University of Oxford

Physics Department

GENERAL RELATIVITY AND COSMOLOGY

EXAM PAPER

2014

SOLUTION NOTES

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2705

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B5: GENERAL RELATIVITY & COSMOLOGY

TRINITY TERM 2014

Saturday, 21 June, 9.30 am – 11.30 am

10 minutes reading time

Answer two *questions.*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. Consider the space-time metric

$$ds^2 = -c^2\,dt^2 + \left(\frac{R^2}{r^2+\alpha^2}\right) dr^2 + R^2\,d\theta^2 + (r^2+\alpha^2)\sin^2\theta\,d\phi^2,$$

where $R^2 = r^2 + \alpha^2\cos^2\theta$ and α is a constant. Find the geodesic equation for ϕ and show that there is an integral of motion given by

$$\dot{\phi} = \frac{J}{(r^2+\alpha^2)\sin^2\theta},$$

where derivatives are with respect to t. [6]

Show that the geodesic equation for θ can be solved by $\theta = \pi/2$ and $\dot{\theta} = 0$. For that value of θ, show that the geodesic equation for r has the following integral form

$$\frac{r^2}{r^2+\alpha^2}\dot{r}^2 + \frac{J^2}{r^2+\alpha^2} = B^2$$

and that $r = \sqrt{D^2 + (vt)^2}$, where D and v are constants, is a solution. Find B and J in terms of v, D and α. [9]

Consider the coordinate transformation

$$\begin{aligned} x &= \sqrt{r^2+\alpha^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2+\alpha^2}\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

while t remains unchanged. Our original metric takes the form

$$ds^2 = -c^2\,dt^2 + f^2\,dx^2 + g^2\,dy^2 + h^2\,dz^2.$$

Find the functions f, g and h. [8]

Given your solutions to the geodesic equations and what you have learnt from the coordinate transformation, interpret your results. [2]

2. Consider the space-time metric

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + A^2(t)\,\mathrm{d}x^2 + B^2(t)\left(\mathrm{d}y^2 + \mathrm{d}z^2\right)$$

Show that the non-vanishing Christoffel symbols are

$$\Gamma^t{}_{xx} = \frac{AA'}{c^2}, \quad \Gamma^t{}_{yy} = \frac{BB'}{c^2}, \qquad \Gamma^t{}_{zz} = \frac{BB'}{c^2},$$
$$\Gamma^x{}_{tx} = \frac{A'}{A}, \quad \Gamma^y{}_{ty} = \frac{B'}{B}, \quad \Gamma^z{}_{tz} = \frac{B'}{B},$$

where $A' = \mathrm{d}A/\mathrm{d}t$ and $B' = \mathrm{d}B/\mathrm{d}t$. [8]

Find the components of the Ricci tensor, given by

$$R_{\nu\beta} \equiv \partial_\mu \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\mu\nu} + \Gamma^\mu{}_{\mu\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\mu}$$

for this space-time. Show that the Einstein field equations for this metric in the presence of an energy-momentum tensor with $T_{00} = \rho c^2$ and all other components vanishing are

$$\begin{aligned} 2\frac{A'B'}{AB} + \left(\frac{B'}{B}\right)^2 &= \frac{8\pi G}{c^2}\rho \\ \frac{A''}{A} + \frac{B''}{B} + \frac{A'B'}{AB} &= 0 \\ 2\frac{B''}{B} + \left(\frac{B'}{B}\right)^2 &= 0 \end{aligned}$$

[8]

Show that, if $\rho \neq 0$, the functions $B = B_0 t^m$ and $A = A_0 t^m + A_1 t^n$ are solutions to these equations, where m and n are constants. Show that $m = 2/3$ and find a possible non-zero value for n. How does the density, ρ evolve as a function of time? Discuss the evolution of this space-time for small and for large values of t and compare to that of the FRW metric. [6]

Is this metric a good description of our observable Universe, and if not, why? What observation might allow you to distinguish this metric from the FRW metric? [3]

3. Write down the metric for a non-Euclidean, homogeneous and isotropic universe with a scale factor $a(t)$. Write down the FRW equations for such a metric in the presence of a perfect fluid with an energy density, ρ_I, and a constant equation of state parameter, w_I. Write down an expression for the density parameter, Ω. Using the FRW equation, find an expression for Ω as a function of the scale factor, a, and the Hubble parameter, H. [7]

Express Ω purely as a function of a in the regime where ρ_I dominates and use it to find the values of w_I for which curvature becomes negligible at late time, i.e. for which the universe becomes arbitrarily close to Euclidean for large a. [3]

Assume that now, through some strange physical process, the universe was completely dominated by an exotic perfect fluid with density ρ_I and curvature was negligible. What values must w_I take for there not to have existed an initial singularity (i.e. no "Big Bang" for a finite value of cosmic time, t)? Write down the solution to a as a function of t for $w_I < -1$, and show that it has an event horizon. [9]

Consider the particular case of a non-Euclidean, hyperspherical universe with dust and ρ_I. Find an expression for the deceleration parameter for such a universe in terms of the values of Ω_M and Ω_I today. Find also an expression correct up to quadratic order in the redshift z for the luminosity distance $D_L(z)$. [6]

4. A massive particle, ζ, with mass, m_ζ, has an occupation number $f(E)$ for states at energy E satisfying the Fermi–Dirac distribution,

$$f(E) = \frac{1}{e^{\frac{E-\mu}{K_B T}} + 1}$$

where E is the energy of the particle and T is the temperature of the universe, set by the thermal bath of photons. When is this an accurate description of the particle ζ and what conditions must be satisfied? [4]

What is the equation of state for particle ζ when $m_\zeta c^2 \gg K_B T$ and when $m_\zeta c^2 \ll K_B T$? If T_0 is the temperature of the photons today, at what redshift, z_ζ, do you expect the transition between the two types of equation of state to occur? Consider a spatially flat universe, with fractional energy density in matter, Ω_M and fractional energy density in photons $\Omega_\gamma = 10^{-4}$. Assume that $\Omega_\zeta \ll \Omega_M$ to find a lower bound on m_ζ such that the transition between the two equations of state occurs before matter-radiation equality comes about at redshift z_*. [5]

Assume ζ can decay into a particle ξ and that $0 < m_\zeta - m_\xi \ll m_\zeta$. Using the fact that a particle with mass M and chemical potential, μ, in the non relativistic regime has a number density given by

$$n^{eq} = \left(\frac{2\pi}{h^2}\right)^{\frac{3}{2}} (M k_B T)^{\frac{3}{2}} \exp\left(-\frac{Mc^2 - \mu}{k_B T}\right)$$

and justifying all your assumptions, find an expression for n_ζ/n_ξ as a function of temperature, where n_X is the number density of particle $X = \zeta, \xi$. Discuss the consequences of your result if the system remains in thermal equilibrium until today. [5]

