GR Collection TT

1. $$\frac{1}{P+\rho c^2}\frac{dP}{dr} = -\frac{1}{2}\frac{d\ln|g_{00}|}{dr} \qquad (*)$$

- This eq. follows from $\nabla_\mu T^{\mu\nu} = 0$ (conservation of the energy-momentum tensor of matter). Einstein's field eqs.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu},$$

must be consistent with it, and it is, since $\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu}\right) = 0$, with $\nabla^\mu g_{\mu\nu} = 0$ separately (for metric connection).

- $$ds^2 = -c^2 d\tau^2 = -B(r)c^2dt^2 + A(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2$$

- the metric inside the star with EOS $P = K\rho^\gamma$.

To find B, use $*$: $\dfrac{1}{P+\rho c^2}\dfrac{dP}{dr} = -\dfrac{1}{2}\dfrac{d\ln B}{dr}$

$$\frac{K\gamma\rho^{\gamma-1}d\rho}{K\rho^{\gamma}+\rho c^2} = -\frac{1}{2}\, d\ln B$$

$$\frac{K\gamma}{c^2}\frac{\rho^{\gamma-2}d\rho}{\frac{K}{c^2}\rho^{\gamma-1}+1} = -\frac{1}{2}\, d\ln B$$

$$x = \rho^{\gamma-1} \Rightarrow dx = (\gamma-1)\rho^{\gamma-2}d\rho$$

$$\Rightarrow \frac{K\gamma}{c^2(\gamma-1)}\frac{dx}{\frac{K}{c^2}x+1} = -\frac{1}{2}\, d\ln B$$

This can be integrated to give

$$\frac{1+\frac{K\rho^{\gamma}}{\rho c^2}}{1+\frac{K\rho_0^{\gamma}}{\rho_0 c^2}} = \left(\frac{B}{B_0}\right)^{\frac{1-\gamma}{2\gamma}},$$

where $B_0 = B(\rho_0)$. Since $P = K\rho^{\gamma}$, we find

$$B = B_0\left(1+P/\rho c^2\right)^{\frac{2\gamma}{1-\gamma}},$$ where

$P(\rho_0)$ – pressure at the surface – is set to zero.

Time intervals:

$$d\tau = B^{1/2}\,dt = B_0^{1/2}\frac{dt}{\left(1+P/\rho c^2\right)^{\gamma/(\gamma-1)}}$$

$$\left.\begin{aligned} d\tau_{surf} &= B_0^{1/2}\,dt \\ d\tau_c &= B_0^{1/2}\frac{dt}{\left(1+\frac{K\rho_c^{\gamma-1}}{c^2}\right)^{\frac{\gamma}{\gamma-1}}} \end{aligned}\right\}$$

$$d\tau_c = \frac{d\tau_{surf}}{\left(1+\frac{K}{c^2}\rho_c^{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}}} \quad ; \ \gamma>1.$$

$\Rightarrow$ time (clocks) run faster at the surface.

- Outside of the (spherically-symmetric) star, the metric is the Schwarzschild metric, with $B = 1 - 2GM/c^2R$ at $r = R$ (surface of the star).

We expect the metric to be continuous (otherwise we encounter infinite forces – recall $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$ and

$\Gamma^{\alpha}_{\beta\gamma}$ depends on $\partial_{\alpha} g_{\beta\gamma}$).

In our solution, $B = B_0$ at $r = R$

$\Rightarrow B_0 = 1 - 2GM/c^2R$.

2. Kerr Black Holes

$$ds^2 = -Bc^2 dt^2 + A dr^2 + U dt\, d\varphi +$$

$$+ R^2 \sin^2\theta \, d\varphi^2 + W d\theta^2$$

B, A, U, R, W are functions of r, θ.

Fix $\theta = \pi/2$. Denoting $\dot{t} \equiv dt/dp$ etc,

$$I = \int dp \left[-Bc^2 \dot{t}^2 + A\dot{r}^2 + U\dot{t}\dot{\varphi} + R^2 \dot{\varphi}^2 \right]$$

E.o.m. follow from $\delta I = 0$ (i.e. from Euler - Lagrange eqs). Here we are asked to show this explicitly by computing

$$\delta I = I[t + \delta t] - I[t] = 0$$

(and similarly for δr and $\delta\varphi$).

