C6 - THEORETICAL PHYSICS

2019 EXAM PAPER

SOLUTION NOTES (INFORMAL)

Not for distribution

andrei.starinets@physics.ox.ac.uk

4. (a) State Goldstone's theorem and prove it in the context of classical scalar fields. [7]

(b) A real scalar field has the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}\mu^2\phi^2 - \lambda\phi^4,$$

with μ^2, $\lambda > 0$. What are the symmetries of this theory? Determine a vacuum solution, $\langle\phi\rangle = \phi_0$. Writing $\phi = \phi_0 + \Phi$, determine the effective Lagrangian for Φ including its mass and interactions. Does the vacuum break symmetry, and are your results for the spectrum consistent with Goldstone's theorem? [6]

(c) For scattering processes involving Φ, write down the Feynman rules for vertices and propagators. For the specific process $\Phi\Phi \to \Phi\Phi$ (at tree-level), draw all relevant Feynman diagrams (you are *not* required to compute the scattering amplitude). [3]

(d) We next consider a complex scalar field Ψ with standard kinetic terms and potential

$$V(\Psi, \Psi^*) = -\alpha(\Psi^*\Psi)^3 + \beta(\Psi^*\Psi)^5.$$

What are the dimensions of α and β? What are the symmetries of the theory? Determine a vacuum solution $\langle\Psi\rangle = \Psi_0$ and the masses of excitations about it. Are your results consistent with Goldstone's theorem? [6]

(e) Denoting heavy and light real scalar fields in your spectrum as H and L respectively, by considering residual discrete symmetries determine the H^2L, H^3L and L^3 couplings. Using these draw all tree-level diagrams contributing to $LL \to LL$ scattering. [3]

Q4: 25
Section:100

5. (a) A real classical scalar field has the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2.$$

Derive the equations of motion for ϕ, and solve them as a mode expansion in plane waves with coefficients $a(k)$. Derive expressions for the conjugate momentum $\pi(x)$ and the Hamiltonian, and evaluate the Hamiltonian explicitly in terms of $a(k)$. [9]

(b) For a quantum scalar field, state appropriate canonical commutation relations for the mode operators $a(k)$ and $a^\dagger(k)$. Using these, evaluate the equal-time correlators $[\phi(x),\, \phi(y)]$ and $[\phi(x),\, \pi(y)]$, and also the 2-point function $\langle 0|[\phi(x_A), \phi(x_B)]|0\rangle$, where the events x_A and x_B can have arbitrary space-like separation. [6]

(c) Now re-consider the classical scalar field of the first part, taking $m = 0$, with general mode amplitudes $a(k)$ but such that the overall energy in the field is finite. Suppose that at time $t = 0$ a rapid external influence causes a rescaling of the spatial dimensions $\mathbf{x} \to \lambda \mathbf{x}$, with the effect on the field ϕ that

$$\phi_{new}(\mathbf{x},\, t = 0) = \phi_{old}\left(\frac{\mathbf{x}}{\lambda},\, t = 0\right).$$

Work out the corresponding transformation in the mode amplitudes $a(k)$. How has the energy in the field changed? [5]

(d) Write down the quantum Hamiltonian for a free massless quantum scalar field (the derivation is not required). An adiabatic transformation takes $\mathbf{x} \to \lambda \mathbf{x}$ and $\mathbf{p} \to \frac{\mathbf{p}}{\lambda}$. How does the overall energy in the field transform? Qualitatively discuss how your results here would be modified if $m \neq 0$. [5]

Q5: 25
Section:125

6. (a) Explain what is meant by the *interaction picture* and derive the Schrödinger equation in this picture. [6]

(b) State and prove Dyson's formula for the computation of scattering amplitudes in quantum field theory. [7]

(c) State Wick's theorem, defining relevant quantities, and illustrate it by obtaining a formal expression for

$$\langle 0| T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi^3(x_4)\right)|0\rangle .$$

