

SECOND PUBLIC EXAMINATION

Honour School of Physics Part C: 4 Year Course

Honour School of Physics and Philosophy Part C

C6: THEORETICAL PHYSICS SOLUTIONS

TRINITY TERM 2021

Monday, XX June, START TIME – END TIME

*Answer **four** questions.*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

*The numbers in the margin indicate the weight that the Examiners anticipate
assigning to each part of the question.*

Do NOT turn over until told that you may do so.

1. A one-dimensional simple harmonic oscillator has mass m and frequency ω .

(a) Write down the path integral for the propagator $\langle x|U(t;0)|x'\rangle$, specifying any conditions on the paths involved in this expression. [3]

(b) Write down the equations of motion obeyed by a classical path x_{cl} . Show that the propagator can be written in the form

$$\langle x|U(t;0)|x'\rangle = A(t)e^{\frac{i}{\hbar}S_{\text{cl}}[x,x',t]}. \quad (1)$$

where $S_{\text{cl}}[x,x',t]$ is the action for a classical path x_{cl} that starts at position x' at time $t=0$ and ends at position x at time t , and

$$A(t) = \int \mathcal{D}y \exp \left[\frac{i}{\hbar} \int_0^t dt' \frac{m}{2} \left\{ \left(\frac{dy}{dt'} \right)^2 - \omega^2 y^2 \right\} \right] \quad (2)$$

is a path integral over paths satisfying $y(0) = y(t) = 0$, and is therefore independent of x and x' . (Hint: express the paths in the path integral as $x(t) = x_{\text{cl}}(t) + y(t)$ and use the equations of motion obeyed by x_{cl} .) [6]

(c) By considering the most general solution to the classical equations of motion and imposing appropriate boundary conditions, show that

$$S_{\text{cl}}[x,x',t] = \frac{m\omega}{2\sin\omega t} [(x^2 + x'^2)\cos\omega t - 2xx']. \quad (3)$$

[6]

(d) Using the results of parts (b) and (c) and the fact that the ground state wavefunction

$$\psi_0(x) = \left(\frac{m\omega}{\sqrt{\pi\hbar}} \right)^{1/2} e^{-m\omega x^2/2\hbar} \quad (4)$$

must be an eigenstate of the time evolution operator, deduce an expression for $A(t)$ in terms of m , ω , t , and $\hbar$. (Hint: start from the eigenvalue equation $U(t;0)|\psi_0\rangle = e^{-iE_0 t/\hbar}|\psi_0\rangle$ with $E_0 = \hbar\omega/2$.) [8]

(e) Combining the results above, write down an explicit expression for the path integral of the harmonic oscillator $\langle x|U(t;0)|x'\rangle$ in terms of x, x', t, m, ω , and $\hbar$. Check your answer by verifying that in the limit $\omega \rightarrow 0$, it reduces to the free particle propagator,

$$\langle x|U_{\text{free}}(t;0)|x'\rangle = \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im}{2\hbar t}(x-x')^2}. \quad (5)$$

[2]

You may find the following formula useful: if $\text{Re}(z) > 0$,

$$\int_{-\infty}^{\infty} dy e^{-\frac{1}{2}zy^2 + Jy} = \sqrt{\frac{2\pi}{z}} e^{J^2/2z}.$$

2. In some systems, the low-temperature ordered phase changes from spatially uniform (the “ferromagnet”, denoted F) to spatially modulated, (e.g. with $\phi(x)$ periodic in space, denoted M) as a function of some experimentally tunable parameter. Including the high-temperature phase disordered phase (the “paramagnet”, P), there are three distinct phase boundaries, P-F, P-M, and F-M, which meet at a *Lifshitz point*, LP. A transition from a spatially uniform to a periodically modulated phase is first-order if the period becomes nonzero discontinuously across the transition.

A Landau-Ginzburg theory that captures this behaviour for a real or complex scalar order parameter ϕ in one spatial dimension takes the form

$$\beta\mathcal{H} = \int dx [\alpha_2(T)|\phi|^2 + \rho_2|\partial_x\phi|^2 + \rho_4|\partial_x^2\phi|^2 + \alpha_4|\phi|^4], \quad (1)$$

where α_2 changes sign from positive to negative as the temperature is lowered, and $\alpha_4 > 0$ as usual, but we allow ρ_2 to be either negative or positive as some external parameter is varied. We ignore any temperature dependence in α_4 , ρ_2 , and ρ_4 .

(a) Explain why we can ignore ρ_4 when $\rho_2 > 0$, but not when $\rho_2 < 0$. [2]

(b) When ϕ is a complex scalar, consider a mean-field *ansatz* in which a single Fourier mode $\phi(x) = A_k e^{ikx}$ with $k = \pm k_0$ is non-vanishing. Show that it has a free energy density that depends on the choice of k_0 :

$$f = (\alpha_2 + \rho_2 k_0^2 + \rho_4 k_0^4)|A_k|^2 + \alpha_4|A_k|^4. \quad (2)$$

By solving the saddle-point equations, determine the values of k_0 and A_k that minimize f , and hence compute the free energy densities f_P, f_F, f_M of the three phases, as functions of $\alpha_2, \rho_2, \rho_4 > 0$ and $\alpha_4 > 0$. Use these to determine the phase boundaries in the $\rho_2 - \alpha_2$ plane, identifying the order of each transition. Sketch the phase diagram. [12]

(c) The Lifshitz point LP is at $\alpha_2 = \rho_2 = 0$. Using the Gaussian approximation and assuming that $\alpha_2(T) = At$ where $t = \frac{T-T_c}{T_c}$, determine the mean-field correlation length exponent as this point is approached from phase P along the line $\rho_2 = 0$. (You should not need to explicitly perform any integrals.) [4]

When ϕ is a real scalar, we must consider a modified mean-field *ansatz* $\phi(x) = A_k \cos(kx)$ with $k = \pm k_0$. In this case, the F-M phase boundary changes its location and its order; the other phase boundaries are unchanged. Therefore, for the remainder of this question, you can focus on $\alpha_2 < 0$.

(d) For $k_0 \neq 0$, (2) is no longer valid. Determine the free energy density for $k_0 \neq 0$, and solve the resulting saddle-point equations to determine the modified free energy density $\tilde{f}_M$. (*Hint: it may help to divide the system up into unit cells of length $2\pi/k_0$.*) [4]

(e) The free energy f_F of the ferromagnet (with $k_0 = 0$) is unchanged from part (b). By comparing f_F and $\tilde{f}_M$ for $\alpha_2 < 0$ as ρ_2 is varied, determine the new F-M phase boundary and the order of the transition. [3]

You may find the following integrals useful:

$$\int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \cos^2 x dx = \pi, \quad \int_0^{2\pi} \sin^4 x dx = \int_0^{2\pi} \cos^4 x dx = \frac{3\pi}{4}.$$

Q2: 25
Section:50

3. Consider a charged particle of unit mass $m = 1$ and unit charge $q = 1$ traveling along a one-dimensional tube in an electric field $\mathcal{E}$. The particle is subject to a viscous damping force proportional to its velocity ($F_{\text{vis}}(t) = -\gamma v(t)$) as well a random force $\eta(t)$ due to collisions with other particles. The collisions are assumed to be isotropic and instantaneous, so that the force they generate vanishes on average $\langle \eta(t) \rangle = 0$, and the forces at different times are uncorrelated, $\langle \eta(t)\eta(t') \rangle = \Gamma \delta(t - t')$.

