\lambda_k Y^k = 0$ for <u>generic</u> $\lambda_k \Rightarrow Y^k = 0$.

If $Y^1 = 0$ but $Y^{2,3} \neq 0$, a rotation (part of Lor. Transf.) of axes can make $Y'^1 \neq 0$. Same with boosts.

For tensors of higher rank, one can have some components vanishing in all frames. E.g. $F^{\mu\nu} = -F^{\nu\mu}$ has diag. comp. zero in all frames.

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} F^{\rho\lambda} \Rightarrow$$

if $\mu = \nu \Rightarrow \lambda_\rho \lambda_\varkappa F^{\rho\varkappa} = 0$ with generic λ_ρ does not necessarily imply $F^{\rho\varkappa} = 0$, can be $F^{\rho\varkappa} = -F^{\varkappa\rho}$.

(c) $A^\mu = (\Phi/c, \bar{A})$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\bar{B} = \bar{\nabla} \times \bar{A} \qquad \bar{E} = -\nabla\phi - \partial_t \bar{A}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Sometimes, $F^{\mu}{}_{\nu}$ is used as well:

$$F^{\mu}{}_{\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Note that the signs are the ones corresp. to $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$.

In general, $F'^{\mu\nu} = \dfrac{\partial x'^{\mu}}{\partial x^{\rho}} \dfrac{\partial x'^{\nu}}{\partial x^{\lambda}} F^{\rho\lambda}$.

Lor. transf. are linear: $x' = \Lambda x$,

with e.g. $x'^0 = \gamma(x^0 - \beta x^1)$,

$x'^1 = \gamma(x^1 - \beta x^0)$ for motion along OX

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In matrix form: $F' = \Lambda F \Lambda^T$

Multiplying 4×4 matrices, we get

$F' = A + B$, where

$$A = \begin{bmatrix} 0 & E_x/c & \gamma E_y/c & \gamma E_z/c \\ -\frac{E_x}{c} & 0 & -\frac{\beta\gamma}{c} E_y & -\frac{\beta\gamma}{c} E_z \\ -\frac{\gamma E_y}{c} & \frac{\beta\gamma}{c} E_y & 0 & 0 \\ -\frac{\gamma E_z}{c} & \frac{\beta\gamma E_z}{c} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -\beta\gamma B_z & \beta\gamma B_y \\ 0 & 0 & \gamma B_z & -\gamma B_y \\ \beta\gamma B_z & -\gamma B_z & 0 & B_x \\ -\beta\gamma B_y & \gamma B_y & -B_x & 0 \end{bmatrix}$$

We can compare F' with the standard form of $F'^{\mu\nu}$ to get

$$E'_x = E_x \quad E'_y = \gamma E_y - \gamma\beta c B_z ,$$

$$E'_z = \gamma E_z + \gamma \beta c B_y .$$

Since the motion is along OX,

$$E_x = E_{\parallel} , \quad E_{y,z} = E_{\perp}$$

$$\Rightarrow \bar{E}'_{\parallel} = \bar{E}_{\parallel} , \quad \bar{E}'_{\perp} = \gamma(\bar{E}_{\perp} + \bar{V} \times \bar{B})$$

Similarly, $B'_x = B_x$, $B'_y = \gamma B_y + \gamma\beta E_z/c$,

$$B'_z = \gamma B_z - \gamma \beta E_y / c \Rightarrow$$

$$\Rightarrow \bar{B}'_{\parallel} = \bar{B}_{\parallel} , \quad \bar{B}'_{\perp} = \gamma\left(\bar{B}_{\perp} - \bar{V} \times \bar{E}/c^2\right).$$

(d) Note that $F_{\mu\nu} F^{\mu\nu} = -F_{\mu\nu} F^{\nu\mu} =$

$= -\mathrm{tr}\, F_{\mu\nu} F^{\nu\sigma}$: so, one can multiply two matrices ($F_{\mu\nu}$ and $F^{\nu\sigma}$) given above and then take a trace. Alternatively, this can be summed component by component. Either way,

$$F^{\mu\nu}F_{\mu\nu} = 2\bar{B}^2 - \frac{2}{c^2}\bar{E}^2 = 2D,$$

where $D = \bar{B}^2 - \bar{E}^2/c^2$ is one of two invariants (clearly, $F_{\mu\nu}F^{\mu\nu}$ is a Lor-invariant).

Now, $\tilde{F}_{ab} = \frac{1}{2}\,\varepsilon_{ab\mu\nu}F^{\mu\nu}$. Note $\tilde{F}_{ab} = -\tilde{F}_{ba}$. One has e.g. $\tilde{F}_{01} =$

$$= \frac{1}{2}\,\varepsilon_{01\mu\nu}F^{\mu\nu} = \frac{1}{2}\,\varepsilon_{0123}F^{23} + \frac{1}{2}\,\varepsilon_{0132}F^{32}$$

$$= \frac{1}{2}\,\varepsilon_{0123}F^{23} - \frac{1}{2}\,\varepsilon_{0123}F^{32} = \frac{1}{2}\left(F^{23} - F^{32}\right) =$$

$= F^{23} = B_x$, and similarly for other components:

$$\tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{bmatrix}$$

Then: $\tilde{F}_{\mu\nu} F^{\mu\nu} = -\operatorname{tr} \tilde{F}_{\mu\nu} F^{\nu\sigma} =$

$= \frac{4}{c} \bar{E} \cdot \bar{B} \equiv 4\alpha$, where

$\alpha = \bar{B} \cdot \bar{E}/c$ is the other invariant.

Note that D is a scalar, whereas α is a pseudoscalar (it changes sign under $\bar{x} \to -\bar{x}$).

$F_{\mu\nu} F^{\mu\nu}$ and $\tilde{F}_{\mu\nu} F^{\mu\nu}$ have no free indices and thus are Lor-invar. (a scalar and a pseudoscalar, resp.).

- Note that $\varepsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) = 4\, \varepsilon_{\mu\nu\rho\sigma}\, \partial^\mu A^\nu \partial^\rho A^\sigma =$

$= 4\, \partial^\mu (\varepsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma) = 4\, \partial^\mu K_\mu$,

where $K_\mu \equiv \varepsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma$.

(Here, we used antisymmetry of $\varepsilon_{\mu\nu\rho\sigma}$ and also $\varepsilon_{\mu\nu\rho\sigma} \partial_\rho \partial_\sigma = 0$)

One can also write $K_\mu = \frac{1}{2} J_\mu$, where

$$J_\mu = \varepsilon_{\mu\nu\rho\sigma} A^\nu F^{\rho\sigma}.$$

Thus, $\dfrac{\bar{E}\cdot\bar{B}}{c} = \dfrac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} =$

$$= \frac{1}{4}\cdot\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) =$$

$$= \frac{1}{2}\partial^\mu K_\mu = \frac{1}{4}\partial^\mu J_\mu.$$

Now, $\displaystyle\int d^4x\, \frac{\bar{E}\cdot\bar{B}}{c} = \frac{1}{4}\int d^4x\, \partial^\mu J_\mu =$

$= \dfrac{1}{4}\displaystyle\int_{\Sigma_3} d\Sigma_\mu \cdot J^\mu$ by Ostrogradsky–Gauss theorem.

Assuming the fields vanish on the 3-dim boundary of 4-dim volume, we have $\int d^4x\, \bar{E}\cdot\bar{B} = 0$.

4. The material of Question 4 is off-syllabus for 2019-2020 year. These questions are properly treated within Quantum Field Theory courses.