Is this expression finite? [6]

(d) We now consider the decays of a massive scalar field Φ with mass m. This field interacts with both electromagnetism and an additional scalar a via the Lagrangian

$$\mathcal{L} = \mathcal{L}_{kinetic} + \mathcal{L}_{int},$$

with

$$\mathcal{L}_{kinetic} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu a \partial^\mu a$$

and

$$\mathcal{L}_{int} = \frac{1}{2}m^2\Phi^2 + \frac{1}{2}m_a^2 a^2 + \frac{\lambda\Phi}{M_P}F_{\mu\nu}F^{\mu\nu} + g\Phi a^2.$$

You can assume $M_P \gg m_\Phi$, g^{-1}. By deriving estimates for the possible decay modes, plot how the Φ lifetime will depend on its mass m_Φ, indicating how the different decay modes contribute. [6]

Q6: 25
Section:150

Total Marks in Paper 150

C6 2019 Exam Part 2

Q4. Goldstone's Theorem states that breaking a global symmetry spontaneously leads to massless excitations (scalar particles) around vacuum state. The number of the excitations equals - at least - to the number of the broken symmetry generators.

For scalar fields $\varphi = (\varphi_1 \cdots \varphi_n)$ vac. state corresponds to

$$\frac{\partial V}{\partial \varphi}\Big|_{\varphi=\varphi_0} = 0 .$$

For fluctuations around φ_0, $\varphi = \varphi_0 + \delta\varphi$, we have

$$V(\varphi) = V(\varphi_0) + \frac{1}{2}\frac{\partial^2 V}{\partial\varphi_a \partial\varphi_b}(\varphi_0)\,\delta\varphi_a\,\delta\varphi_b + \ldots$$

Masses of excitations are determined by the eigenvalues of $M_{ab} = \frac{\partial^2 V}{\partial\varphi_a \partial\varphi_b}(\varphi_0)$.

If $\mathcal{L}$ (and thus V) is invariant under a global symmetry group with $\delta\varphi^a = \varepsilon^\alpha T_\alpha^{ab}\phi_b$, then $\mathcal{L}(\varphi+\delta\varphi)-\mathcal{L}(\varphi) = \delta\mathcal{L} = 0 \Rightarrow \frac{\partial V}{\partial\varphi^a} T_\alpha^{ab}\phi_b\,\varepsilon^\alpha = 0.$

This applies also to $\varphi_a = \varphi_a^0 + \delta\varphi_a$:

$$\frac{\partial V}{\partial\varphi^a}(\varphi_0^a+\delta\varphi^a)\, T_\alpha^{ab}(\varphi^b+\delta\varphi^b)\,\varepsilon^\alpha = 0$$

Expanding around φ_0^a and taking into account that $\frac{\partial V}{\partial\varphi^a}(\varphi_0) = 0$, we have

$$\frac{\partial^2 V}{\partial\varphi^a\partial\varphi^c}(\varphi_0^a)\,\delta\varphi^c\, T_\alpha^{ab}(\varphi_0^b+\delta\varphi^b)\,\varepsilon^\alpha = 0$$

$$+ \dots$$

$$\Rightarrow M_{ac}\,\delta\varphi^c\, T_\alpha^{ab}\varphi_0^b\varepsilon^\alpha + O(\delta\varphi^2) = 0.$$

Suppose we have N generators ($\alpha = 1,\dots N$), and for some of them $T_\alpha^{ab}\varphi_0^b \neq 0$ (symmetry is spontaneously broken by vac. state). Then we should have

$M_{ac}\,\delta\varphi^c = 0$, i.e. M_{ab} should have zero eigenvalue $\Rightarrow$ excitation is massless, and their number is at least equal to the number of such „broken" generators.

(b) $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \lambda\phi^4$,

$\mu^2, \lambda > 0$

- <u>Symmetries</u>: Poincaré invariance

  $\mathbb{Z}_2$ invar: $\phi \to -\phi$.