(a) Write down a system of two stochastic first-order differential equations for the velocity $v(t)$ and position $x(t)$ of the particle. Which of $x(t), v(t)$ corresponds to a Markov process when considered independently of the other? Explain. [4]

(b) Calculate the average velocity and show that for long times $t \gg \gamma^{-1}$ it tends to a constant v_d , termed the *drift velocity*. Determine v_d in terms of the external field $\mathcal{E}$ and the damping coefficient γ . [5]

(c) Compute the variance of the velocity, and use this to establish how Γ and γ must be related in order to be consistent with thermal equilibrium at temperature T . [5]

For the rest of this question, we will take $\bar{x}(t) = \langle x(t) \rangle$ to represent the *average* position, so that $\delta x(t) = x(t) - \langle x(t) \rangle$ describes fluctuations of x about its average.

(d) Show that for $t \gg \gamma^{-1}$ the average position grows ballistically, $\bar{x}(t) \sim \text{const} + ct$. Determine c in terms of v_d . [4]

(e) Use the fact that

$$\frac{d(\delta x(t))}{dt} = v(t) - \langle v(t) \rangle = \int_0^t dt' \eta(t') e^{-\gamma(t-t')} \quad (1)$$

to argue that that when observed on time scales large compared to γ^{-1} , $\delta x(t)$ satisfies a particular limit of the Langevin equation. Write down (without proof) the partial differential equation satisfied by the probability distribution $P_1(\delta x, t)$. [4]

(f) Combining the results of parts (d) and (e), discuss the behaviour of the probability distribution $P_1(x, t)$. On physical grounds, what kind of random walk process do you expect will lead to a similar probability distribution? [3]

Q3: 25
Section:75

① (Path integral of a harmonic oscillator)

(a) The propagator may be written as the path integral

$$\langle x | U(t; 0) | x' \rangle = \int \mathcal{D}x(t) e^{i S[x(t)]/\hbar} \quad [1]$$

where

$$S[x(t')] = \int_0^t dt' \left\{ \frac{1}{2} m \left(\frac{dx(t')}{dt'} \right)^2 - \frac{1}{2} m \omega^2 x(t')^2 \right\} \quad [1]$$

with $x(t'=0) = x'$; $x(t'=t) = x$ [1]

(i.e. paths start at x' and end at x)

(b) Let us write $x(t') = x_{cl}(t') + y(t')$,
(it follows that $y(0) = y(t) = 0$),

Classical equations of motion: $\ddot{x}_{ce} = -\omega^2 x_{ce}$

$$\begin{aligned}
 S[x(t')] &= \int_0^t dt' \frac{1}{2} m \{ (\dot{x}_{cl} + \dot{y})^2 - \omega^2 (x_{cl} + y)^2 \} \\
 &= \int_0^t dt' \frac{1}{2} m \{ \dot{x}_{cl}^2 - \omega^2 x_{cl}^2 \} + \int_0^t dt' \frac{1}{2} m \{ \dot{y}^2 - \omega^2 y^2 \} \\
 &\quad + \int_0^t dt' m \{ \dot{x}_{cl} \dot{y} - \omega^2 x_{cl} y \} \\
 &= S_{cl}[x, x', t] + \int_0^t dt' \frac{1}{2} m \{ \dot{y}^2 - \omega^2 y^2 \} \\
 &\quad - \int_0^t dt' m y \{ \dot{x}_{cl} + \omega^2 x_{cl} \}
 \end{aligned}$$

Now, $\mathcal{D}x(t') = \mathcal{D}y(t')$, since $x = x_{ce} + y$ has Jacobian 1. [1]

$$\therefore \langle x | u(t, 0) | x' \rangle = e^{i S_{ce}[x, x', t]/\hbar} \int \mathcal{D}y \exp \left\{ \frac{i}{\hbar} \int_0^t dt' \frac{1}{2} m (\dot{y}^2 - \omega^2 y^2) \right\}$$

Now since the paths y satisfy $y(0) = y(t) = 0$ independent of x, x' , $\int \mathcal{D}(y) (\dots) = A(t)$ is indep. of x, x' , so

$$\langle x | u(t, 0) | x' \rangle = A(t) e^{i S_{ce}[x, x', t]/\hbar}$$

(c) We have $\ddot{x}_{ce} = -\omega^2 x_{ce}$

general soln:

$$x_{ce}(t') = A \cos \omega t' + B \sin \omega t'.$$

A, B set by boundary conditions on x_{ce} :

$$x' = x_{ce}(t'=0) = A; \quad x = x_{ce}(t'=t) = A \cos \omega t + B \sin \omega t.$$

$$\therefore A = x'; \quad B = \frac{x - x' \cos \omega t}{\sin \omega t} \quad [1]$$

$$\text{Now } \dot{x}_{ce}(t') = \omega (B \cos \omega t' - A \sin \omega t')$$

$$\begin{aligned} \therefore S_{ce}[x, x', t] &= \int_0^t dt' \frac{1}{2} m (\dot{x}_{ce}^2(t') - \omega^2 x_{ce}^2(t')) \\ &= \frac{1}{2} m \omega^2 \int_0^t dt' \left[(B \cos \omega t' - A \sin \omega t')^2 - (A \cos \omega t' - B \sin \omega t')^2 \right] \\ &= \frac{1}{2} m \omega^2 \int_0^t dt' \left[(B^2 - A^2) (\cos^2 \omega t' - \sin^2 \omega t') - 4AB \sin \omega t' \cos \omega t' \right] \\ &= \frac{1}{2} m \omega^2 \int_0^t dt' \left[(B^2 - A^2) \cos 2\omega t' - 2AB \sin 2\omega t' \right] \end{aligned}$$

$$= \frac{m\omega^2}{2} \left[(B^2 - A^2) \frac{\sin 2\omega t}{2\omega} - \frac{2AB(1 - \cos 2\omega t)}{2\omega} \right]$$

$$= \frac{m\omega}{2} \left[\left\{ \left(\frac{x - x' \cos \omega t}{\sin \omega t} \right)^2 - x'^2 \right\} \sin \omega t \cos \omega t - 2 \left(\frac{x - x' \cos \omega t}{\sin \omega t} \right) x' \sin^2 \omega t \right]$$

$$= \frac{m\omega}{2 \sin \omega t} \left[(x^2 + x'^2 \cos^2 \omega t - 2xx' \cos \omega t - x'^2 \sin^2 \omega t) \cos \omega t - 2xx' \sin^2 \omega t + 2x'^2 \cos \omega t \sin^2 \omega t \right]$$

$$= \frac{m\omega}{2 \sin \omega t} \left[\{x^2 + x'^2 (\cos^2 \omega t - \sin^2 \omega t + 2 \sin^2 \omega t)\} \cos \omega t - 2xx' \{\cos^2 \omega t + \sin^2 \omega t\} \right]$$

$$S_{ee}[x, x', t] = \frac{m\omega}{2 \sin \omega t} \left[(x^2 + x'^2) \cos \omega t - 2xx' \right] \quad [5]$$

(d) We have

$$\langle x | u(t; 0) | x' \rangle = A(t) e^{\frac{im\omega}{2\hbar \sin \omega t} [(x^2 + x'^2) \cos \omega t - 2xx']}$$

Now we require

$$u(t; 0) | \psi_0 \rangle = e^{\frac{-i\omega t}{2}} | \psi_0 \rangle \quad (E_0 = \frac{\hbar\omega}{2})$$

