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# Problem set 2
# Groups and the Lorentz transformation

## Problem 1: Derivation of the rotation formula

Consider a vector $\mathbf{v} = (v_x, v_y, v_z)$. A rotation by an angle $\theta$ around an axis defined by the unit vector $\hat{\mathbf{u}} = (u_x, u_y, u_z)$ transforms $\mathbf{v}$ into $\mathbf{v}' = R\mathbf{v}$ where:

$$R\left(\theta, \hat{\mathbf{u}}\right) = I + \left(1 - \cos\theta\right)\left(\hat{\mathbf{u}} \cdot \mathbf{J}\right)^2 + \sin\theta\ \hat{\mathbf{u}} \cdot \mathbf{J},$$

with:

$$\hat{\mathbf{u}} \cdot \mathbf{J} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}.$$

Derive the rotation formula:

$$\mathbf{v}' = \mathbf{v}\cos\theta + \left(\mathbf{u}\times\mathbf{v}\right)\sin\theta + \mathbf{u}\left(\mathbf{u}\cdot\mathbf{v}\right)\left(1 - \cos\theta\right).$$

This formula may also be derived directly from geometric arguments.

## Problem 2: Lorentz transformation for a boost in an arbitrary direction

Consider a frame (S) with origin $O$ and another frame (S') with origin $O'$ moving with velocity $\mathbf{v} = v\hat{\mathbf{n}}$ relative to (S), where $\hat{\mathbf{n}}$ is a unit vector. At time $t = 0$ (as measured in frame S), the origins $O$ and $O'$ coincide.
The transformation matrix for a boost with velocity $\mathbf{v}$ is given by:

$$\Lambda = \mathrm{e}^{-\boldsymbol{\zeta}\cdot\mathbf{K}},$$

where $\boldsymbol{\zeta} = \zeta\hat{\mathbf{n}}$ is the boost vector and $\mathbf{K} = K_x\hat{\mathbf{x}} + K_y\hat{\mathbf{y}} + K_z\hat{\mathbf{z}}$, with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ being unit vectors along the $x$-, $y$- and $z$-axes, respectively. The $K_i$ represent the generators of the boost:

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

a) Show that $\left(\hat{\mathbf{n}} \cdot \mathbf{K}\right)^3 = \hat{\mathbf{n}} \cdot \mathbf{K}$, and use this result to derive:

$$\Lambda = I - \left(\sinh\zeta\right)\hat{\mathbf{n}} \cdot \mathbf{K} + \left(\cosh\zeta - 1\right)\left(\hat{\mathbf{n}} \cdot \mathbf{K}\right)^2.$$

b) Given that the trajectory of $O'$ is $\mathbf{r} = \mathbf{v}t$ in frame (S) and $\mathbf{r}' = \mathbf{0}$ in frame (S'), where $\mathbf{r}$ and $\mathbf{r}'$ are the spatial position vectors, show that $\tanh\zeta = \beta$, where $\boldsymbol{\beta} \equiv \mathbf{v}/c$. Using this result, express $\Lambda$ in terms of $\gamma = \left(1-\beta^2\right)^{-1/2}$ and the components of $\boldsymbol{\beta}$.

c) Show that the transformation matrix $\Lambda$ can also be derived by expressing a boost along an arbitrary direction $\hat{\mathbf{n}}$ as a sequence of operations: a rotation, a boost along one of the coordinate axes, followed by another rotation. [*Hint:* It might be useful to recall that for a rotation by an angle $\theta$ around an axis aligned with the unit vector $\hat{\mathbf{u}}$, the rotation matrix is given by $R(\theta, \hat{\mathbf{u}}) = I + (1-\cos\theta)(\hat{\mathbf{u}}\cdot\mathbf{J})^2 + \sin\theta\, \hat{\mathbf{u}}\cdot\mathbf{J}$].

