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Problem set 1
Symmetries

Problem 1: Conserved quantity under a Galilean transformation

Consider the infinitesimal Galilean transformation $t' = t$ and $\mathbf{r}' = \mathbf{r} + \epsilon \mathbf{v} t$, between two reference frames (F) and (F'), where (F') moves with a constant velocity $-\epsilon \mathbf{v}$ relative to (F).

a) Show that the action for a free particle is not conserved under this transformation. However, show that the variation δL of the Lagrangian under this transformation is a total time derivative, which ensures the transformation is a symmetry.

b) The version of Noether's theorem derived in the lectures applies when the action is conserved. For $t' = t$, this requires $\delta L = \text{const.}$ Now, consider an infinitesimal transformation $q_i(t) \to q_i'(t') = q_i(t) + \epsilon \eta_i (q_k, t)$ and $t' = t$, such that δL is not constant but equals $\epsilon \mathrm{d} f / \mathrm{d} t$, where $L(q_i, \dot{q}_i, t)$ and $f(q_i, t)$. Show that this leads to another form of Noether's theorem:

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \eta_i - f = \text{const.}$$

c) Apply this result to the case of a Galilean transformation for a system of N particles.

Problem 2: Conservation of the Laplace-Runge-Lenz vector

Consider a non-relativistic particle of mass m and position $\mathbf{r}$, moving in a central potential of the form $V(r) = -k/r$, with $k > 0$. Let $\mathbf{L}$ denote its angular momentum.

a) Consider an infinitesimal transformation of the form $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \boldsymbol{\epsilon} \times \mathbf{L}$, where $\boldsymbol{\epsilon}$ represents an infinitesimal vector. Show that the change in the Lagrangian due to this transformation can be written as:

$$\delta L = -\frac{mk}{r^3} \left[\mathbf{r}^2 (\boldsymbol{\epsilon} \cdot \dot{\mathbf{r}}) - (\boldsymbol{\epsilon} \cdot \mathbf{r}) (\mathbf{r} \cdot \dot{\mathbf{r}}) \right].$$

Further, prove that:

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t} \left(-mk \boldsymbol{\epsilon} \cdot \frac{\mathbf{r}}{r} \right)$$

b) Use Noether's theorem, as established in Problem 2, to show that the following vector, known as the Laplace-Runge-Lenz vector, is a constant of motion:

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - km\hat{\mathbf{r}},$$

where $\mathbf{p}$ is the momentum and $\hat{\mathbf{r}}$ is the unit vector in the radial direction. This is a useful result, as the equation of planetary motion can be derived directly from the fact that $\mathbf{A}$ is constant.

Problem 3: Conservation of energy for fields

Consider a Lagrangian density $\mathcal{L}$ that depends on a field $\phi(\mathbf{r}, t)$, its time derivative $\partial_t\phi$ and its spatial derivatives $\partial_i\phi$, but not explicitly on $\mathbf{r}$ or t. The action over an arbitrary spatial domain Ω and between arbitrary times t_A and t_B is given by:

$$S = \int_{t_A}^{t_B} \mathrm{d}t \int_{\Omega} \mathrm{d}v \, \mathcal{L}\left(\phi, \partial_t\phi, \partial_i\phi\right),$$

where $\mathrm{d}v = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is a small volume element.

a) Derivation of Noether's theorem for fields: Consider a time translation $t \to t' = t + \epsilon$ by an infinitesimal amount ϵ. Since the Lagrangian density $\mathcal{L}$ does not explicitly depend on time, this transformation leaves the action invariant. By following the derivation from section 1.3.2 of the lecture notes (or otherwise), derive Noether's theorem for the field, which takes the form:

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\phi\right)}\partial_t\phi - \mathcal{L}\right) + \partial_i\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_i\phi\right)}\partial_t\phi\right) = 0. \tag{1}$$

Here, ∂_i denotes a spatial derivative, where $i = 1, 2, 2$, and repeated indices imply summation over i.