Inserting resolutions of identity

$$e^{\frac{-i\omega t}{2}} \langle x | \psi_0 \rangle = \int dx' \langle x | u(t; 0) | x' \rangle \langle x' | \psi_0 \rangle$$

$$e^{\frac{-i\omega t}{2}} \psi_0(x) = A(t) \int dx' e^{\frac{im\omega}{2\hbar \sin \omega t} [(x^2 + x'^2) \cos \omega t - 2xx']} \psi_0(x') \quad [2]$$

let us write $\psi_0(x) = N e^{-mw x^2/2\hbar}$. Defining $\xi = \sqrt{\frac{\hbar}{mw}}$
and $y = \frac{x}{\xi}$, $y' = \frac{x'}{\xi}$ we have

$$e^{-i\omega t/2} (A(t))^{-1} e^{-y^2/2} = \sqrt{\frac{\hbar}{mw}} \int dy' e^{i \frac{(y^2 + y'^2) \cos \omega t - 2yy'}{2 \sin \omega t}} e^{-y'^2/2}$$

$$= \sqrt{\frac{\hbar}{mw}} e^{i \frac{y^2}{2} \cot \omega t} \int dy' \exp \left\{ - \overbrace{(1 - i \cot \omega t)}^{\equiv z} \frac{y'^2}{2} + \overbrace{\left(\frac{-iy}{\sin \omega t} \right)}^{\equiv J} y' \right\}$$

$$z = 1 - i \cot \omega t = -i \frac{(\cos \omega t + i \sin \omega t)}{\sin \omega t} = -i \frac{e^{i\omega t}}{\sin \omega t} \quad \& \quad J = \frac{-iy}{\sin \omega t}.$$

Use provided identity: ($\text{Re}(z) > 0$ so applicable)

$$e^{-i\omega t/2} (A(t))^{-1} e^{-y^2/2} = \sqrt{\frac{\hbar}{mw}} e^{i \frac{y^2}{2} \cot \omega t} \sqrt{\frac{2\pi}{z}} \exp \left(\frac{J^2}{2z} \right)$$

$$= \sqrt{\frac{2\pi i \hbar \sin \omega t}{mw}} e^{-i\omega t/2} \exp \left\{ i \frac{y^2}{2} \frac{\cos \omega t}{\sin \omega t} - \frac{iy^2}{2 \sin \omega t} e^{-i\omega t} \right\} \quad [2]$$

$$\cancel{e^{-i\omega t/2}} (A(t))^{-1} \cancel{e^{-y^2/2}} = \sqrt{\frac{2\pi i \hbar \sin \omega t}{mw}} \cancel{e^{-i\omega t/2}} \cancel{e^{-y^2/2}}$$

$$\therefore A(t) = \left(\frac{mw}{2\pi i \hbar \sin \omega t} \right)^{1/2} \quad [2]$$

(c)

$$\langle x | U(t; 0) | x' \rangle = \left(\frac{mw}{2\pi i \hbar \sin \omega t} \right)^{1/2} \exp \left(\frac{imw}{2\hbar} \{ (x^2 + x'^2) \cos \omega t - 2xx' \} \right)$$

As $\omega \rightarrow 0$, we see that $\frac{\omega}{\sin \omega t} \rightarrow \frac{1}{t}$, $\cos \omega t \rightarrow 1$,

where the limit follows.

$$x \text{ ————— } x$$

② (Lifshitz point)

We have

$$F = \int dx [\alpha_2(\tau) \phi^2 + \rho_2 |\partial_x \phi|^2 + \rho_4 |\partial_x^2 \phi|^2 + \alpha_4 \phi^4]$$

- (a) Landau theory is applied in the vicinity of a critical point where we can expand in powers of the order parameter + its gradients. We keep the leading nontrivial order consistent w/ thermodynamic stability. For $\rho_2 > 0$ we can drop the ρ_4 term b/c F is stable. [1]
 For $\rho_2 < 0$, gradients in ϕ are energetically favored $\Rightarrow$ system is unstable to having $\phi(x)$ vary very quickly (violating Landau assumptions) $\Rightarrow$ need ρ_4 to cut this off & restore stability. (accept reasonable alternatives) [1]

- (b) For complex scalar, need to work with $\phi_k = A_k e^{ikx}$ w/ $k = \pm k_0$. (note that for $k_0 = 0$, corresponds to uniform soln; also as soln is uniform in $D-1$ dimensions we can basically ignore those).

$$f = (\alpha_2 + \rho_2 k_0^2 + \rho_4 k_0^4) A_k^2 + \alpha_4 A_k^4 \quad (\text{can fix } A_k \in \mathbb{R} \text{ as phase is unimportant})$$

Minima: $\frac{\delta f}{\delta k_0} = 0 \Rightarrow 2k_0 A_k^2 (\rho_2 + 2\rho_4 k_0^2) = 0$ [1]

$$\frac{\delta f}{\delta A_k} = 0 \Rightarrow 2A_k [(\alpha_2 + \rho_2 k_0^2 + \rho_4 k_0^4) + 2\alpha_4 A_k^2] = 0$$
 [1]

Solutions:

Ⓐ $k_0 = 0, A_k = 0, \boxed{f_P = 0}$ (always) [1]

Ⓑ $k_0 = 0, A_k^2 = -\frac{\alpha_2}{2\alpha_4}, \boxed{f_F = -\frac{\alpha_2^2}{4\alpha_4}}$ (if $\alpha_2 < 0$) [1]

Ⓜ $k_0^2 = -\frac{\rho_2}{2\rho_4}, A_k^2 = \frac{1}{2\alpha_4} \left(\frac{\rho_2^2}{4\rho_4} - \alpha_2 \right), \boxed{f_M = -\frac{1}{4\alpha_4} \left(\alpha_2 - \frac{\rho_2^2}{4\rho_4} \right)^2}$ (if $\rho_2 < 0$ and $\alpha_2 < \frac{\rho_2^2}{4\rho_4}$) [1]

Consider the 4 quadrants: what is lowest-energy soln?

$\alpha_2 > 0, \rho_2 > 0$: only P is a soln.

$\alpha_2 < 0, \rho_2 > 0$: F has lower energy than P

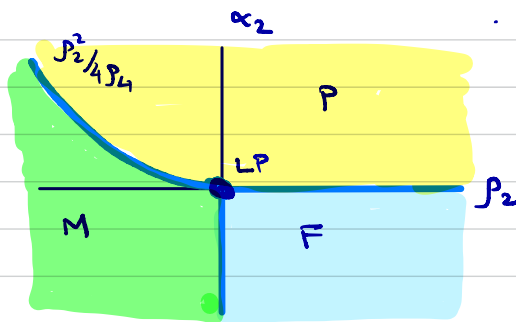
$\alpha_2 > 0, \rho_2 < 0$: P if $\alpha_2 > \frac{\rho_2^2}{4\rho_4}$, M if $\alpha_2 < \frac{\rho_2^2}{4\rho_4}$ → [1]

$\alpha_2 < 0, \rho_2 < 0$: M has lower energy than either P or F.