## Problem 3: Decomposing a Lorentz transformation into boost and rotation

We consider a matrix $L$ that belongs to $\mathrm{SO}(1,3)$, meaning it satisfies $L^\intercal g L = g$.

a) Express the inverse matrix $L^{-1}$ in terms of the components of $L$, and show that:

$$\begin{aligned} L_{00}^2 - L_{01}^2 - L_{02}^2 - L_{03}^2 &= 1, \\ L_{00}L_{j0} - L_{0k}L_{jk} &= 0, \quad \text{for} \quad j = 1,2,3. \end{aligned}$$

b) Let $O$ and $O'$ denote the origins of frames (S) and (S'), respectively, and let $\mathbf{v}$ represent the velocity of $O'$ relative to frame (S). We assume that $O$ and $O'$ coincide at $t=0$. By relating the coordinates of $O'$ in frame (S) to those in frame (S'), show that $\beta_i \equiv v_i/c = -L_{0i}/L_{00}$. Then, show that $L_{00} = \gamma \equiv \left(1-\beta^2\right)^{-1/2}$.

c) Let $\Lambda(\boldsymbol{\beta})$ denote the transformation matrix corresponding to a boost with velocity $\mathbf{v} = \boldsymbol{\beta}c$, and define the matrix $R = L\Lambda^{-1}(\boldsymbol{\beta}) = L\Lambda(-\boldsymbol{\beta})$. Show that $R$ belongs to $\mathrm{SO}(1,3)$. Using the expression for $\Lambda(\boldsymbol{\beta})$ in terms of $\gamma$ and the components of $\boldsymbol{\beta}$ from problem 2, show that $R_{00} = 1$ and $R_{i0} = R_{0i} = 0$ for $i = 1,2,3$. Argue that $R$ represents a rotation. This demonstrates that $L = R\Lambda(\boldsymbol{\beta})$, decomposing the Lorentz transformation into a boost and a spatial rotation.

## Problem 4: Lorentz transformations and their expressions through SU(2)

SU(2) is the group of unitary $2\times 2$ complex matrices with determinant 1. This means that if $M \in \mathrm{SU}(2)$, then $M^\dagger M = MM^\dagger = I$ and $\det(M) = 1$, where $M^\dagger$ is the Hermitian adjoint (complex conjugate transpose) of $M$.

a) Show that the group has three independent generators.

b) Let $J_i$, for $i = 1, 2, 3$, be three generators of SU(2). A matrix $M$ in SU(2) can be written to first order in some small (complex) parameters $z_1$, $z_2$ and $z_3$ as $M = I + \mathrm{i}z_1 J_1 + \mathrm{i}z_2 J_2 + \mathrm{i}z_3 J_3$. Show that the matrices $K_i = z_i J_i$ are Hermitian, meaning $K_i^\dagger = K_i$. Express the components of $K_i$ in terms of real numbers, and show that $M$ can be written as $M = I - \mathrm{i}\,(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$, where the $a_i$ are real numbers and the $\sigma_i$ are the Pauli spin matrices, given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[*Hint*: Consider first the case where $z_2 = z_3 = 0$]

c) We now establish a connection between SU(2) and SO(1, 3). Show that any $2 \times 2$ Hermitian complex matrix $S$ can be expressed as $ctI + x\sigma_1 + y\sigma_2 + z\sigma_3$, where $ct$, $x$, $y$ and $z$ are real numbers. Calculate the determinant of $S$. Now, consider a $2 \times 2$ matrix $L$ such that $\det(L) = 1$, and define the transformation $S' = LSL^\dagger$. Show that $S'$ is Hermitian and can be interpreted as representing a new spacetime position vector $X'$, which is a Lorentz transformation of the original spacetime position vector $X = (ct, x, y, z)^\intercal$.

d) We first consider the case where the transformation corresponds to a boost. Assume that $L$ is of the form:

$$L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

with $a$ and $b$ real, and $\det(L) = ab = 1$. Show that the transformation $LSL^\dagger$ represents a boost along the $z$-axis. Further, demonstrate that $L$ can be written as $L = (\cosh q)\, I - (\sinh q)\, \sigma_3$, where $q$ is the rapidity.

e) Next, we consider the case where the transformation is a pure rotation. Under a rotation of the spatial coordinates, the time-dependent term remains unchanged, so we can focus on matrices of the form $S = x\sigma_1 + y\sigma_2 + z\sigma_3$. Show that the condition $\mathrm{tr}(S') = \mathrm{tr}(S) = 0$ for all $x$, $y$ and $z$, together with $\det(L) = 1$, implies that $L$ is unitary. Hence, $L$ can be expressed as:

$$L = \mathrm{e}^{-\mathrm{i}(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)},$$

where the $a_i$ are real. Consider the case where $a_1 = a_2 = 0$ and let $a_3 = \theta$. Show that:

$$L\,(0, 0, \theta) = \mathrm{e}^{-\mathrm{i}\theta\sigma_3} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}.$$

Which rotation in SO(3) does the transformation $LSL^\dagger$ correspond to? Is there another element of SU(2) that corresponds to the same rotation in SO(3)?