The reaction rate for the conversion between ζ and ξ is given by $\Gamma = \Gamma_0 (T/T_0)^6$ where Γ_0 is a constant. Use the Boltzman equation

$$\frac{\mathrm{d}\ln N_\zeta}{\mathrm{d}\ln a} = -\frac{\Gamma}{H}\left[1 - \left(\frac{N_\zeta^{eq}}{N_\zeta}\right)^2\right]$$

to discuss the different regimes in the evolution of $N_\zeta = n_\zeta a^3$. Explain why freeze-out may lead to a non-negligible n_ζ today. [5]

Assume that $\Omega_M = \Omega_\zeta + \Omega_\xi \simeq 1$, that $z_\zeta \gg z_*$, and $\mu_\zeta = 0$ at freeze-out. Show that

$$\Omega_\zeta \simeq 0.1\Omega_\gamma \left(\frac{M_\zeta c^2}{k_B T_0}\right)^{\frac{5}{2}} \exp\left(-\frac{Mc^2 a_F}{k_B T_0}\right) a_F^{\frac{3}{2}}$$

where $a_F \simeq 3.3(\Gamma_0/H_0)^{1/4}$ and $\rho_\gamma = \frac{\pi^2}{15}(k_B T_0)(k_B T_0/\hbar c)^3$. [6]

1

1.

$$ds^2 = -c^2 dt^2 + \frac{R^2}{r^2+\alpha^2} dr^2 + R^2 d\theta^2 +$$

$$+ (r^2+\alpha^2)\sin^2\theta\, d\varphi^2$$

$$R^2 = r^2 + \alpha^2 \cos^2\theta$$

a) geodesic for φ

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = \frac{\partial \mathcal{L}}{\partial \varphi} \Rightarrow \frac{d}{d\lambda}\left((r^2+\alpha^2)\sin^2\theta\,\dot{\varphi}\right) = 0$$

$$\Rightarrow \boxed{(r^2+\alpha^2)\sin^2\theta\,\dot{\varphi} \equiv J = \mathrm{const}}$$

$(x^0)^2$

$$-\frac{d}{d\lambda}(c\dot{t}) = 0$$

$$\dot{t} = \mathrm{const} \Rightarrow t = \alpha\lambda + \beta$$

$r \to \infty \quad t = \lambda$

Can choose $\lambda = t$.

$\alpha = 1, \beta = 0$

$$\Rightarrow \boxed{\dot{\varphi} = \frac{J}{(r^2+\alpha^2)\sin^2\theta}}$$

with $\lambda = t$.

$$\frac{d}{d\lambda}\left(R^2\dot{\theta}\right) = \frac{-\dot{r}^2}{r^2+\alpha^2}\,\alpha^2\,\cancel{2}\cos\theta\sin\theta\ +$$

$$+\ \left(r^2+\alpha^2\right)\cancel{2}\sin\theta\cos\theta\ \dot{\varphi}^2$$

$$-\ \cancel{2}\alpha^2\sin\theta\cos\theta\ \dot{\theta}^2$$

$\theta = \pi/2$, $\dot{\theta} = 0$ is a solution.

Geodesic for r:

$$\frac{d}{d\lambda}\left(\frac{R^2}{r^2+\alpha^2}\dot{r}\right) = \frac{\dot{r}^2}{2}\frac{d}{dr}\left(\frac{R^2}{r^2+\alpha^2}\right) + \underbrace{\left(\frac{d}{dr}R^2\right)\frac{\dot{\theta}^2}{2}}_{0} +$$

$$+\ \cancel{2}r\sin^2\theta\ \dot{\varphi}^2$$

$$\frac{R^2}{r^2+\alpha^2}\ddot{r} + \dot{r}^2\cancel{\frac{d}{dr}\left(\frac{R^2}{r^2+\alpha^2}\right)} = \frac{\dot{r}^2}{2}\cancel{\frac{d}{dr}\left(\frac{R^2}{r^2+\alpha^2}\right)} +$$

$$+\ \cancel{2}r\cancel{\sin^2\theta}\ \frac{J^2}{\left(r^2+\alpha^2\right)^2}$$

$R^2 = r^2$ for $\theta = \pi/2$

$$\frac{d}{d\lambda}\left(\frac{r^2}{r^2+\alpha^2}\dot{r}\right) = \frac{\dot{r}^2}{2}\frac{d}{dr}\left(\frac{r^2}{r^2+\alpha^2}\right) + r\dot{\varphi}^2$$

$$\frac{d}{dr} = \frac{d}{d\lambda}\frac{d\lambda}{dr} = \frac{1}{\dot{r}}\frac{d}{d\lambda}$$

$$\frac{r^2}{r^2+\alpha^2}\ddot{r} + \frac{\dot{r}^2}{2}\frac{d}{dr}\left(\frac{r^2}{r^2+\alpha^2}\right) =$$

$$= \frac{rJ^2}{(r^2+\alpha^2)^2}$$

$$\frac{\dot{r}}{2}\frac{d}{d\lambda}\frac{r^2}{r^2+\alpha^2}$$

$$r\ddot{r} + \frac{r\dot{r}^2}{2}\frac{F'}{F} = \frac{J^2}{(r^2+\alpha^2)}$$

$$r\dot{r}\frac{d\dot{r}}{dr} + \frac{r\dot{r}^2}{2}\frac{F'}{F}$$

$$\mathcal{L} = -\frac{c^2 \dot{t}^2}{2} + \frac{R^2}{r^2+\alpha^2}\frac{\dot{r}^2}{2} + \frac{R^2}{2}\dot{\theta}^2 +$$

$$+ (r^2+\alpha^2)\sin^2\theta\,\frac{\dot{\varphi}^2}{2}$$

$$\mathcal{L} = -\frac{c^2}{2} + \frac{R^2}{r^2+\alpha^2}\frac{\dot{r}^2}{2} + \frac{\cancel{(r^2+\alpha^2)}\,J^2}{2(r^2+\alpha^2)^{\cancel{2}}}$$

$$\mathcal{L} = -\frac{c^2}{2}\ ; \qquad \frac{r^2}{r^2+\alpha^2}\frac{\dot{r}^2}{2} + \frac{J^2}{2(r^2+\alpha^2)} =$$

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu \qquad = \frac{c^2}{2} + \mathcal{L}$$

$$+\frac{ds^2}{2} = -\frac{c^2 d\tau^2}{2}$$

$$\frac{r^2}{r^2+\alpha^2}\dot{r}^2 + \frac{J^2}{r^2+\alpha^2} = c^2 + 2\mathcal{L}$$

$$r^2 = A^2 + v^2 t^2 \qquad r^2\dot{r}^2 + J^2 = B^2(r^2+\alpha^2)$$

$$\cancel{2} r\dot{r} = \cancel{2} v^2 t \qquad \dot{r}^2 + \frac{J^2}{r^2} = B^2\left(1+\frac{\alpha^2}{r^2}\right)$$

$$\dot{r}^2 + \frac{J^2 - \alpha^2 B^2}{r^2} = B^2$$

$$\frac{ds^2}{d\lambda^2} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\left(\frac{d\tau}{d\lambda}\right)^2$$

$$g_{\mu\nu}u^\mu u^\nu = -c^2$$

$$g_{\mu\nu}\,dx^\mu dx^\nu = -c^2 d\tau^2$$

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\frac{c^2}{2}\left(\frac{d\tau}{d\lambda}\right)^2$$

λ : affine param.