- <u>Vacuum solution</u>

E.o.m. $\quad \Box\phi + V'(\phi) = 0$

Also, the energy:

$$P^0 = \int d^3x\, T^{00} = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V\right)$$

where $\pi = \partial_0\phi$. Min energy $\Rightarrow \phi = $ const and min of V $\Rightarrow$ vac. solution is $\phi = \phi_0 = $ const such that $V'(\phi) = 0$.

$\Rightarrow \mu^2\phi_0 - 4\lambda\phi_0^3 = 0 \Rightarrow \phi_0^2 = \mu^2/4\lambda$.

$V'' = -\mu^2 + 12\lambda\phi^2 \Rightarrow \phi_0^2 = \mu^2/4\lambda$

is a min. $\qquad \phi_0 = \pm\,\mu/2\sqrt{\lambda}$.

Another solution, $\phi_0 = 0$, is a max.

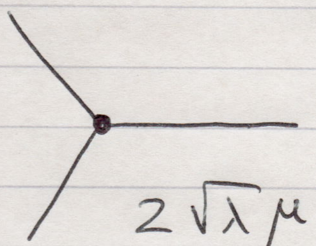
- With $\phi = \phi_0 + \Phi$, we find the effective Lagrangian for Φ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \mu^2\Phi^2 - 2\sqrt{\lambda}\,\mu\Phi^3 - \lambda\Phi^4 + \mu^4/16\lambda$$

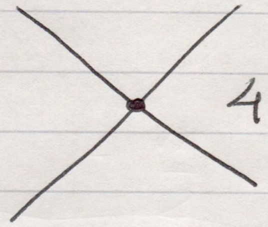
Excitations Φ around $\phi_0 = \pm\mu/2\sqrt{\lambda}$ are <u>massive</u>, with $m = \sqrt{2}\,\mu$. Interaction terms are $\mp 2\sqrt{\lambda}\,\mu\,\Phi^3 - \lambda\Phi^4$.

- The vac. breaks $\mathbb{Z}_2$ symmetry
- This is consistent with Goldstone's theorem (we have massive excitations around vac but the broken symmetry is <u>not a continuous</u> symmetry).

(c) The form of interaction terms in the effective Lagrangian implies the following Feynman rules for vertices:



$2\sqrt{\lambda}\,\mu\, 3!$



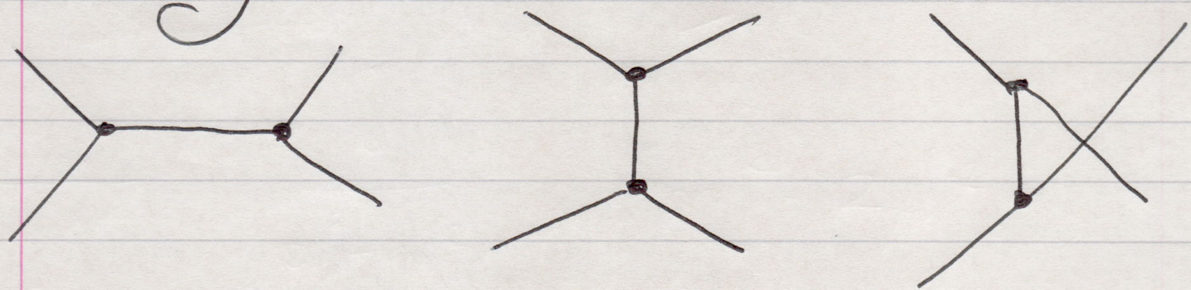
$4!\,\lambda$

For the propagator we have:

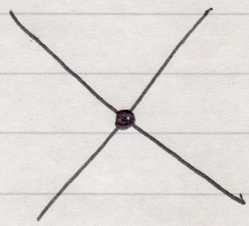
$$\frac{i}{p^2 - m^2},$$

where $m^2 = 2\mu^2$.