$$I[t+\delta t] = \int dp \left[-Bc^2 (\dot{t} + \delta\dot{t})^2 + \right.$$

$$\left. + U(\dot{t} + \delta\dot{t})\dot{\varphi} + A\dot{r}^2 + R^2\dot{\varphi}^2 \right]$$

Now, $(\dot{t} + \delta\dot{t})^2 = \dot{t}^2 + 2\dot{t}\delta\dot{t} + \cancel{O(\delta\dot{t}^2)}$

$$\delta I = \int dp \left[-2Bc^2 \mathring{t}\, \delta \mathring{t} + U\mathring{\varphi}\, \delta \mathring{t} \right] =$$

$$= \left(-2Bc^2 \mathring{t} + U\mathring{\varphi}\right) \delta t \Big| - \int dp \left[\frac{d}{dp}\left(-2Bc^2 \mathring{t} + U\mathring{\varphi}\right)\right] \delta t = 0$$

$$\Rightarrow \quad \frac{d}{dp}\left(2Bc^2 \mathring{t} - U\mathring{\varphi}\right) = 0$$

(Note that the coefficients B, A, U, R do not depend on t, so there is no need to consider $B(t+\delta t)$ etc.)

In varying w.r.t. r, note $B(r+\delta r) = B(r) + B_r'(r)\,\delta r + O(\delta r^2)$ and similarly for other coefficients.

Also, $\mathring{B} = B_r' \mathring{r}$ (since θ is fixed), similarly for other coefficients.

We should find, after integrating by parts:

$$\delta I = \int dp \Big[-\delta r B' c^2 \dot{t}^2 + \delta r A' \dot{r}^2$$

$$- 2\,\delta r \frac{d}{dp}(A\dot{r}) + \delta r U' \dot{t}\dot{\varphi} + 2RR'\dot{\varphi}^2 \delta r \Big]$$

$\delta I = 0 \Rightarrow$ e.o.m.

$$\ddot{r} + \frac{A'}{2A}\dot{r}^2 + \frac{B'c^2}{2A}\dot{t}^2 - \frac{U'}{2A}\dot{t}\dot{\varphi} - \frac{RR'}{A}\dot{\varphi}^2 = 0$$

Finally, var. w.r.t φ gives

$$\frac{d}{dp}\Big[R^2\dot{\varphi} + \frac{U}{2}\dot{t} \Big] = 0 .$$

Of course, these e.o.m. follow from

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \qquad x^\mu = t, r, \varphi, \theta$$

but here we are asked to compute the var. explicitly.

The Schwarzschild eoms are recovered by setting $U = 0$, $R = r$.

3. Photon orbits in a Schwarzschild geometry.

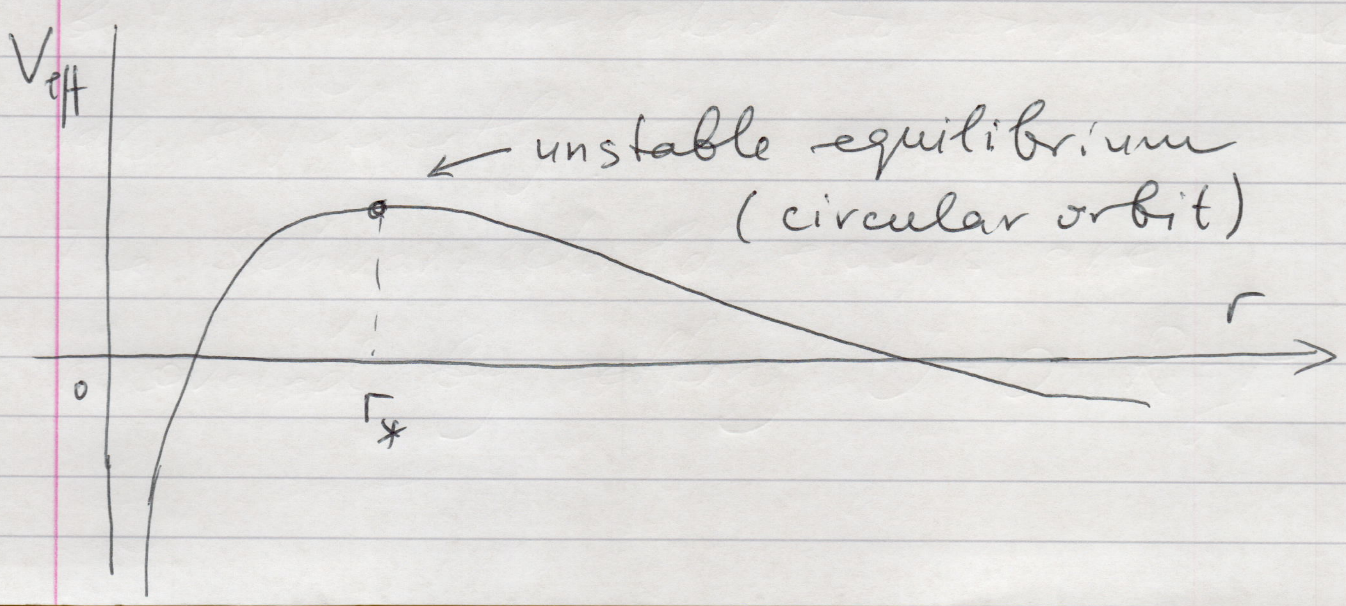
$$\begin{cases} \dot{r}^2 + \dfrac{J^2}{r^2}\left(1 - \dfrac{2GM}{c^2 r}\right) = c^2 \\ r^2 \dot{\varphi} = J \end{cases}$$