$\tau = \alpha\lambda + \beta.$

$$-c^2\dot{t} = \text{const}$$

$$\dot{t} = \bar{B}$$

$$t = \bar{B}\lambda + \bar{C}$$

$$\frac{r^2}{r^2+\alpha^2}\dot{r}^2 + \frac{J^2}{r^2+\alpha^2} = c^2\left(1 - \left(\frac{d\tau}{d\lambda}\right)^2\right)$$

$$\equiv B^2.$$

$$\dot{r}^2 = \frac{B^2(r^2+\alpha^2)}{r^2} - \frac{J^2}{r^2}$$

$$\dot{r}^2 + \frac{J^2 - \alpha^2 B^2}{r^2} = B^2$$

$$\dot{r} r = v^2 t \qquad \dot{r}^2 = \frac{v^4 t^2}{r^2}$$

$$\underline{(R^2 + v^2 t^2)\, v^4 t^2}$$

$$\frac{v^4 t^2}{r^2} + \frac{J^2 - \alpha^2 B^2}{r^2} = B^2$$

$$\underline{v^4 t^2} + J^2 - \alpha^2 B^2 = B^2 R^2 + \underline{B^2 v^2 t^2}$$

$$v^4 = B^2 v^2 \qquad \boxed{B = v}$$

$$J^2 = \alpha^2 B^2 + B^2 R^2$$

$$\boxed{J^2 = B^2(\alpha^2 + R^2)} \qquad B = 0?$$

$$\boxed{J = v\sqrt{\alpha^2 + R^2}}$$

$$\frac{\mu \dot{r}^2}{2} + \frac{J^2}{2\mu r^2} = 0$$

$$\dot{\varphi} = 0$$

$$r'' + \frac{\dot{r}^2}{2}\frac{F'}{F} = \frac{J^2}{r(r^2+\alpha^2)}$$

$$F = \frac{r^2}{r^2+\alpha^2}$$

$$r'' + \frac{\alpha^2}{r(r^2+\alpha^2)} = \frac{J^2}{r(r^2+\alpha^2)}$$

$$\boxed{r'' + \frac{\alpha^2 - J^2}{r(r^2+\alpha^2)} = 0}$$

$$\dot{r} = \frac{v^2 t}{\sqrt{R^2+v^2t^2}} = \frac{v^2 t}{r}$$

$$r'' = \frac{v^2}{r} - \frac{v^2 t}{r^2}\frac{v^2 t}{r} =$$

$$= \frac{v^2}{r}\left(1 - \frac{v^2 t^2}{r^2}\right)$$

$$v^2\left(1-\frac{v^2t^2}{r^2}\right)+\frac{\alpha^2-J^2}{r^2+\alpha^2}=0$$

$$r^2=R^2+v^2t^2$$

$$v^2\left(\frac{\overset{R^2}{\cancel{r^2-v^2t^2}}}{R^2+v^2t^2}\right)+\frac{\alpha^2-J^2}{R^2+\alpha^2+v^2t^2}=0$$

$$\frac{v^2R^2}{R^2+v^2t^2}+\frac{\alpha^2-J^2}{R^2+v^2t^2+\alpha^2}=0$$

$$\left(R^2+v^2t^2+\alpha^2\right)v^2R^2+\left(\alpha^2-J^2\right)\left(R^2+v^2t^2\right)=0$$

$$\cancel{v^2t^2}\,v^2R^2+\left(\alpha^2-J^2\right)\cancel{v^2t^2}=0$$

$$\boxed{v^2R^2+\alpha^2-J^2=0}$$

$$\left(R^2+\alpha^2\right)v^2R^2+\left(\alpha^2-J^2\right)R^2=0$$

(9)

$$\left(\alpha^2 - J^2\right) R^2 = -v^2 R^4$$

$$\left(R^2 + \alpha^2\right) \cancel{v^2 R^2} - \cancel{v^2 R^2} = 0$$

$\alpha^2 = 0$ only.

$$dz = \cos\theta \, dr - r \sin\theta \, d\theta$$

$$dx = \frac{r}{\sqrt{}} dr \sin\theta \cos\varphi +$$

$$+ \cos\theta \, d\theta \sqrt{} \cos\varphi -$$

$$- \sin\varphi \, d\varphi \sin\theta \sqrt{}$$

$$dy = \frac{r}{\sqrt{}} dr \sin\theta \sin\varphi +$$

$$+ \cos\theta \, d\theta \sqrt{} \sin\varphi +$$

$$+ \cos\varphi \, d\varphi \sin\theta \sqrt{}$$

$$dr^2\left\{h^2\cos^2\theta + \frac{f^2 r^2}{r^2+\alpha^2}\sin^2\theta\, \cancel{\cos^2\varphi}\right\}$$

$$+ \frac{g^2 r^2}{r^2+\alpha^2}\sin^2\theta \sin^2\varphi\Big\}$$

$$\Big/\!\!\Big/$$

$$\frac{r^2+\alpha^2\cos^2\theta}{r^2+\alpha^2}$$

$$(r^2+\alpha^2)h^2\cos^2\theta + f^2r^2\sin^2\theta\cos^2\varphi + g^2r^2\sin^2\theta\sin^2\varphi =$$

$$=(r^2+\alpha^2)h^2\cos^2\theta + f^2r^2\sin^2\theta =$$

$$= \underline{r^2+\alpha^2\cos^2\theta} \qquad f=g$$

$$h^2\cos^2\theta + f^2\sin^2\theta\cos^2\varphi + g^2\sin^2\theta\sin^2\varphi = 1$$