<u>Tree-level</u> diagrams for $\phi\phi \to \phi\phi$ scattering are:



and



(d) $V(\psi, \psi^*) = -\alpha(\psi^*\psi)^3 + \beta(\psi^*\psi)^5$

- In 4d, since the action in units $\hbar = 1$ is dimensionless, we have $[\psi] = M$. Thus, $[\alpha] = M^{-2}$, $[\beta] = M^{-6}$.

- Symmetries: - Poincaré symmetry
  - $U(1)$ global symmetry:

$$\psi \to e^{i\alpha}\psi, \text{ where } \alpha = \text{const}.$$

- Vacuum is determined by the condition $V'_\psi = 0$:

$$-3\alpha(\psi^*\psi)^2\psi^* + 5\beta(\psi^*\psi)^4\psi^* = 0$$

$$\Rightarrow (\psi^*\psi)^2 = 3\alpha/5\beta$$

($V'_{\psi^*} = 0$ gives the same result.)

It is a circle of vacua $\psi_0 = |\psi_0| e^{i\delta}$, parametrised by δ, with $|\psi_0|^2 = \sqrt{\frac{3\alpha}{5\beta}}$.

Choose one of them, e.g. the one corresp. $\delta = 0$, and consider small fluctuations around it : $\psi = |\psi_0| + X + iY$.
Expanding V to quadratic order in X, Y, we find the terms

$$\frac{36\alpha^2}{5\beta} X^2 + 0 \cdot Y^2$$

$\Rightarrow$ we have one massive scalar with $m_X^2 = \frac{72\alpha^2}{5\beta}$ and one massless scalar.

Since $U(1)$ symmetry is spontaneously broken by the vac. solution (there is one „broken" generator), we indeed expect one massless excitation by Goldstone's theorem.

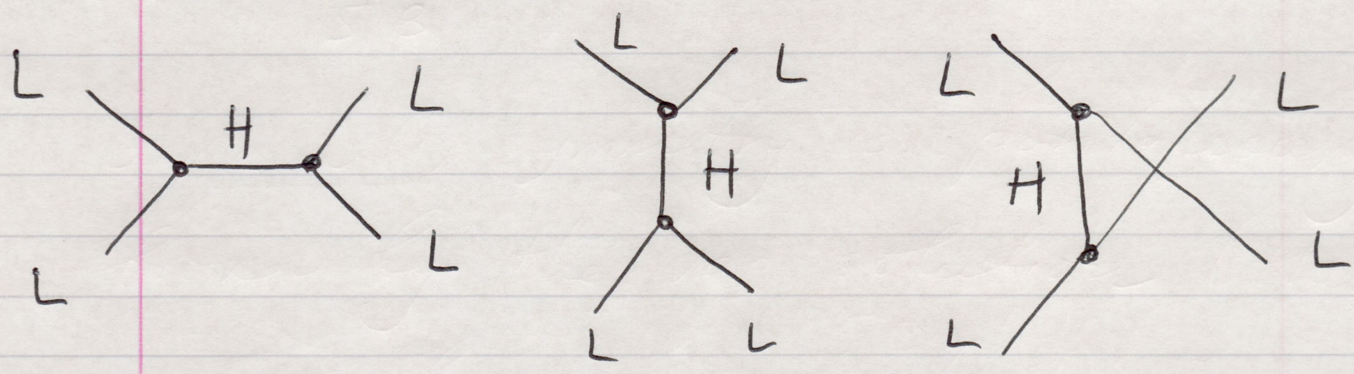
(e) We now redefine $L \equiv Y$ and $H \equiv X$. The potential depends on

$\psi^*\psi = (|\psi_0|^2 + X)^2 + Y^2$ and thus has a symmetry $Y \to -Y$

$\Rightarrow$ in the expansion of $\bar{V}$ in X, Y, terms H^2L, H^3L, L^3 must vanish.