This is effectively a 1-dim motion with $\frac{m\dot{x}^2}{2} + V_{eff} = E$, one can take $V_{eff} = \frac{J^2}{r^2} - \frac{2GMJ^2}{c^2 r^3} - c^2$

and $E = 0$.

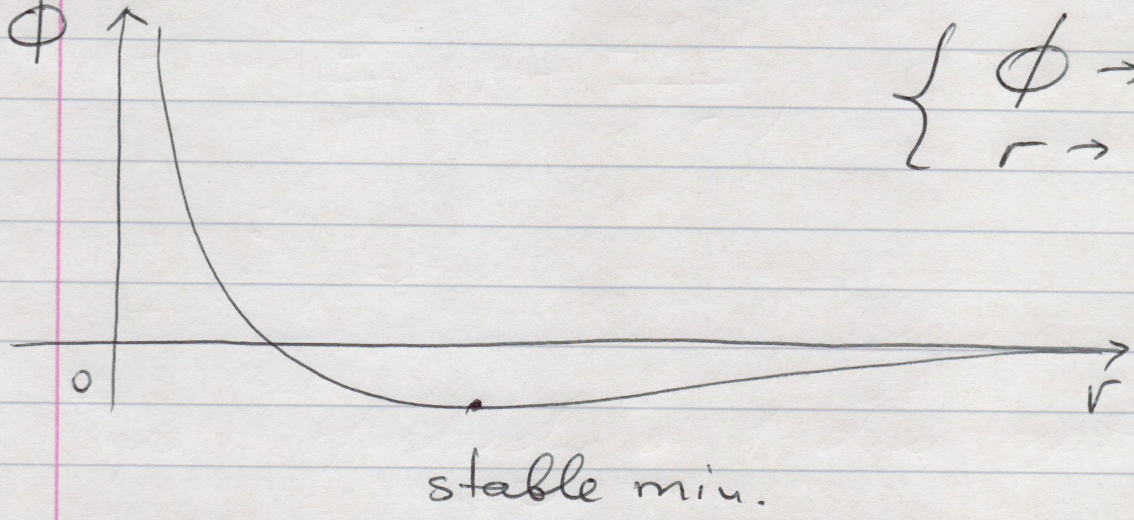
At $r \to 0$, $V_{eff} \to -\infty$; at $r \to \infty$, $V_{eff} \to -c^2$

Also, $V'_{eff} = 0 \Rightarrow$ max at $r = r_* = \frac{3GM}{c^2}$.



In the Newtonian case, with massive objects,

$$\Phi(r) = \frac{L^2}{2r^2} - \frac{GM}{r}$$



$$\begin{cases} \Phi \to +\infty \\ r \to 0 \end{cases}$$

- Use $u = 1/r \Rightarrow du = -dr/r^2$

$$d\varphi = \frac{J\,dp}{r^2} \Rightarrow dp = r^2 d\varphi / J$$

So, $\dot{r} = \frac{dr}{dp} = -\frac{r^2 du}{r^2 d\varphi} J = -J \frac{du}{d\varphi}$

$\Rightarrow$ the e.o.m. becomes $(R_s = 2GM/c^2)$

$$\boxed{u'^2 + u^2 - R_s u^3 = c^2/J^2}$$

Taking deriv. w.r.t. φ, we get

$$2u'u'' + 2uu' - 3R_s u^2 u' = 0$$

$$\Rightarrow \boxed{u'' + u - \frac{3}{2} R_s u^2 = 0}$$

$u = \text{const} \Rightarrow u_c = \dfrac{2}{3R_s} \Rightarrow r_* = \dfrac{3}{2} R_s$

as before.