$$\boxed{h^2\cos^2\theta + f^2\sin^2\theta = 1}$$

$$(r^2+\alpha^2)h^2\cos^2\theta + f^2r^2 - f^2r^2\cos^2\theta$$

$$= r^2 + \alpha^2\cos^2\theta$$

$$r^2h^2\cancel{\cos^2}\theta + \alpha^2h^2\cos^2\theta -$$

$$- f^2\cancel{r^2}\cos^2\theta + f^2r^2 =$$

$$= r^2 + \alpha^2\cos^2\theta \qquad f = h = 1$$

$$\boxed{f = h = 1 = g}$$

12

$$x^2+y^2=(r^2+\alpha^2)\sin^2\theta$$

$$z^2=r^2\cos^2\theta$$

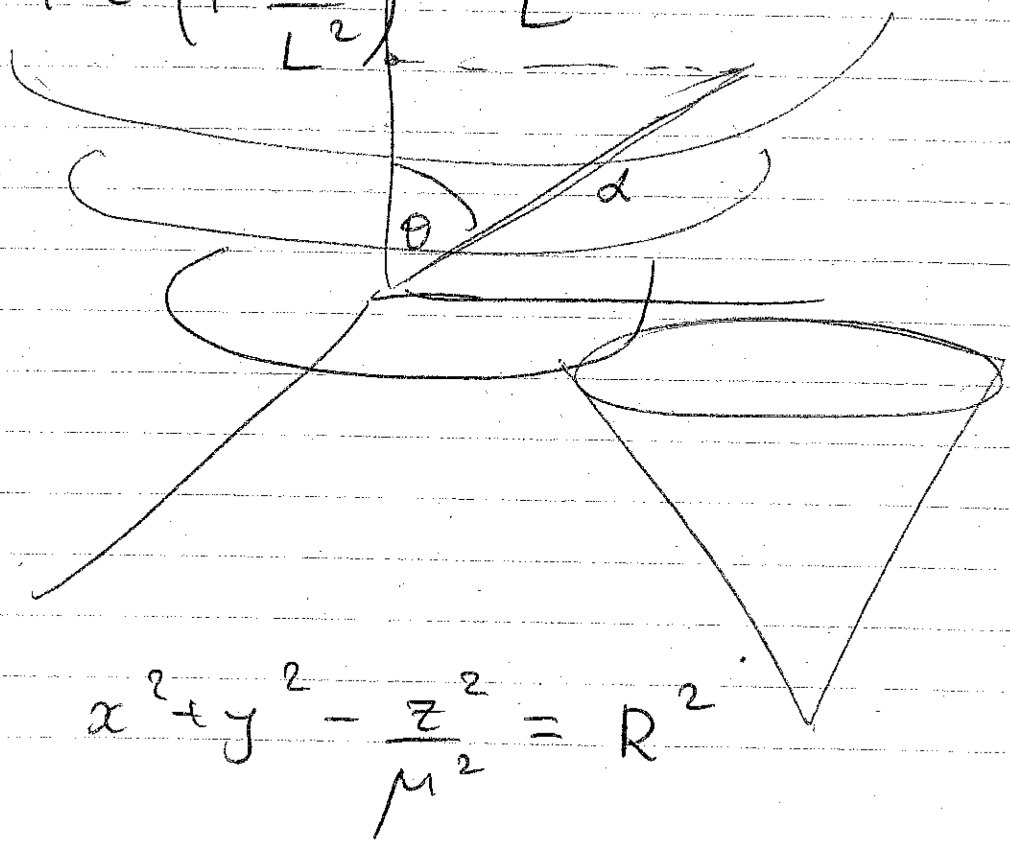
$$x^2+y^2+z^2=r^2\sin^2\theta+\alpha^2\sin^2\theta+r^2\cos^2\theta$$

$$=r^2+\alpha^2\cos^2\theta$$

$$x^2+y^2+z^2=L^2+\frac{\alpha^2 z^2}{L^2}$$

$$\vec{R}^2=r^2+\alpha^2\cos^2\theta$$

$$x^2+y^2+z^2\left(1-\frac{\alpha^2}{L^2}\right)=L^2$$



$$x^2+y^2-\frac{z^2}{\mu^2}=R^2$$

$$\frac{x^2+y^2}{r^2+\alpha^2} + \frac{z^2}{r^2} = 1$$

$$\boxed{\frac{x^2}{1+\frac{\alpha^2}{r^2}} + \frac{y^2}{1+\frac{\alpha^2}{r^2}} + z^2 = r^2}$$

surface of const r

$$\boxed{x^2+y^2 = z^2 \operatorname{tg}^2\theta + \alpha^2 \sin^2\theta}$$

surf. of const θ

$r = a\cosh\xi$

$r^2+\alpha^2 = a^2\cosh^2\xi + \alpha^2 = \operatorname{sh}^2\xi$

$a = 1$ prolate spheroidal

$\alpha^2 = -1$ coord.

Problem 2

$$ds^2 = -c^2 dt^2 + A^2(t)\, dx^2 + B^2(t)\,(dy^2 + dz^2)$$

Non-vanishing Christoffel symbols:

$$\Gamma^t_{xx} = \frac{AA'}{c^2}, \quad \Gamma^t_{yy} = \frac{BB'}{c^2}, \quad \Gamma^t_{zz} = \frac{BB'}{c^2},$$

$$\Gamma^x_{tx} = \Gamma^x_{xt} = \frac{A'}{A}, \quad \Gamma^y_{ty} = \Gamma^y_{yt} = \frac{B'}{B}$$

$$\Gamma^z_{tz} = \Gamma^z_{zt} = B'/B$$

Ricci tensor:

$$R_{tt} = -\frac{BA'' + 2AB''}{AB},$$

$$R_{xx} = \frac{A}{c^2 B}\left(2A'B' + BA''\right),$$

$$R_{yy} = \frac{BA'B' + AB'^2 + ABB''}{c^2 A},$$

$$R_{zz} = R_{yy}.$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} .$$

$$G_{tt} = \frac{B'^2}{B^2} + \frac{2A'B'}{AB} = \frac{8\pi G}{c^4} \rho c^2 ,$$

$$G_{xx} = -\frac{A^2}{c^2 B^2}\left(B'^2 + 2BB''\right) = 0 ,$$

$$G_{yy} = -\frac{B}{c^2 A}\left(A'B' + BA'' + AB''\right) = 0 ,$$

$$G_{zz} = G_{yy} = 0 . \qquad \text{O.K.}$$

With $B = B_0 t^m$, $A = A_0 t^m + A_1 t^n$:

$$2\frac{B''}{B} + \frac{B'^2}{B^2} = \frac{2m(m-1)}{t^2} + \frac{m^2}{t^2} = \frac{3m^2 - 2m}{t^2} = 0$$

$\Rightarrow m = 0 , \ m = 2/3 .$

Then eq for G_{zz}, G_{yy} give:

$$m = \frac{2}{3} \Rightarrow n = \frac{2}{3} \quad \text{or} \quad n = -1/3$$

$$m = 0 \Rightarrow n = 0 , \quad n = 1 .$$

1) $m = 0, n = 0$: $A, B = \text{const}$

$$\Rightarrow \rho = 0$$

<u>Solution</u>: Mink. space ; $\rho = 0$.

2) $m = 0, \quad n = 1$

$$B = B_0 \; ; \quad A = A_1 t + A_0$$

$$\rho = 0$$

3) $m = \frac{2}{3}$; $n = \frac{2}{3}$

$$B = B_0 t^{2/3} \qquad A = A_0 t^{2/3}$$

$\Rightarrow$ <u>FRW</u> with redef. of y, z
e.g.

$$\rho = 4/3t^2$$

4) $m = 2/3$; $n = -1/3$ $\qquad B = B_0 t^{2/3}$

$$A = A_0 t^{2/3} + A_1 t^{-\frac{1}{3}}$$

$$\rho = \frac{4}{3t^2 + 3\frac{A_1}{A_0}t}$$

(FRW for late t.)