Phase boundaries: $P-F$: $\alpha_2 = 0, \rho_2 > 0$; $P-M$: $\alpha_2 = \frac{\rho_2^2}{4\rho_4}, \rho_2 < 0$ } in both cases, $A_k \rightarrow 0$ as we approach P from M or F ⇒ 2nd order. [2]

$F-M$: $\alpha_2 < 0, \rho_2 = 0$; as $\rho_2 \rightarrow 0^-$, $A_k \rightarrow -\frac{\alpha_2}{2\rho_4}$ which is its value in F. $\therefore$ also 2nd order. [2]

Sketch:



(Answers can vary but must contain clear identification of phases in terms of A_k, k_0 , and transitions + rationale for continuity) [2]

(c) Along $\rho_2 = 0$,

$$\beta \mathcal{H} = \int d^D r (\rho_4 (\partial_x \phi)^2 + \alpha_2 \phi^2 + \alpha_4 \phi^4) = \int dk [(\alpha_2 + \rho_4 k^2) \tilde{\phi}(k) \tilde{\phi}(-k) + \text{quartic}] \quad \text{[1]}$$

Gaussian approx: drop quartic terms.

$$\text{Using the standard shortcut, } \langle \tilde{\phi}(p) \tilde{\phi}(q) \rangle = \frac{(2\pi)^D \delta^D(\vec{p} + \vec{q})}{\rho_4 p^4 + \alpha_2}.$$

$$\therefore G(\vec{r}) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{-i p r}}{\rho_4 p^4 + \alpha_2} \sim \frac{e^{-r \cdot (\frac{\alpha_2}{\rho_4})^{1/4}}}{|r|^{-3}} \Rightarrow \zeta \sim \left(\frac{\rho_4}{\alpha_2}\right)^{1/4} \sim t^{-1/4} \quad \text{[2]}$$

$$\therefore \boxed{\nu = \frac{1}{4}}$$

[1]

(d) For real scalar: $\phi_k = A_k \cos(kx)$ ($k = \pm k_0$).

$$\begin{aligned} \frac{\partial f}{\partial k_0} &= \frac{1}{L} \int dx \left[A_k^2 \{ (\alpha_2 + p_4 k_0^2) \cos^2 k_0 x + p_2 \sin^2 k_0 x \} + \alpha_4 A_k^4 \cos^4 k_0 x \right] \\ &= \frac{1}{L} \cdot \frac{L}{\left(\frac{2\pi}{k_0}\right)} \int_0^{2\pi/k_0} dx \left[A_k^2 \{ (\alpha_2 + p_4 k_0^2) \cos^2 k_0 x + p_2 \sin^2 k_0 x \} + \alpha_4 A_k^4 \cos^4 k_0 x \right] \end{aligned}$$

$$f = \frac{1}{2} \left[A_k^2 (\alpha_2 + p_2 k_0^2 + p_4 k_0^4) + \frac{3}{4} \alpha_4 A_k^4 \right]$$

We find $\tilde{f}_F$ similarly to before:

[2]

For $p_2 < 0$, $\alpha_2 > 0$: from part (b) we can guess $k_0 \neq 0$.

$$0 = \frac{\delta f}{\delta k_0} \Rightarrow 2k_0 A_k^2 (p_2 + 2p_4 k_0^2) = 0$$

$$0 = \frac{\delta f}{\delta A_k} \Rightarrow A_k \left\{ (\alpha_2 + p_2 k_0^2 + p_4 k_0^4) + \frac{3}{2} \alpha_4 A_k^2 \right\} = 0$$

$$\left. \begin{aligned} k_0^2 &= \frac{-p_2}{2p_4} \\ \text{and } A_{k,M}^2 &= \frac{2}{3\alpha_4} \left(\frac{p_2^2}{4p_4} - \alpha_2 \right) \\ \text{with } f_M &= -\frac{1}{6\alpha_4} \left(\frac{p_2^2}{4p_4} - \alpha_2 \right)^2 \end{aligned} \right\} \textcircled{M}$$

[2]

$$(e) f_F = \tilde{f}_M \text{ if } -\frac{\alpha_2^2}{4\alpha_4} = -\frac{1}{6\alpha_4} \left(\frac{p_2^2}{4p_4} - \alpha_2 \right)$$

$$\Rightarrow (-\alpha_2) = \sqrt{\frac{2}{3}} \left(\frac{p_2^2}{4p_4} - \alpha_2 \right) \quad \left(\text{recall } \alpha_2 < 0 \text{ so } \sqrt{-\alpha_2^2} = |\alpha_2| = -\alpha_2 \right)$$

$$\alpha_2 = -\frac{(2+\sqrt{6})}{4} \frac{p_2^2}{4p_4} \sim \text{F-M boundary} \quad [2]$$

But from (b), $k_0 \propto p_2 \Rightarrow$ F-M boundary is at

$k_0 \neq 0 \Rightarrow$ first order transition

[1]

③ We have a $q=1, m=1$ particle moving in 1D, subject to 3 forces:

i) $F_{\text{ext}} = qE = E$

ii) $F_{\text{vis}} = -\gamma v(t)$

iii) $F_{\text{coll}} = \eta(t)$. w/ $\langle \eta(t) \rangle = 0$
 $\langle \eta(t) \eta(t') \rangle = \Gamma \delta(t-t')$.

(a) System of stochastic diff. equations:

$$\frac{dx}{dt} = v(t) \quad \dots (1) \quad [1]$$

$$\frac{dv}{dt} = (F_{\text{ext}} + F_{\text{vis}} + F_{\text{coll}}) = E - \gamma v(t) + \eta(t) \quad \dots (2) \quad [1]$$

$v(t+dt)$ only depends on $v(t) \Rightarrow$ Markov process [1]

However need $x(t)$ and $v(t)$ to get $x(t+dt) \Rightarrow x(t)$ is not a Markov process on its own. [1].

(b) Using (2), we have

$$\frac{d}{dt'} [v(t') e^{\gamma t'}] = \left[\frac{dv}{dt'} + \gamma v \right] e^{\gamma t'} = [E + \eta(t')] e^{\gamma t'}. \quad [1]$$

Integrating both sides from 0 to t

$$\begin{aligned} v(t) e^{\gamma t} - v(0) &= \int_0^t dt' [E + \eta(t')] e^{\gamma t'} \\ &= \frac{E}{\gamma} (e^{\gamma t} - 1) + \int_0^t dt' \eta(t') e^{\gamma t'}. \end{aligned} \quad [2]$$

So

$$v(t) = \frac{E}{\gamma} (1 - e^{-\gamma t}) + v(0) e^{-\gamma t} + \int_0^t dt' \eta(t') e^{-\gamma(t-t')} \quad [1]$$

Now since $\eta(t) = 0$

$$\langle v(t) \rangle = v(0)e^{-\gamma t} + \frac{\varepsilon}{\gamma} (1 - e^{-\gamma t}) \underset{t \gg \frac{1}{\gamma}}{\approx} \frac{\varepsilon}{\gamma} \equiv v_d. \quad [1]$$

(c) Recall that the variance is $\langle (v(t) - \langle v(t) \rangle)^2 \rangle \equiv \langle (\delta v(t))^2 \rangle$.

Using our results from (b),

$$\delta v(t) = \int_0^t dt' \eta(t') e^{-\gamma(t-t')} \quad [1]$$

So

$$\begin{aligned} \langle (\delta v(t))^2 \rangle &= \int_0^t dt_1 \int_0^t dt_2 \underbrace{\langle \eta(t_1) \eta(t_2) \rangle}_{\propto \delta(t_1 - t_2)} e^{-\gamma(t-t_1)} e^{-\gamma(t-t_2)} \\ &= \Gamma e^{-2\gamma t} \int_0^t dt_1 e^{-2\gamma t_1} \\ &= \frac{\Gamma}{2\gamma} e^{-2\gamma t} (1 - e^{-2\gamma t}) = \frac{\Gamma}{2\gamma} (1 - e^{-2\gamma t}). \quad [2] \end{aligned}$$

Now at long times, we expect fluctuations around the average to satisfy equipartition.

$$\Rightarrow \frac{1}{2} m \langle (\delta v)^2 \rangle \underset{t \rightarrow \infty}{=} \frac{1}{2} k_B T \Rightarrow \frac{\Gamma}{2\gamma} = k_B T. \quad [2]$$

[Note: alternative is to take $\varepsilon = 0$; then at long times $\langle v(t) \rangle \rightarrow 0$

$$\text{So } \frac{1}{2} m \langle (\delta v)^2 \rangle \underset{t \rightarrow \infty}{=} \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k_B T$$

~ in case students don't realize it's fluctuations that should obey equipartition ... give full marks for this as well! ┐

From now on: $\bar{x}(t) = \langle x(t) \rangle$, $\delta x(t) = x(t) - \bar{x}(t)$.