This can also be seen explicitly. Thus, the couplings H^2L, H^3L, L^3 vanish.

- Tree-level diagrams contributing to $LL \to LL$ scattering come from Y^2X and Y^4 terms:



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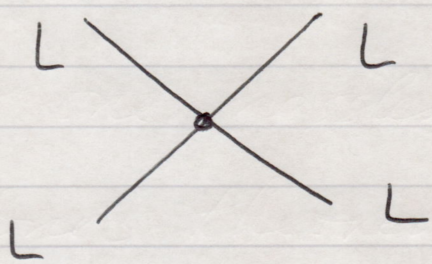


Q5. Classical real scalar field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 .$$

- $\mathcal{E}$-L eqs: $\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = \frac{\partial \mathcal{L}}{\partial \phi}$

$$\Rightarrow \partial_\mu (\partial^\mu \phi) = -m^2 \phi \Rightarrow (\Box + m^2)\phi = 0$$

$$\Rightarrow (\partial^2_{tt} - \partial^2_i + m^2)\phi = 0$$

- Eom can be solved by using Fourier decomposition for $\phi(t, \bar{x})$:

$$\phi(t,\bar{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i\bar{k}\bar{x}} \phi(t,\bar{k})$$

$$\Rightarrow \ddot{\phi}(t,\bar{k}) + \omega^2_{\bar{k}} \phi(t,\bar{k}) = 0,$$

where $\omega_{\bar{k}} = \sqrt{|\bar{k}|^2 + m^2}$.

General solution:

$$\phi(t,\bar{k}) = a^{(1)}_{\bar{k}} e^{-i\omega_{\bar{k}} t} + a^{(2)}_{\bar{k}} e^{i\omega_{\bar{k}} t}.$$

Since $\phi(t,\bar{x})$ is real, $a^{(2)}_{\bar{k}} = a^{(1)*}_{\bar{k}}$.

We can write $a^{(1)}_{\bar{k}} = N_{\bar{k}} a_{\bar{k}}$,

$a^{(2)}_{\bar{k}} = N_{\bar{k}}\, a^*_{\bar{k}}$, where $N_{\bar{k}} \in \mathbb{R}$ is a normalisation constant. Then

$$\phi(t,\bar{x}) = \int \frac{d^3k}{(2\pi)^3} N_{\bar{k}} \left(a_{\bar{k}} e^{-ikx} + a^*_{\bar{k}} e^{ikx} \right),$$

where $kx = k^0x^0 - \bar{k}\bar{x}$, is a general solution to the e.o.m. A convenient choice of normalisation is $N_{\bar{k}} = 1/2\omega_{\bar{k}}$.

• $\pi(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(t,\bar{x}) =$

$$= \int \frac{d^3k}{(2\pi)^3} (-i\omega_{\bar{k}}) N_{\bar{k}} \left(a_{\bar{k}} e^{-ikx} - a^*_{\bar{k}} e^{ikx} \right).$$

The Hamiltonian is $H = \int d^3x\, \mathcal{H}$, where the Hamiltonian density is defined as

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \Big|_{\dot{\phi} \to \pi} =$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\,\partial_i\phi + \frac{1}{2}m^2\phi^2.$$

One can use $\displaystyle\int \frac{d^3x}{(2\pi)^3} e^{i(\bar{k}+\bar{q})\bar{x}} = \delta^{(3)}(\bar{k}+\bar{q})$

to integrate over $\bar{x}$.