Early times: $\rho \sim \frac{4 A_0}{3 A_1} \frac{1}{t}$

$$ds^2 \approx -c^2 dt^2 + A_1^2 t^{-2/3} dx^2 + B_0^2 t^{4/3} (dy^2 + dz^2)$$

Late times: $\rho \sim \frac{4}{3t^2}$:

$$ds^2 \simeq -c^2 dt^2 + A_0^2 t^{4/3} dx^2 + B_0^2 t^{4/3} (dy^2 + dz^2)$$

With re-def. e.g. x:

FRW with $a(t) \sim t^{2/3}$

The metric is anisotropic at early times — no ~~observ.~~ indications but can look e.g. at possible anisotropies in CMB.

3.

$$\boxed{ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)}$$

$$\boxed{d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \; ; \quad k = 0, \pm 1}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_I - \frac{kc^2}{a^2} + \frac{c^2\Lambda}{3}$$

$$\dot{\rho}_I + 3\frac{\dot{a}}{a}\left(\frac{P}{c^2} + \rho_I\right) = 0$$

$$P = w_I \rho_I c^2.$$

<u>Without Λ:</u>

$$\boxed{\left.\begin{array}{l} H^2 = \dfrac{\dot{a}^2}{a^2} = \dfrac{8\pi G}{3}\rho_I - \dfrac{kc^2}{a^2} \\[2ex] \dot{\rho}_I + 3H(t)\left(1 + w_I\right)\rho_I = 0 \end{array}\right\}}$$

$$\boxed{\Omega \equiv \frac{8\pi G \rho_I}{3H^2}} \; ; \quad 1 = \Omega - \frac{kc^2}{a^2H^2}$$

$$\boxed{\Omega - 1 = \frac{kc^2}{a^2H^2}}$$

$$a\dot{\rho}_I + 3\dot{a}\,\rho_I\,(1+w_I) = 0$$

$$\frac{d}{dt}\left[\rho_I a^{3(1+w_I)}\right] = 0$$

$$\rho_I \sim \text{const}/a^{3(1+w_I)} = \rho_{0,I}/a^{3(1+w_I)}$$

$$H^2 = \frac{8\pi G}{3}\,\frac{\rho_{0,I}}{a^{3(1+w_I)}} - \frac{kc^2}{a^2}$$

$$a^2H^2 = \frac{8\pi G\rho_0}{3}\,\frac{1}{a^{3w_I+1}} - kc^2$$

$$\Omega - 1 = \frac{kc^2}{\alpha\, a^{-1-3w_I} - kc^2} \qquad \alpha = \frac{8\pi G\rho_{0,I}}{3}$$

Need: $-1-3w_I > 0$ to have $\Omega \to 1$

for large a: $\Rightarrow$ $\boxed{w_I < -1/3}$

Now, ignore curvature term.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho_{0,I}}{3} \frac{1}{a^{3(1+w)}}$$

$$\Rightarrow \dot{a}^2 = \frac{\alpha}{a^{3w+1}} \Rightarrow da \cdot a^{\frac{3w+1}{2}} = \sqrt{\alpha}\, dt$$

$$\left(\alpha \equiv \frac{8\pi G \rho_{0,I}}{3} \right)$$

1) $$\int_0^a a^{\frac{3w+1}{2}} da = \sqrt{\alpha} \int_{t_{in}}^{t} dt \Rightarrow$$

$$a(t) = \varkappa \, (t - t_{in})^{\frac{2}{3(w+1)}},$$

$$\varkappa = \left[\frac{3}{2} (w+1) \sqrt{\alpha} \right]^{\frac{2}{3(w+1)}}$$

Assumes $w > -1$, Big Bang at $t = t_{in}$ $(a(t_{in}) = 0)$.

2) $w = -1$: $$\frac{\dot{a}^2}{a^2} = \alpha \Rightarrow a(t) = a_0 e^{\sqrt{\alpha}(t - t_0)}$$

No BB at finite time.

3) Now for $w < -1$:

anticipate $a(t) \sim \dfrac{1}{(t_f - t)^{\nu}}$, $\nu > 0$.

Careful integration:

$$\int_a^{\infty} a^{\frac{3w+1}{2}} \, da = \sqrt{\alpha} \int_t^{t_f} dt$$

$$\Rightarrow a(t) = \left[\left(-\frac{3}{2}(w+1) \right) \sqrt{\alpha} \, (t_f - t) \right]^{\frac{2}{3(w+1)}}$$

i.e. $\nu = -\dfrac{2}{3(w+1)} > 0$. Big Rip.

Event horizon: what part of the Universe (considered at t_*) will ever be accessible for observations?

$$d_{EH}(t_*) = c\, a(t_*) \int_{t_*}^{t_f} \frac{dt}{a(t)} \Rightarrow$$

$$d_{EH}(t_*) = \frac{c}{\nu+1} (t_f - t_*) \cdot \text{ finite value.}$$

Event hor. shrinks to zero at $t_* \to t_f$; $a(t_f) \to \infty$: Big Rip scenario.

The deceleration parameter q :

Einstein's eqs

$$\dot{a}^2 + kc^2 - \frac{c^2}{3}\Lambda a^2 = \frac{8\pi G}{3}\rho a^2$$

$$2a\ddot{a} + \dot{a}^2 + kc^2 - c^2\Lambda a^2 = -\frac{8\pi G}{c^2} P a^2$$

$$\Rightarrow 2a\ddot{a} = -\frac{8\pi G}{c^2} P a^2 - \frac{8\pi G}{3}\rho a^2 + \frac{2}{3}c^2\Lambda a^2$$