(d) Since $\frac{dx}{dt} = v(t)$

$$x(t) = \int_0^t dt' v(t').$$

[1]

Averaging both sides,

$$\bar{x}(t) = \langle x(t) \rangle = \int_0^t dt' \langle v(t') \rangle$$

$$= \int_0^t dt' \left(v(0) e^{-\gamma t'} + \frac{\xi}{\gamma} (1 - e^{-\gamma t'}) \right).$$

$$= \frac{\xi}{\gamma} t + \frac{1}{\gamma} \left(v(0) - \frac{\xi}{\gamma} \right) (1 - e^{-\gamma t}).$$

[2]

So for $t \gg \gamma^{-1}$, $\bar{x}(t) \sim \text{const} + \frac{\xi}{\gamma} t \propto t$.

w/ proportionality constant $c = \frac{\xi}{\gamma} = v_d$.

[1].

(e) Consider

$$\frac{d}{dt} (\delta x(t)) = \frac{dx}{dt} - \frac{d\bar{x}}{dt}$$

$$= v(t) - \langle v(t) \rangle$$

(given in question)

$$= \zeta(t)$$

where.

$$\zeta(t) = \int_0^t dt' \eta(t') e^{-\gamma(t-t')}$$

Now $\langle \zeta(t) \rangle = 0$

$$\langle \zeta(t) \zeta(t') \rangle = \int_0^t dt_1 \int_0^{t'} dt_2 \langle \eta(t_1) \eta(t_2) \rangle e^{-\gamma(t-t_1)} e^{-\gamma(t'-t_2)}.$$

$$= e^{-\gamma t} e^{-\gamma t'} \cdot \int_0^t dt_1 \int_0^{t'} dt_2 \Gamma \delta(t_1 - t_2) e^{-\gamma(t_1 + t_2)}.$$

$$= \Gamma e^{-\gamma(t+t')} \cdot \begin{cases} \int_0^t dt_1 e^{-2\gamma t_1} & t < t' \\ \int_0^{t'} dt_1 e^{-2\gamma t_1} & t' < t. \end{cases}$$

$$= \frac{\Gamma}{2\gamma} e^{-\gamma(t+t')} \cdot e^{+2\gamma \min(t, t')} = \frac{\Gamma}{2\gamma} e^{-\gamma|t-t'|}. \quad [2].$$

Since $e^{-\gamma|t-t'|} > 0$ only for $|t-t'| \lesssim \gamma^{-1}$ and $\int_{-\infty}^{\infty} dt \frac{\Gamma}{2\gamma} e^{-\gamma|t-t'|} = \frac{\Gamma}{\gamma^2}$, we can replace RHS of $\langle \zeta(t) \zeta(t') \rangle$ by δ -fn:

$$\langle \zeta(t) \zeta(t') \rangle \underset{\text{scales} \Rightarrow \gamma^{-1}}{\approx} \frac{\Gamma}{\gamma^2} \delta(t-t'); \quad \langle \zeta(t) \rangle = 0.$$

and

$$\frac{d(\delta x(t))}{dt} = \zeta(t).$$

corresponding to Langevin eq. w/ no viscous damping.

[1].

$\therefore P_1(\delta x, t)$ should satisfy the diffusion equation

accept answers w/ right dimensions even if factors are off.

$$\frac{\partial P_1(\delta x, t)}{\partial t} = D \frac{\partial^2 P_1(\delta x, t)}{\partial x^2} \quad \text{w/ } D = \frac{\Gamma}{2\gamma^2} \quad [1]$$

(f) $\bar{x}(t)$ grows ballistically; fluctuations $\delta x(t)$ around the average grow diffusively.

$\Rightarrow P_1(x, t)$ should describe a gaussian w/ width $\propto \sqrt{Dt}$, centered at $\bar{x}(t) \sim v_d t$.

(explicit answer not necessary [2]. as long as explain well).

$$[\text{explicitly: } P_1(x, t) = \frac{1}{\sqrt{2\pi D|t|}} \exp\left(-\frac{(x - \bar{x}(t))^2}{4D|t|}\right)]$$

We expect on physical grounds that this will come from a biased random walk [1]. (accept reasonable equivalents).

x ————— x₁

4. The Lagrangian density of a three-component real scalar field $\phi_i = (\phi_1, \phi_2, \phi_3)$ is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + \frac{\mu^2}{2} \phi_i \phi_i - \frac{\lambda}{4} (\phi_i \phi_i)^2 - \frac{\mu^4}{4\lambda},$$

where $\mu^2 > 0$ and $\lambda > 0$.

(a) Find the equations of motion for the scalar field ϕ_i .

[2]

The Euler - Lagrange equations,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right] = \frac{\partial \mathcal{L}}{\partial \phi_i},$$

lead to the three equations of motion

$$\square \phi_i - \mu^2 \phi_i + \frac{\lambda}{2} (\phi_k \phi_k) \phi_i = 0.$$

These equations are non-linear. They also couple the three fields ϕ_1, ϕ_2, ϕ_3 together.

(b) Show that the Lagrangian density is invariant under the global continuous symmetry group $O(3)$, acting on the fields via $\phi_i = O_{ij} \phi_j$, whose real 3×3 matrices satisfy $O^T O = \mathbb{1}$.

[2]

What other symmetries does the Lagrangian density have?

[1]

With $\phi' = O_{ij} \phi_j$ we have $\phi'_i \phi'_i = O_{ij} O_{ik} \phi_j \phi_k = O_{ji}^T O_{ik} \phi_j \phi_k$. The Lagrangian density is invariant if $O_{ji}^T O_{ik} = \delta_{jk}$, i.e. $O^T O = \mathbb{1}$. This is the condition the elements of the orthogonal group $O(3)$ satisfy.

The Lagrangian density is also Poincaré-invariant since ϕ_i are scalars which transform as $\phi'_i(x') = \phi_i(x)$ under space-time transformations.

(c) Write the infinitesimal $O(3)$ transformations in the form $\phi'_i = \left(\delta_{ij} + \epsilon_a T_{ij}^a \right) \phi_j$, where $a = 1, \dots, N$, N being the number of independent real parameters ϵ_a . What condition must the matrices T_{ij}^a satisfy, and what is the value of N ?

[3]

The condition $O^T O = \mathbb{1}$ implies that for $M_{ij} \equiv \epsilon_a T_{ij}^a$ we have $M_{ij} = -M_{ji}$, i.e. the matrices must be antisymmetric. Real antisymmetric 3×3 matrices have 3 independent components, i.e. the corresponding linear space is three-dimensional, with three independent T_{ij}^a forming a basis. Thus, $N = 3$.

(d) Apply Noether's theorem and construct the conserved $O(3)$ currents in terms of T_{ij}^a and ϕ_i .

[3]

As discussed in lectures, in the case of an action invariant with respect to a global symmetry, Noether theorem is particularly simple. The Lagrangian density itself is invariant. Thus,

$$\delta\mathcal{L} = \mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta\partial_\mu\phi) - \mathcal{L}(\phi, \partial_\mu\phi) = \frac{\partial\mathcal{L}}{\partial\phi_i} \delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} \epsilon_a T_{ij}^a \partial_\mu\phi_j = 0,$$

where we used $\delta\phi_i = \epsilon_a T_{ij}^a \phi_j$. Replacing the first term using the Euler-Lagrange equations, and remembering that the parameters ϵ^a are independent, we find

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} T_{ij}^a \phi_j \right) = 0.$$

Thus, on shell, we have 3 conserved currents given by

$$j_a^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} T_{ij}^a \phi_j = \partial^\mu\phi_i T_{ij}^a \phi_j,$$

where T_{ij}^a are the generators of $O(3)$.