Note: then the original eom gives J.

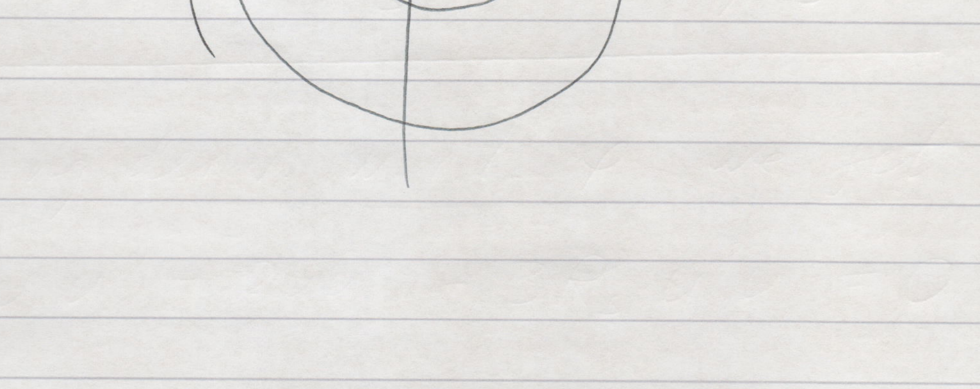
Stability analysis: $u = u_c + v$

$$\cancel{u_c''} + v'' + \underline{u_c} + v - \frac{3}{2} R_s \left(\underline{u_c^2} + 2u_c v + \cancel{v^2} \right) = 0$$

$$\Rightarrow v'' + v - 3 R_s u_c v = 0$$

i.e. $v'' + v - 2v = 0$

or $\underline{\underline{v'' - v = 0}}$ $\Rightarrow$ unstable exp.



4. FRW metric:

$$ds^2 = -c^2 dt^2 + R^2(t)\left(dr^2 + r^2 d\Omega^2\right)$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- c: speed of light

  t: cosmological time

  $R(t)$: scale factor

  r: comoving radial coord.

  θ, φ: comov. angular coord.

  Here $k = 0$: space (3d) is flat

  Friedmann eq.

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - \cancel{\frac{kc^2}{R^2}} + \cancel{\frac{\Lambda c^2}{3}}$$

ρ: energy density of all matter contributions. Note: may include cosmol. constant Λ, not shown explicitly.

- For $R(t) = A t^q$ Assuming Non-Rel

matter only - i.e. $\rho = \rho_{0,M} / R^3$

we get

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G \rho_{0,M}}{3} \frac{1}{R^3}$$

$$\Rightarrow \dot{R} = \left(\frac{8\pi G \rho_{0,M}}{3}\right)^{1/2} R^{-1/2}, \text{ with}$$

$$R(t_0) = 1$$

$$\int_0^R R^{1/2} dR = \left(\frac{8\pi G \rho_{0,M}}{3}\right)^{1/2} \int_0^t dt$$

$$\Rightarrow R(t) = \left(\frac{3}{2}\right)^{2/3} \left(\frac{8\pi G \rho_{0,M}}{3}\right)^{1/3} t^{2/3}$$

$$R(t_0) = 1 \Rightarrow t_0^{-2/3} = \left(\frac{3}{2}\right)^{2/3} \left(\frac{8\pi G \rho_{0,M}}{3}\right)^{1/3}$$

$$\Rightarrow R(t) = (t/t_0)^{2/3}.$$

Compare with $R(t) = A t^q$

$\Rightarrow q = 2/3$, $A = t_0^{-2/3}$.