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = \frac{1}{2}\left[\frac{8\pi G}{3H^2}\left(\frac{3P}{c^2} + \rho\right) - \frac{2}{3}\frac{c^2\Lambda}{H^2}\right]$$

$$P = w\rho c^2$$

$$\boxed{q = \frac{1}{2}\cdot\frac{8\pi G\rho}{3H^2}(1+3w) - \frac{\Lambda c^2}{3H^2}}$$

$$q_0 = \frac{1}{2}\sum \Omega_i (1+3w_i) - \Omega_\Lambda$$

For dust: $w = 0$; also, $\Lambda = 0$. $\Rightarrow$

$$\boxed{q_0 = \frac{1}{2}\left(\Omega_M + \Omega_I (1 + 3w_I)\right)}$$

Luminosity distance (for small z):

$$d_L = \frac{c}{H_0}\left(z + \frac{1}{2}(1 - q_0) z^2 + \dots\right).$$

Now set $w_I = -1$ $\quad (\Lambda = 0)$

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_{0,I} + \frac{8\pi G}{3}\frac{\rho_{0,M}}{a^3} - \frac{kc^2}{a^2}$$

$$\boxed{\frac{\dot{a}^2}{a^2} = H_0^2\left(\Omega_I + \frac{\Omega_M}{a^3} + \frac{\Omega_k}{a^2}\right)},$$

$$\Omega_I = \Omega_{I,0} = \frac{8\pi G}{3H_0^2}\rho_{I,0}$$

$$\Omega_M = \Omega_{M,0} = \frac{8\pi G}{3H_0^2}\rho_{M,0}$$

$$\Omega_k = -kc^2/H_0^2 < 0$$

$$-\frac{a\cdot\ddot{a}}{\dot{a}^2} = \frac{1}{2}\frac{8\pi G\rho_M}{3H^2} - \frac{8\pi G\rho_I}{3H^2}\cancel{(1+3w_I)}$$

$$\Rightarrow -\frac{\ddot{a}}{a} = \frac{1}{2}\frac{8\pi G\rho_{M,0}}{3a^3} - \frac{8\pi G\rho_{I,0}}{3} =$$

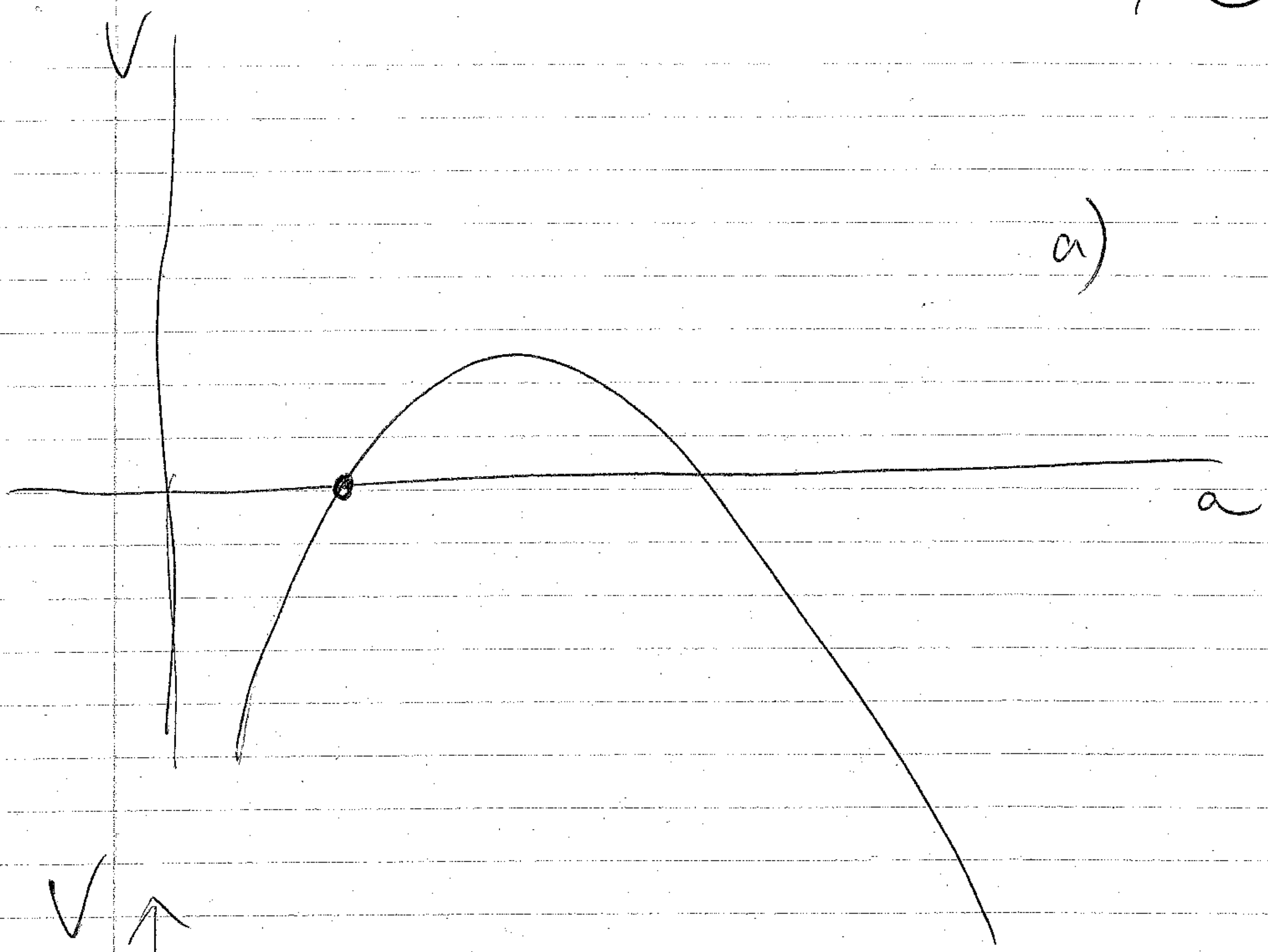
$$= \frac{\Omega_{M,0}H_0^2}{2a^3} - \Omega_{I,0}H_0^2$$