(e) Describe the space of classical vacua of the theory. By choosing the vacuum $\phi_i^0 = (0, v/\sqrt{2}, v/\sqrt{2})$ and considering small fluctuations, $\phi_i = \phi_i^0 + \delta\phi_i$, identify massless and massive excitations. [Hint: you may need to re-define the fields to diagonalize the mass matrix.]

[3]

Explain the number of massless and massive excitations in terms of Goldstone's theorem.

[1]

Classical vacua are minima of the potential V , where

$$V = -\frac{\mu^2}{2} \phi_i \phi_i + \frac{\lambda}{4} (\phi_i \phi_i)^2 + \frac{\mu^4}{4\lambda}.$$

taking the derivative w.r.t. $X \equiv \phi_i \phi_i$, we find a minimum at $X = v^2 = \mu^2/\lambda$, i.e. at

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = v^2 = \mu^2/\lambda > 0.$$

Thus, the space of vacua is a sphere S^2 of radius v . Choosing a vacuum as a point on the sphere, for example, at $\phi_i^0 = (0, v/\sqrt{2}, v/\sqrt{2})$, breaks the $O(3)$ symmetry but leaves the residual $O(2)$ symmetry. Choosing the vacuum $\phi_i^0 = (0, v/\sqrt{2}, v/\sqrt{2})$ and substituting $\phi_i = \phi_i^0 + \delta\phi_i$ in the Lagrangian density, one finds

$$\mathcal{L} = \frac{1}{2} \partial_\mu \delta\phi_1 \partial^\mu \delta\phi_1 + \frac{1}{2} \partial_\mu \delta\phi_2 \partial^\mu \delta\phi_2 + \frac{1}{2} \partial_\mu \delta\phi_3 \partial^\mu \delta\phi_3 - \frac{\mu^2}{2} (\delta\phi_2 + \delta\phi_3)^2.$$

The mode $\delta\phi_1$ is clearly massless. To determine the fate of the other two modes, we can make the mass matrix diagonal by redefining the fields as

$$\delta\phi_X = (\delta\phi_2 + \delta\phi_3)/\sqrt{2}, \quad \delta\phi_Y = (\delta\phi_2 - \delta\phi_3)/\sqrt{2},$$

with $\delta\phi_2^2 + \delta\phi_3^2 = \delta\phi_X^2 + \delta\phi_Y^2$. Then

$$\mathcal{L} = \frac{1}{2} \partial_\mu \delta\phi_1 \partial^\mu \delta\phi_1 + \frac{1}{2} \partial_\mu \delta\phi_X \partial^\mu \delta\phi_X + \frac{1}{2} \partial_\mu \delta\phi_Y \partial^\mu \delta\phi_Y - \mu^2 \delta\phi_X^2.$$

Thus, we have 2 massless modes and one massive mode with the mass $M = \sqrt{2}\mu$. This is exactly what Goldstone's theorem predicts: we have symmetry breaking $O(3) \rightarrow O(2)$, the space of vacua is the coset $O(3)/O(2) \cong S^2$, the number of "broken" generators is $3 - 1 = 2$ which is the number of Goldstone's massless modes.

Now consider a theory of n complex scalar fields $f_i(x)$, $i = 1, \dots, n$ with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu f_i^* \partial^\mu f_i + \frac{\mu^2}{2} f_i^* f_i - \frac{\lambda}{4} (f_i^* f_i)^2 - \frac{\mu^4}{4\lambda},$$

where $\mu^2 > 0$, $\lambda > 0$, and f_i^* is the complex conjugate of f_i .

(f) Show that the group $U(n)$ of $n \times n$ complex matrices satisfying $U^\dagger U = \mathbb{1}$ is a global symmetry of the theory. Find the number of generators of this symmetry group.

[3]

For $f'_i = U_{ik} f_k$ we have $f_i'^* f'_i = U_{ij}^* f_j^* U_{ik} f_k = U_{ji}^{*T} U_{ik} f_j^* f_k$. The Lagrangian density is invariant if $U_{ji}^{*T} U_{ik} = \delta_{jk}$, i.e. if $U^\dagger U = \mathbb{1}$. Infinitesimal unitary transformation can be written in the form $U = \mathbb{1} + \epsilon \tau$. Then the condition $U^\dagger U = \mathbb{1}$ implies that $\tau^\dagger = -\tau$ (matrices τ must be anti-Hermitian). Writing $\tau_{ij} = \alpha_{ij} + i\beta_{ij}$ with real α_{ij} and β_{ij} , we have $\tau_{ij} = -\tau_{ji}^*$ (i.e. $\alpha_{ij} = -\alpha_{ji}$ and $\beta_{ij} = \beta_{ji}$). The number of independent components of a symmetric $n \times n$ matrix is $n(n+1)/2$, of an antisymmetric one - $n(n-1)/2$. Together we have n^2 independent components of an anti-Hermitian matrix τ , i.e. the number of generators of $U(n)$ is n^2 .

(g) Describe the space of vacua of the theory. Choose the vacuum $f_i^0 = (0, 0, \dots, 0, v)$. What is the symmetry group leaving this vacuum state invariant?

[2]

The minima of the potential V are the vacua of the theory. They satisfy the condition $V' = 0$, i.e.

$$f_i^* f_i = \mu^2 / \lambda.$$

In terms of $f_i = x_i + iy_i$, this equation reads

$$x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2 = \mu^2 / \lambda > 0,$$

which defines a sphere S^{2n-1} . This is the space of vacua. Choosing $f_i^0 = (0, 0, \dots, 0, v)$ breaks $U(n)$ symmetry but leaves $U(n-1)$ unaffected. We have $U(n)/U(n-1) \cong S^{2n-1}$.

(h) Apply Goldstone's theorem to predict (without an explicit calculation) the number of massless fields in the theory with the vacuum f_i^0 .

[5]

The group $U(n)$ has n^2 generators, and the group $U(n-1)$ has $(n-1)^2$. The number of "broken" generators is $n^2 - (n-1)^2 = 2n - 1$. According to Goldstone's theorem, this is the number of massless modes in the theory with the vacuum f_i^0 .

Q4: 25
Section:100

5. Consider a real scalar field in four space-time dimensions with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 - \frac{\kappa}{6} \phi^6.$$

(a) List the symmetries of $\mathcal{L}$.

[2]

By construction, the Lagrangian density is Poincaré-invariant. It is also invariant under the $\mathbb{Z}_2$ symmetry $\phi \rightarrow -\phi$.

(b) Find the (mass) dimensions of the parameters m^2 , λ and κ (in units with $\hbar = 1$, $c = 1$).

[3]

The action $S = \int d^4x \mathcal{L}$ must be dimensionless (in units with $\hbar = 1$). The kinetic term implies that the (mass) dimension of the field is $[\phi] = 1$. Then $[m] = 1$, $[\lambda] = 0$ and $[\kappa] = -2$.

(c) Derive the equation of motion for the field ϕ .