- If $\Lambda \neq 0$, at late times the Friedmann eq becomes

$$\frac{\dot{R}^2}{R^2} = \frac{\Lambda c^2}{3} \Rightarrow \frac{dR}{R} = \sqrt{\frac{\Lambda c^2}{3}}\, dt$$

$$R(t) \sim \exp\left(\sqrt{\frac{\Lambda}{3}}\, ct\right)$$

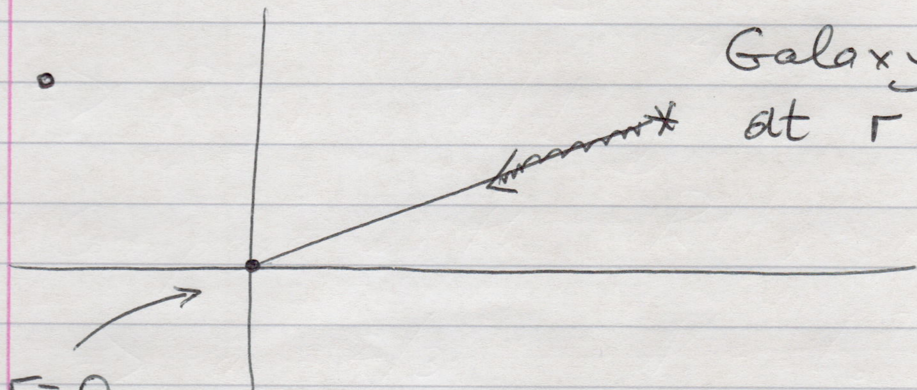
This is clearly different from

$R(t) \sim t^q \sim \exp(q \ln t)$.

- $R(t) = A t^q$
  $\dot{R} = q A t^{q-1}$ $\Bigg\}$ $\frac{\dot{R}}{R} = \frac{q}{t} \equiv H(t)$

$H(t_0) \equiv H_0 = q/t_0$

$\Rightarrow R(t) = \left(t/t_0\right)^q = \left(\frac{H_0 t}{q}\right)^q$.

- 

Galaxy emits light at r, t

$r = 0$
$t = t_0$ (i.e. us now)

Photons : $ds^2 = 0 \Rightarrow -c^2 dt^2 + R^2 dr^2 = 0$
$(k=0)$

$$\Rightarrow dr = \pm \frac{c\,dt}{R(t)}$$

choose $\ominus$ according to direction of light propagation: $\int_r^0 = -\int_t^{t_0}$

$$\Rightarrow r = c\int_t^{t_0} \frac{dt}{R} = c\,t_0^q \int_t^{t_0} \frac{dt}{t^q} =$$

$$= \frac{c\,t_0^q}{1-q}\left[t_0^{1-q} - t^{1-q}\right] =$$

$$= \frac{c\,t_0}{1-q}\left[1 - \left(\frac{t}{t_0}\right)^{1-q}\right] =$$

$$= \frac{c\,t_0}{1-q}\left[1 - R^{\frac{1-q}{q}}\right].$$

Recall: $1+z = 1/R$

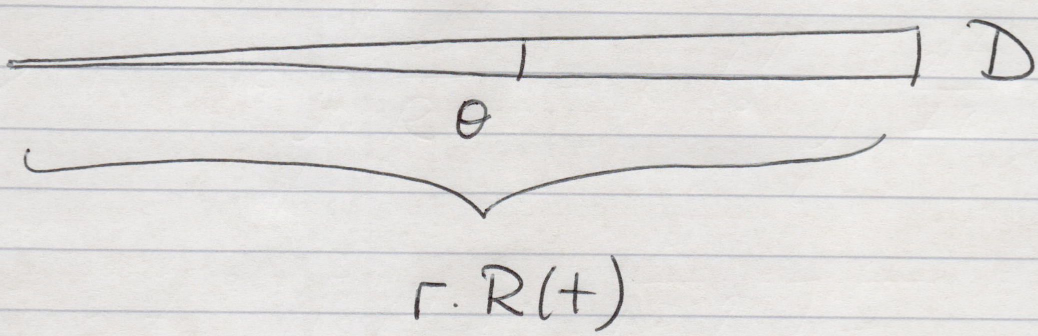
$H_0 t_0 = q \Rightarrow$

$$r = \frac{c}{H_0\alpha}\left[1 - (1+z)^{-\alpha}\right], \quad \alpha = \frac{1-q}{q}.$$

Horizon size: $z \to \infty$

$\Rightarrow D_H = {}^{c}/_{H_0 \alpha}$.

Angular size:



$\tan\theta \sim \theta$

$$\theta \sim \frac{D}{rR(t)} = \frac{D H_0 \alpha (1+z)}{c\left(1-(1+z)^{-\alpha}\right)}.$$

- $T(t)\,R(t) = \text{const}$

$T_\gamma \cdot 1 = T \cdot (1+z)^{-1} \Rightarrow T = T_\gamma (1+z)$.

i.e. $T = 1001 \cdot T_\gamma$

This is independent of q.