b) Conservation of energy and the Hamiltonian: Integrate equation (1) over the entire volume $\mathcal{V}$ of the system. Assuming that the field vanishes at the boundaries, show that:

$$H \equiv \int_{\mathcal{V}} \mathrm{d}v \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\phi\right)}\partial_t\phi - \mathcal{L}\right) = \text{const.} \tag{2}$$

Here, H is the Hamiltonian and represents the total energy of the system, so the expression within the integrand is an energy density. In light of this result, provide a physical interpretation of equation (1).

c) Hamiltonian for a one-dimensional crystal: Calculate the Hamiltonian for a one-dimensional crystal, where the Lagrangian density was derived in the lecture notes:

$$\mathcal{L} = \frac{1}{2}\rho\left(\partial_t\phi\right)^2 - \frac{1}{2}T\left(\partial_x\phi\right)^2 - \mathcal{V}(\phi). \tag{3}$$

Here, $\phi(x,t)$ represents the displacement of the atom located at x from its equilibrium position, ρ is the mass density, T is the tension, and $\mathcal{V}(\phi)$ is the potential energy density.

Problem 4: Lagrangian for the electromagnetic field

Consider the following Lagrangian density for the electromagnetic field:

$$\mathcal{L} = \frac{\epsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2.$$

The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are related to the scalar potential ϕ and vector potential $\mathbf{A}$ as follows:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5}$$

The Lagrangian density can be viewed as a function of the fields ϕ, A_x, A_y and A_z.

a) Apply the Euler-Lagrange equations for fields to show that this Lagrangian yields two of Maxwell's equations in vacuum. How can the other two Maxwell's equations be derived?

b) The results from problem 3 can be generalised to a Lagrangian density that depends on multiple fields ψ_k and their derivatives. Invariance under time translation then yields Noether's theorem in the form $\partial_t\mathcal{E} + \partial_i S_i = 0$, where:

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial\left(\partial_t \psi_k\right)}\partial_t\psi_k - \mathcal{L}, \quad S_i = \frac{\partial \mathcal{L}}{\partial\left(\partial_i \psi_k\right)}\partial_t\psi_k, \tag{6}$$

with summation over k implied. Show that:

$$\mathcal{E} = \frac{\epsilon_0}{2}\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}^2 - \epsilon_0\boldsymbol{\nabla}\cdot(\phi\mathbf{E}).$$

The last term is a divergence, representing a flux of energy through a surface when a volume integral is carried out, and therefore should belong to the flux $\partial_i S_i$ rather than the energy density $\mathcal{E}$. Recall that $\mathcal{E}$ was derived from the definition of H, which is an integral over the entire space, under the assumption that divergence terms do not contribute, as the fields were assumed to vanish at the boundaries. To remove this term from the energy density, we can apply a gauge transformation $\phi \to \tilde{\phi} = \phi - \partial_t\Gamma$ and $\mathbf{A} \to \tilde{\mathbf{A}} = \mathbf{A} + \boldsymbol{\nabla}\Gamma$, where we choose the function Γ such that $\partial_t\Gamma = \phi$, i.e. $\tilde{\phi} = 0$ (this is called the Weyl, or temporal, gauge). Show that $\mathbf{E}$ and $\mathbf{B}$ are invariant under this gauge transformation.
Using the fields $\tilde{\phi}$ and $\tilde{\mathbf{A}}$ as variables for the Lagrangian density, show that:

$$\mathcal{E} = \frac{\epsilon_0}{2}\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}^2 \quad \text{and} \quad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}.$$

Problem 5: Lagrangian for the Schrödinger equation

Show that, if ψ and $\psi^\star$ are treated as two independent field variables, the Lagrangian density:

$$\mathcal{L} = \frac{h^2}{8\pi^2 m}\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\psi^\star + V\psi\psi^\star - \frac{\mathrm{i}h}{4\pi}\left(\psi^\star\partial_t\psi - \psi\partial_t\psi^\star\right),$$

leads to the Schrödinger equation:

$$\frac{-h^2}{8\pi^2 m}\boldsymbol{\nabla}^2\psi + V\psi = \frac{\mathrm{i}h}{2\pi}\partial_t\psi.$$

Calculate the energy density.