Revision Lecture: C6 (QFT part)

1) Classical field theory

- Lagrangian density: field content, E-L equations of motion

ex: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \lambda\phi^4$

E-L: $$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right] = \frac{\partial\mathcal{L}}{\partial\phi_i}$$

$i = 1, 2 \ldots N$ lists components of the field (in the example above, $i=1$ only)

Note type of equations (linear or non-linear; coupled or decoupled - in case of several field components).

- Symmetries of $\mathcal{L}$.

— space-time symmetries (e.g. $\mathcal{L}$ above is Poincaré-invariant)

– global symmetries (continuous or discrete): e.g. $\mathcal{L}$ above is invar. under $\phi \to -\phi$ (this is $\mathbb{Z}_2$ discrete symmetry). Continuous symmetries are of the form $\phi_i = M_{ik}\phi_k$, where M realise rep. of a symmetry group. Note that fields ϕ_i may be complex (this is usually specified explicitly).

<u>Note</u>: you need to show that $\mathcal{L}$ is invariant (explain why).

- Infinitesimal field transformations

Usually of the form:

$$\phi_i' = \phi_i + \lambda_a T^a_{ij}\phi_j$$

Need to know how to deduce properties of T^a_{ij} from those of M_{ik}.

Exercise: try standard groups such as $SU(N)$, $SO(N)$, $Sp(2N)$ etc.

- Noether theorem and currents:

For continuous symmetries, should be able to apply explicitly (i.e. effectively prove) Noether's theorem and derive expressions for Noether's currents.

- Vacuum solutions

Energy $P^0 = \int d^3x\, T^{00} = \int d^3x \left(\frac{1}{2}\pi^2 + \right.$ $\left. + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right)$ in the example of $\mathcal{L}$ above. Min energy $\Rightarrow V'(\phi_0) = 0$

<u>Explain</u> all steps.

Here, $\phi_0^2 = \mu^2/4\lambda$; $\phi_0 = 0$

Check min or max. Describe space of solutions explicitly and in full detail.

Note: ϕ may be <u>complex</u>.

- Fluctuations around vac. solution

Choose vac. sol. ϕ_0. Consider $\phi = \phi_0 + \delta\phi$.
Terms quadratic in $\delta\phi$ give mass terms.
Higher orders give interactions.
Note that the mass term can be of the form $\delta\phi^T M \delta\phi$ and you may want to diagonalise M (or just find its eigenvalues) to see which fields are massless. In the example above, we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \mu^2 \Phi^2 - 2\sqrt{\lambda}\mu\Phi^3 - \lambda\Phi^4 + \mu^4/16\lambda$$

for $\phi = \phi_0 + \Phi$.

$\Rightarrow$ excitations around $\phi_0 = \pm\mu/2\sqrt{\lambda}$ are <u>massive</u>, with $m = \sqrt{2}\mu$.

(Need to count number of massive and massless fluctuations.)

- Goldstone theorem

Spontaneous breaking of a global contin. symmetry => massless excitations around vac. state (the number of such excit. is equal at least to the number of the broken symmetry generators).

<u>Note</u>: in the example above, excitations are massive since $\mathbb{Z}_2$ (which is broken) is not a continuous symmetry.

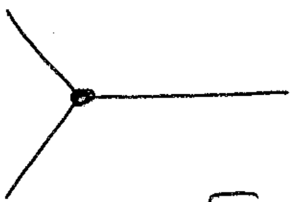
<u>Note</u>: symmetry may be broken only partially - need to analyze if the vac. state still has a residual symmetry. Count the number of generators. See lectures and lecture notes for examples.

- Interaction terms

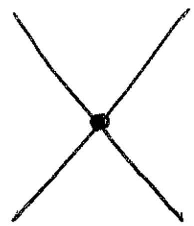
In the example above :

$$\mathcal{L}_{int} = \mp 2\sqrt{\lambda}\,\mu\,\phi^3 - \lambda\,\phi^4$$

$\Rightarrow$ Feynman rules



$2\sqrt{\lambda}\,\mu\,3!$



$4!\,\lambda$

- Classical dimension of fields

ex: $V(\psi,\psi^*) = -\alpha\,(\psi^*\psi)^3 + \beta\,(\psi^*\psi)^5$

with standard kinetic term $\partial_\mu\psi^*\partial^\mu\psi$.

In $d=4$, since the action in units $\hbar=1$ is dimensionless $\sim \int d^4x\, \partial_\mu\psi^*\partial^\mu\psi$

$\Rightarrow [\psi] = M = 1/L$. $\Rightarrow [\alpha] = M^{-2}$,

$[\beta] = M^{-6}$.

Note that V is invar. under $U(1)$: $\psi \to e^{i\alpha}\psi$.

2) Canonical quantisation

- canonical momenta:

$$\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$$

- Hamiltonian $H = \int d^3x \, \mathcal{H}$, where

$$\mathcal{H} = \pi_i \dot{\phi}_i - \mathcal{L} \Big|_{\dot{\phi}_i = \dot{\phi}_i(\pi)}$$

<u>Note</u>: relation $\dot{\phi}_i \longleftrightarrow \pi_i$ is not always 1-1. Most interesting systems have singular $\mathcal{L}$s : $\det\left(\partial^2 \mathcal{L} / \partial \dot{\phi}_i \partial \dot{\phi}_j\right) = 0$.

Can have e.g. $\pi_i = 0$. Then Hamilton's e.o.m.

$$\dot{\pi}_i = \{\pi_i, H\}_{PB} = 0$$

may give more constraints.

Reminder: Poisson Bracket of two fields

$$\{F, G\}_{PB} = \int d^3x' \left(\frac{\delta F}{\delta \phi_i(t, \bar{x}')} \frac{\delta G}{\delta \pi_i(t, \bar{x}')} - \frac{\delta F}{\delta \pi_i} \frac{\delta G}{\delta \phi_i} \right)$$

Here $$\frac{\delta \phi_i(t,\bar{x})}{\delta \phi_j(t,\bar{x}')} = \delta^{(3)}(\bar{x}-\bar{x}')\,\delta_{ij}$$

and $$\frac{\delta \phi_i(t,\bar{x})}{\delta \pi_j(t,\bar{x}')} = 0 .$$

Exercise: compute $\{F, G\}_{PB}$ with different F, G, both elementary fields such as $\pi_i(x)$ and composites such as $F = \int d^3x\, \pi^2(x)$.

Canonical quantization

Dirac rul: $[\hat{A}, \hat{B}] = i\hbar \{A, B\}_{PB}$

- Solving <u>free</u> eom via Fourier

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\bar{p}}} \left(a_{\bar{p}}\, e^{-ipx} + a^*_{\bar{p}}\, e^{ipx} \right)$$

Need to know how to show this in detail (see lectures)

- All aspects of two-point functions for free scalar fields: integral repres., properties such as causality, equations they satisfy etc.
- Review in full the scheme of canonical quantization: how to construct operators such as $\hat{P}^\mu$ etc in terms of $a, a^\dagger$, their commut. properties etc. Should be able to do this without hints.
- Should be able to find Green's function of a linear operator using Fourier transform method. Need to know everything about Fourier repres. of $G_F(x-y)$: see lecture notes.