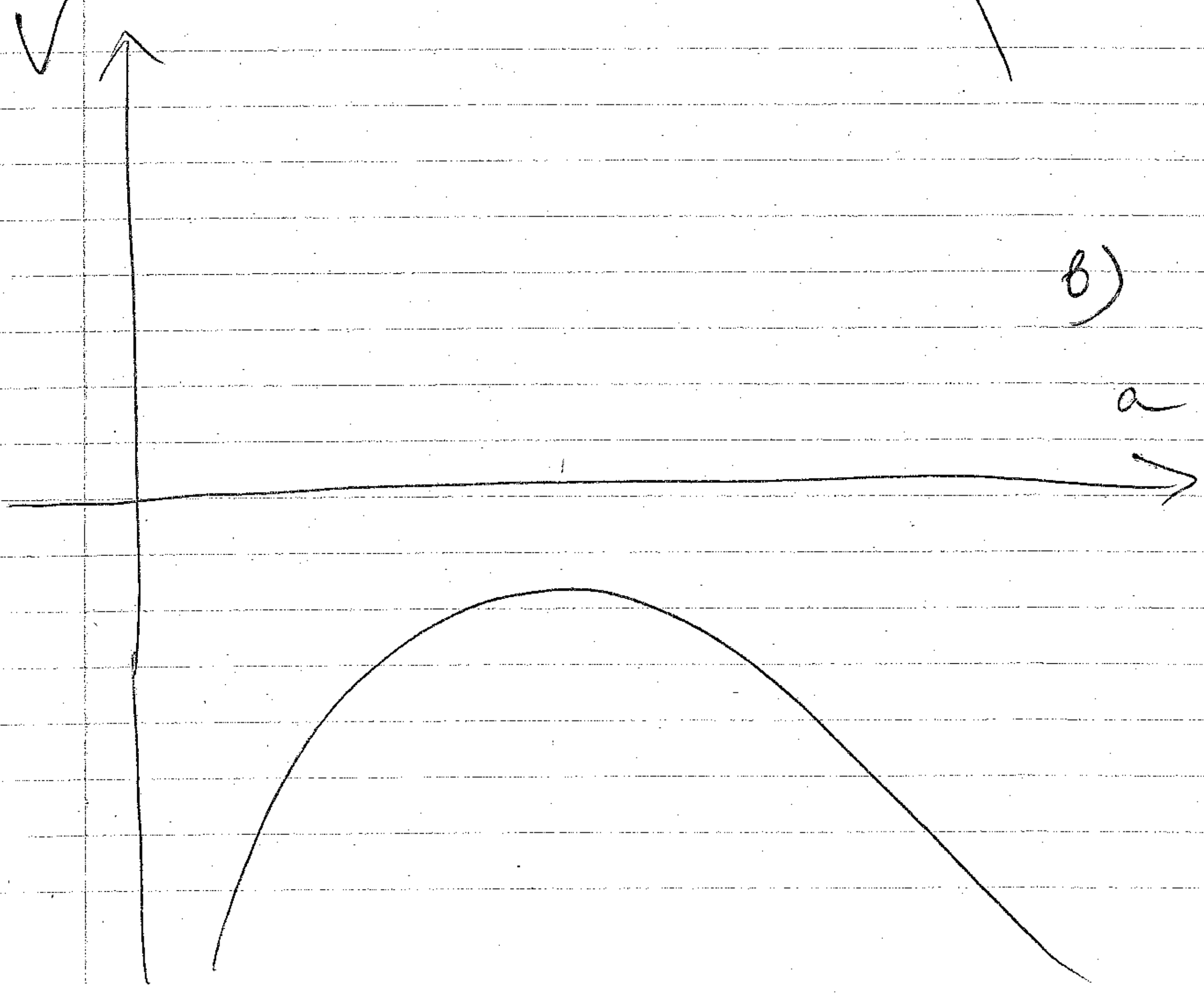
$$\boxed{\ddot{a} = -\frac{H_0^2}{2}\left[\frac{\Omega_{M,0}}{a^2} - 2\Omega_{I,0}\,a\right]}$$

$$\frac{\dot{a}^2}{H_0^2} - \left(\Omega_k + \frac{\Omega_M}{a} + \Omega_I a^2\right) = 0$$

$$\Omega_I + \Omega_M + \Omega_k = 1 \qquad (a = a_0 = 1)$$

$$V(a) = -\left(\frac{\Omega_M}{a} + \Omega_I a^2 + 1 - \Omega_I - \Omega_M\right).$$


V
a)
a


V
b)
a

$\dot{a} = 0$

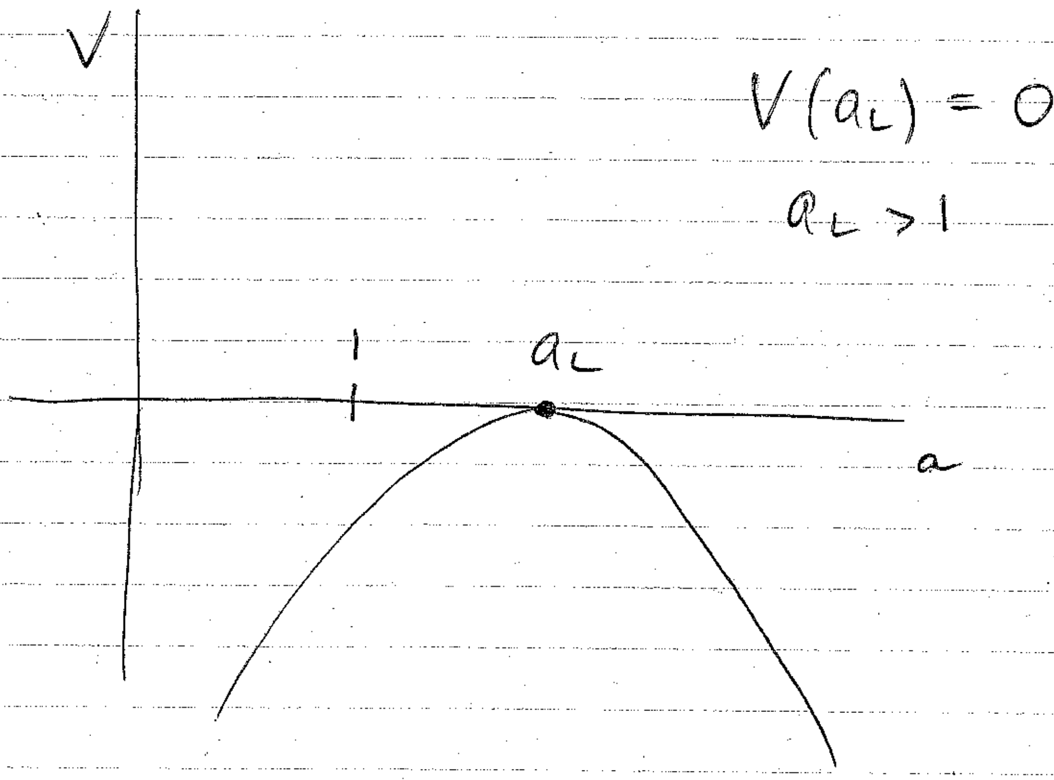
$$\Omega_I a_L^2 + \frac{\Omega_M}{a_L} + 1 - \Omega_I - \Omega_M = 0$$

Solution with $a_L > 1$?

a) if $a = a_L$ is to the right of $a = 1$ in a), the Univ. recollapses.

b) no recollapse but also no $\dot{a} = 0$

c)