[2]

Applying the Euler-Lagrange equations,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right] = \frac{\partial \mathcal{L}}{\partial \phi_i},$$

we find

$$\square \phi + m^2 \phi + \lambda \phi^3 + \kappa \phi^5 = 0.$$

(d) Compute the momentum $\pi(x)$ conjugate to $\phi(x)$ and find the Hamiltonian density $\mathcal{H}$ of the theory. Show (explaining all details) that the solution to the free equation of motion (with $\lambda = 0$ and $\kappa = 0$) is given by

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} (a_p e^{-ipx} + a_p^* e^{ipx}),$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

[3]

By definition, the momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$. The Hamiltonian density is $\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$, where $\dot{\phi}$ must be expressed through π . We have

$$\mathcal{H} = \frac{1}{2}\pi^2 + \partial_i \phi \partial_i \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\kappa}{6}\phi^6.$$

The free equation of motion is $(\square + m^2)\phi = 0$. As discussed in detail in lectures, solution to this equation can be found by using Fourier transform

$$\phi(t, \mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\mathbf{x}} \phi(t, \mathbf{p}).$$

Then $\phi(t, \mathbf{x})$ obeys the equation for the harmonic oscillator,

$$\ddot{\phi}(t, \mathbf{x}) + \omega_{\mathbf{p}}^2 \phi(t, \mathbf{x}) = 0,$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, the solution of which is

$$\phi(t, \mathbf{x}) = a_{\mathbf{p}}^{(1)} e^{-i\omega_{\mathbf{p}} t} + a_{\mathbf{p}}^{(2)} e^{i\omega_{\mathbf{p}} t},$$

where $a_{\mathbf{p}}^{(1,2)}$ are integration constants. Since the field $\phi(t, \mathbf{x})$ is real, we have $a_{\mathbf{p}}^{(2)} = a_{\mathbf{p}}^{(1)*}$. We can write $a_{\mathbf{p}}^{(1)} \equiv N_{\mathbf{p}} a_{\mathbf{p}}$, where the (real) normalisation constant $N_{\mathbf{p}}$ can be chosen to be $1/2\omega_{\mathbf{p}}$ but can also be left unspecified. We obtain then

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^* e^{ipx}),$$

which is the required result.

(e) Now promote a_p and a_p^* to operators a_p and $a_p^\dagger$ obeying the commutation relations $[a_p, a_q^\dagger] = (2\pi)^3 2\omega_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$, with all other commutators vanishing. Find the equal time commutation relations between the operators ϕ and π .

[4]

First, we find the expression for $\pi(x) = \dot{\phi}$:

$$\pi(t, \mathbf{x}) = -i \int \frac{d^3 q}{(2\pi)^3 2\omega_{\mathbf{q}}} \omega_{\mathbf{q}} (a_{\mathbf{q}} e^{-iqx} - a_{\mathbf{q}}^* e^{iqx}).$$

Promoting a_p and a_p^* to operators a_p and $a_p^\dagger$, we have

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}),$$

$$\pi(t, \mathbf{x}) = -i \int \frac{d^3 q}{2(2\pi)^3} (a_{\mathbf{q}} e^{-iqx} - a_{\mathbf{q}}^\dagger e^{iqx}).$$

Then, taking into account the commutation relations $[a_p, a_q^\dagger] = (2\pi)^3 2\omega_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$, with all other commutators vanishing, and the Fourier representation of the delta-function, $\delta(\alpha) = \frac{1}{2\pi} \int e^{i\alpha\tau} d\tau$, we find

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}).$$

(f) Show that the integral representation for the Pauli - Jordan function $\Delta(x-y) = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$ can be written as

$$\Delta(x-y) = \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \epsilon(p_0) e^{-ip(x-y)},$$

where $\epsilon(p_0) = p_0/|p_0|$. Explain all steps in detail. [4]

Using the definition, inserting the expansions for $\phi(x)$ and $\phi(y)$ in terms of the operators a_p and $a_p^\dagger$ given in part e), and computing the commutators, we find

$$\Delta(x-y) = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right).$$

Now the integration over p_0 can be trivially added by remembering that

$$\delta(p^2 - m^2) = \frac{1}{2\omega_{\mathbf{p}}} [\delta(p_0 + \omega_{\mathbf{p}}) + \delta(p_0 - \omega_{\mathbf{p}})].$$

Another useful point to remember is that one can change the sign of the three-dimensional momenta in the integrand by doing the standard change of variables. We thus arrive at

$$\Delta(x-y) = \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \epsilon(p_0) e^{-ip(x-y)},$$

where $\epsilon(p_0) = p_0/|p_0|$.

(g) Show explicitly that $\Delta(x-y)$ is Lorentz-invariant. Explain why $\Delta(x-y) = 0$ for space-like separated x and y . [3]

The invariance of the Lorentz scalars and the measure in the integrand is self-evident. The only non-trivial part is to show the invariance of the sign of the p_0 . Indeed, as explained explicitly in lectures, the sign of the component A^0 of a time-like 4-vector A^μ is invariant under the transformations Λ of the orthochronous component $L^\uparrow$ of the Lorentz group. We have $A'^\mu = \Lambda^\mu_\nu A^\nu$. Thus,

$$A'^0 = \Lambda^0_0 A^0 + \Lambda^0_i A^i.$$

For $\Lambda \in L^\uparrow$, we have $\Lambda^0_0 \geq 1$, so $\Lambda^0_0 > 0$. The sign of A^0 is then preserved if $|\Lambda^0_i A^i| < |\Lambda^0_0 A^0|$. By Cauchy-Bunyakovsky inequality,

$$(\Lambda^0_i A^i)^2 \leq (\Lambda^0_i \Lambda^0_i) (A^i A^i).$$

Also, $A^i A^i < (A^0)^2$ (since A^μ is time-like) and $\Lambda^0_i \Lambda^0_i = (\Lambda^0_0)^2 - 1$ (this follows from $\Lambda^\mu_\rho \Lambda^\nu_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$). Therefore,

$$(\Lambda^0_i A^i)^2 < [(\Lambda^0_0)^2 - 1] (A^0)^2 < (\Lambda^0_0)^2 (A^0)^2,$$

and we get the required inequality $|\Lambda^0_i A^i| < |\Lambda^0_0 A^0|$. Thus, $\epsilon(p_0)$ is Lorentz-invariant.

At equal times, the commutator $[\phi(x), \phi(y)]$ vanishes (this is one of the fundamental commutation relations), so at equal times (which corresponds to a space-like interval $(x-y)^2 = -(\mathbf{x}-\mathbf{y})^2 < 0$) we have $\Delta(x-y) = 0$. We have shown that $\Delta(x-y)$ is Lorentz-invariant. But for space-like separated x and y , with $(x-y)^2 < 0$, one can always find a Lorentz transformation to a frame where these two events are simultaneous (i.e. to the "equal time" frame). Thus, $\Delta(x-y) = 0 \forall (x-y)^2 < 0$.

(h) Show that in the full theory (with $\lambda \neq 0$ and $\kappa \neq 0$) the following equation holds

$$(\Box_x + m^2) \langle 0|T\phi(x)\phi(y)|0\rangle = -\lambda\langle 0|T\phi^3(x)\phi(y)|0\rangle - \kappa\langle 0|T\phi^5(x)\phi(y)|0\rangle - i\delta^{(4)}(x-y)$$

Hint: recall that $\theta'(x) = \delta(x)$.

[4]

Taking the first derivative of the two-point function gives

$$\partial_x^\mu \langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\partial^\mu\phi(x)\phi(y)|0\rangle + \langle 0|[\phi(x), \phi(y)]|0\rangle\delta(x^0 - y^0),$$

where the second term (being the equal-time commutator) vanishes. The second derivative then gives

$$\Box_x \langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\Box_x\phi(x)\phi(y)|0\rangle + \langle 0|[\partial_0\phi(x), \phi(y)]|0\rangle\delta(x^0 - y^0).$$

We now use the equal times canonical commutation relation

$$\langle 0|[\partial_0\phi(x), \phi(y)]|0\rangle\delta(x^0 - y^0) = -i\delta^{(4)}(x-y)$$

and the equations of motion

$$\Box\phi + m^2\phi + \lambda\phi^3 + \kappa\phi^5 = 0$$

to arrive at

$$(\Box_x + m^2) \langle 0|T\phi(x)\phi(y)|0\rangle = -\lambda\langle 0|T\phi^3(x)\phi(y)|0\rangle - \kappa\langle 0|T\phi^5(x)\phi(y)|0\rangle - i\delta^{(4)}(x-y).$$

Q5: 25
Section:125

6. Consider a massive vector field A_μ described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}J^\mu A_\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $J^\mu = A^\mu$.