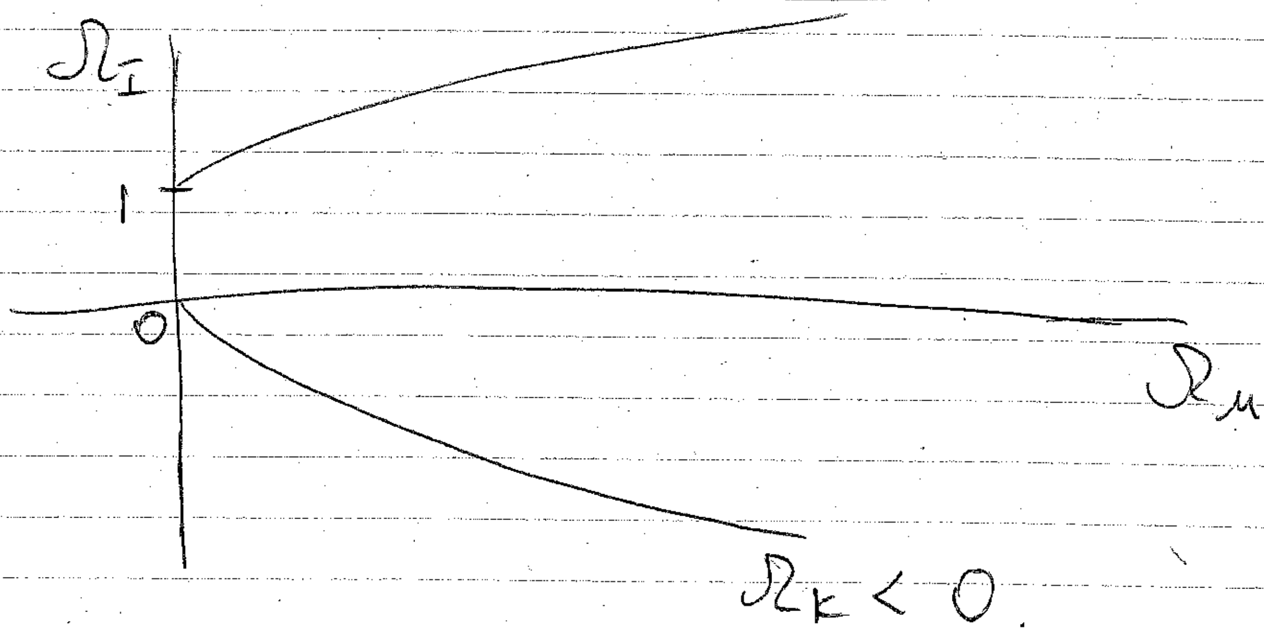
$$V' = \frac{\Omega_M}{a^2} - 2a\Omega_I$$

$$V'(a_L) = 0 \Rightarrow \Omega_M = 2\Omega_I a_L^3$$

$V(a_L) = 0$:

$$\Omega_I a_L^3 + \Omega_M + (1 - \Omega_I - \Omega_M) a_L = 0$$

$$\boxed{\frac{3\Omega_M}{2} + (1 - \Omega_I - \Omega_M)\left(\frac{\Omega_M}{2\Omega_I}\right)^{\frac{1}{3}} = 0}$$



4. • ζ is in thermal eq. with photons (bath): should be able to thermalize via interactions with photons and self-inter., without disturb. heat bath.

• $m_\zeta c^2 \gg kT$: non-rel. matter

$$P \simeq 0 .$$

• $m_\zeta c^2 \ll kT$: rel. massless gas,

$$P = \frac{1}{3} \rho c^2 .$$

• $T_0 a_0 = Ta$:

$$m_\zeta c^2 = kT = \frac{kT_0}{a} = kT_0 (1+z_\zeta).$$

$$\boxed{1 + z_\zeta = \frac{m_\zeta c^2}{kT_0}}$$

- $k = 0$, $\Omega_\zeta \ll \Omega_m$;

$$\frac{\Omega_m}{a_*^3} = \frac{\Omega_r}{a_*^4} \Rightarrow \frac{1}{a_*} = \frac{\Omega_m}{\Omega_r} = 1 + z_*$$

Before: $1 + z_\zeta > 1 + z_*$

$$\boxed{\frac{m_\zeta c^2}{k T_0} > \frac{\Omega_m}{\Omega_r}}$$

$$\Rightarrow \boxed{m_\zeta c^2 > k T_0 \frac{\Omega_m}{\Omega_r}}$$

- $\zeta \rightarrow \xi$ $\quad m_\zeta > m_\xi$ but

$$m_\zeta - m_\xi \ll m_\zeta$$

$\Rightarrow$ non-rel.

Equilibrium : $\mu_\xi = \mu_\zeta$

$$\frac{n_\zeta}{n_\xi} \simeq e^{-\left(\frac{m_\zeta c^2}{kT} - \frac{m_\xi c^2}{kT}\right)} = e^{-\frac{\Delta m c^2}{kT}}.$$

approx. 1 for $m_\zeta \sim m_\xi$.

$$\Rightarrow \quad \frac{n_S}{n_\xi} \simeq e^{-\frac{\Delta m c^2}{kT}}$$

with $T \sim 1/a$: $n_S/n_\xi \to 0$ i.e.
all heavier particles disappear if
remain in eq.

- $$\frac{d \ln N_\xi}{d \ln a} = -\frac{\Gamma}{H}\left[1 - \frac{(N_\xi^{eq})^2}{N_\xi^2}\right]$$

with $\Gamma/H \sim 1$ at $\frac{\Gamma_0 T^6}{T_0^6} \frac{a_F^2}{H_0 \sqrt{\Omega_r}} \sim 1$
(freeze out condition)

$\left(H^2 = H_0^2 \frac{\Omega_r}{a^4} \text{ for rad. only}\right)$, $T_0 a_0 = Ta$,

$$\sim \frac{\Gamma_0}{a_F^4 H_0 \sqrt{\Omega_r}} \sim 1 \quad \text{i.e.} \quad \boxed{a_F \sim \left(\frac{\Gamma_0}{H_0\sqrt{\Omega_r}}\right)^{1/4}}$$

After freeze out the eq:

$$\frac{d \ln N_\xi}{d \ln a} \simeq 0 \Rightarrow \text{comoving}$$

particle number frozen rather than approaching equilibrium (tiny) value.

Compute $\Omega_\zeta / \Omega_\gamma$ assuming that $\Omega_M = \Omega_\zeta + \Omega_3 \simeq 1$, $Z_\zeta \gg Z_*$, $\mu_\zeta = 0$ at freeze-out:

$$\frac{\Omega_\zeta}{\Omega_\gamma} = \frac{\rho_\zeta}{\rho_\gamma}, \quad \rho_\gamma = \frac{\pi^2}{15}(k_B T_0)\left(\frac{k_B T_0}{\hbar c}\right)^3.$$

$$\rho_\zeta = n_\zeta a_F^3 m_\zeta c^2 / a_0^3, \quad \text{where}$$

$$n_\zeta = \left(\frac{2\pi}{h^2}\right)^{3/2}\left(m_\zeta k_B T\right)^{3/2} e^{-m_\zeta c^2/k_B T}$$

and $T a_F = T_0 a_0$, $a_0 = 1$.

Combining these, we get

$$\frac{\Omega_\zeta}{\Omega_\gamma} = \frac{15}{(2\pi)^{3/2}\pi^2}\left(\frac{m_\zeta c^2}{k_B T_0}\right)^{5/2} a_F^{3/2} e^{-\frac{m_\zeta c^2 a_F}{k_B T_0}},$$

with $\frac{15}{(2\pi)^{3/2}\pi^2} \simeq 0.1$ and

$$a_F \sim \left(\frac{T_0}{H_0\sqrt{\Omega_r}}\right)^{1/4} \sim 3.3 \left(\frac{T_0}{H_0}\right)^{1/4}$$ with

$\Omega_r = 10^{-4}$.