(a) Derive the equations of motion for the field A_μ . Show that the equation $\partial_\mu A^\mu = 0$ is a consequence of the equations of motion. How many degrees of freedom (independent components) does the massive vector field have? Is the Lagrangian density gauge-invariant? [5]

Applying the Euler-Lagrange equations,

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right] = \frac{\partial \mathcal{L}}{\partial A_\nu},$$

we find the equations of motion

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0.$$

Taking a derivative of this expression, we have

$$\partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = 0.$$

Due to antisymmetry of $F_{\mu\nu}$, the first term vanishes identically, and we find $\partial_\mu A^\mu = 0$ as a consequence of the equations of motion. This means that the four components A_μ are not independent, there is one constraint on them (in Fourier space, it can be written as an algebraic equation, $k^\mu A_\mu = 0$). Thus, only three components are independent, so a massive vector field has 3 degrees of freedom. The Lagrangian density is not invariant under gauge transformations $A_\mu \rightarrow A_\mu - \partial_\mu f$: the field strength tensor $F^{\mu\nu}$ is invariant but the term $A^\mu A_\mu$ is clearly not.

(b) Write the equations of motion in the form $L^{\mu\nu} A_\nu = 0$, where $L^{\mu\nu}$ is a matrix linear differential operator. Find the momentum-space Green's function of the operator $L^{\mu\nu}$, i.e. a Fourier image of the function $G_{\mu\nu}(x-y)$ obeying the equation

$$L^{\mu\nu}(x)G_{\nu\rho}(x-y) = \delta^{(4)}(x-y)\delta^\mu_\rho.$$

Does the function has a smooth limit as $m \rightarrow 0$? [Hint: introduce $G_{\mu\nu}(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}_{\mu\nu}(p)$, assume the ansatz $\tilde{G}_{\mu\nu}(p) = A p^2 \eta_{\mu\nu} + B p_\mu p_\nu$, and find the coefficients A and B .] [5]

Using the definition of $F^{\mu\nu}$, we can write the equations of motion as

$$(\square + m^2) A^\mu - \partial^\mu (\partial_\nu A^\nu) = 0$$

or

$$[(\square + m^2) \eta^{\mu\nu} - \partial^\mu \partial^\nu] A_\nu \equiv L^{\mu\nu} A_\nu = 0$$

Following the hint, we introduce the ansatz $\tilde{G}_{\mu\nu}(p) = A p^2 \eta_{\mu\nu} + B p_\mu p_\nu$ and act by $L_x^{\mu\nu}$ on $G_{\mu\nu}(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}_{\mu\nu}(p)$. The result is

$$[(-p^2 + m^2) \eta^{\mu\nu} + p^\mu p^\nu] \tilde{G}_{\mu\sigma}(p) = \delta_\sigma^\nu,$$

or, explicitly,

$$(-p^2 + m^2) A p^2 \delta_\sigma^\nu + (A p^2 + B(-p^2 + m^2) + B p^2) p^\nu p_\sigma = \delta_\sigma^\nu.$$

This is satisfied with $(-p^2 + m^2) A p^2 = 1$ and $A p^2 + B m^2 = 0$, so we get A and B , and

$$\tilde{G}_{\mu\nu}(p) = -\frac{1}{p^2 - m^2} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right),$$

which is the desired result. Clearly, it does not have a smooth limit when $m \rightarrow 0$.

(c) Show that in the case of a massless vector field, the corresponding operator $L^{\mu\nu}$ has no inverse. [Hint: Follow the same steps as in the massive case. Show that $L^{\mu\nu}(p) p_\nu = 0$ and explain why the matrix $L^{\mu\nu}(p)$ cannot be inverted.] [4]

Following the same steps as in the massive case, we find that the constant B now remains undetermined, while for A we have the equations $A p^2 = 0$ and $A p^4 = -1$ which are not compatible. The matrix

$$L^{\mu\nu}(p) = [-p^2 \eta^{\mu\nu} + p^\mu p^\nu]$$

is not invertible: we obviously have $L^{\mu\nu}(p) p_\nu \equiv 0$ for a generic non-zero p_ν , i.e. the equation $L^{\mu\nu}(p) p_\nu = 0$ has a nontrivial solution, hence the determinant of $L^{\mu\nu}(p)$ vanishes and it cannot be inverted.

(d) We now consider a scalar theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J(x) \phi(x),$$

where $J(x)$ is an external source. Find the equation of motion satisfied by the field ϕ and show that its (classical) solution is given by

$$\phi_c(x) = \int d^4 y G_F(x-y) J(y),$$

where $G_F(x-y)$ is the Feynman Green's function for the scalar field,

$$G_F(x-y) = - \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon},$$

proportional to the Feynman propagator $\Delta_F(x-y) = -i G_F(x-y)$. [5]

The equation of motion is obtained by applying Euler-Lagrange equations. It reads

$$(\square_x + m^2) \phi(x) = J(x).$$

This inhomogeneous linear differential equation can be solved by introducing a Green's function satisfying

$$(\square_x + m^2) G_F(x - y) = \delta^{(4)}(x - y).$$

Then the solution to the inhomogeneous equation reads

$$\phi_c(x) = \int d^4y G_F(x - y) J(y).$$

The integral representation for the Green's function can be easily found by using Fourier transform, with the result

$$G_F(x - y) = i\Delta_F(x - y) = - \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}.$$

All of the above is the standard lecture material.

(e) The generating functional for the free scalar field $\phi(x)$ is given by

$$W_0[J] = \int [d\phi] \exp \left\{ i \int d^4x \mathcal{L} \right\}.$$

Show that by making a change of variables $\phi(x) = \phi_c(x) + \xi(x)$, the generating functional can be written in the form

$$W_0[J] = N \exp \left\{ -\frac{1}{2} \int d^4x d^4y J(x) \Delta_F(x - y) J(y) \right\}.$$

Identify the prefactor N explicitly and show that it is independent of J .

[6]

With $\phi(x) = \phi_c(x) + \xi(x)$, the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_c \partial^\mu \phi_c - \frac{1}{2} m^2 \phi_c^2 + J(x) \phi_c(x) + \frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \frac{1}{2} m^2 \xi^2 + \partial_\mu \phi_c \partial^\mu \xi - m^2 \phi_c \xi + J(x) \xi(x).$$

Integrating by parts in the action $S = \int d^4x \mathcal{L}$ and using the equation of motion, we find

$$S = \int d^4x \frac{1}{2} J(x) \phi_c(x) + \int d^4x \left(\frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \frac{1}{2} m^2 \xi^2 \right).$$

We now use the solution

$$\phi_c(x) = \int d^4y G_F(x - y) J(y)$$

to obtain

$$iS = -\frac{1}{2} \int d^4x d^4y J(x) \Delta_F(x - y) J(y) + i \int d^4x \left(\frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \frac{1}{2} m^2 \xi^2 \right).$$

The generating functional then naturally factorizes

$$W_0[J] = \int [d\phi] \exp \left\{ i \int d^4x \mathcal{L} \right\} = N \exp \left\{ -\frac{1}{2} \int d^4x d^4y J(x) \Delta_F(x - y) J(y) \right\},$$

where the prefactor N is independent of J :

$$N = \int [d\xi] \exp \left\{ \frac{i}{2} \int d^4x (\partial_\mu \xi \partial^\mu \xi - m^2 \xi^2) \right\}.$$