We have

$$\frac{1}{2}\pi^2 = -\frac{1}{2}\int\frac{d^3k}{2(2\pi)^3}\int\frac{d^3q}{2(2\pi)^3}\left(a_{\bar{k}}e^{-ikx} - \right.$$

$$\left. - a^*_{\bar{k}}e^{ikx}\right)\left(a_{\bar{q}}e^{-iqx} - a^*_{\bar{q}}e^{iqx}\right) =$$

$$= -\frac{1}{8}\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[a_{\bar{k}}a_{\bar{q}}e^{-ikx-iqx} - \right.$$

$$- a_{\bar{k}}a^*_{\bar{q}}e^{-ikx+iqx} - a^*_{\bar{k}}a_{\bar{q}}e^{ikx-iqx} +$$

$$\left. + a^*_{\bar{k}}a^*_{\bar{q}}e^{ikx+iqx}\right]$$

$$\frac{1}{2}\partial_i\phi\,\partial_i\phi = \frac{1}{2}\int\frac{d^3k}{2\omega_k(2\pi)^3}\int\frac{d^3q}{2\omega_q(2\pi)^3}\times$$

$$\times\left(ik_i a_{\bar{k}}e^{-ikx} - ik_i a^*_{\bar{k}}e^{ikx}\right)\times$$

$$\times\left(iq_i a_{\bar{q}}e^{-iqx} - iq_i a^*_{\bar{q}}e^{iqx}\right)$$

$$\frac{1}{2}m^2\phi^2 = \frac{m^2}{2}\int\frac{d^3k}{2\omega_k(2\pi)^3}\int\frac{d^3q}{2\omega_q(2\pi)^3}\times$$

$$\times\left(a_{\bar{k}}e^{-ikx} + a^*_{\bar{k}}e^{ikx}\right)\left(a_{\bar{q}}e^{-iqx} + a^*_{\bar{q}}e^{iqx}\right).$$

Combining this, we find

$$\frac{1}{2}\pi^2 + \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 =$$

$$= \frac{1}{2}\int\frac{d^3k}{2\omega_k(2\pi)^3}\int\frac{d^3q}{2\omega_q(2\pi)^3}\Big[\left(-\omega_{\bar k}\omega_{\bar q} - k_i q_i + m^2\right)$$

$$\times\left(a_{\bar k}a_{\bar q}e^{-ikx-iqx} + a^*_{\bar k}a^*_{\bar q}e^{ikx+iqx}\right)$$

$$+\left(\omega_{\bar k}\omega_{\bar q} + k_i q_i + m^2\right)\left(a_{\bar k}a^*_{\bar q}e^{-ikx+iqx} +\right.$$

$$\left. + a^*_{\bar k}a_{\bar q}e^{ikx-iqx}\right)\Big].$$

For $H = \int d^3x\,\mathcal{H}$, integrating over $\bar x$ and then over $\bar q$, we find $\bar q = -\bar k$ in the first half of the above expression and $\bar q = \bar k$ in the second. In both cases, $\omega_{\bar k} = \omega_{\bar q}$, since they depend on $|\bar k|$ or $|\bar q|$. Thus, we have $(-\omega^2_{\bar k} + \bar k^2 + m^2)$ in the first case and $(\omega^2_{\bar k} + \bar k^2 + m^2)$ in the second.

But $\omega_k^2 = \bar{k}^2 + m^2$, so the first bracket vanishes. We obtain (B)

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \frac{a_{\bar{k}} a^*_{\bar{k}} + a^*_{\bar{k}} a_{\bar{k}}}{2}.$$

Therefore, for classical fields

$$H = \int \frac{d^3k}{2\omega_k (2\pi)^3} \omega_{\bar{k}} a^*_{\bar{k}} a_{\bar{k}}.$$

(b) For a quantum scalar field, we have $[\hat{a}_{\bar{k}}, \hat{a}^+_{\bar{k}'}] = (2\pi)^3 2\omega_{\bar{k}} \delta^{(3)}(\bar{k}-\bar{k}')$,
$[\hat{a}_{\bar{k}}, \hat{a}_{\bar{k}'}] = 0$, $[\hat{a}^+_{\bar{k}}, \hat{a}^+_{\bar{k}'}] = 0$.
The quantum field is written as

$$\phi(t,\bar{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(\hat{a}_k e^{-ikx} + \hat{a}^+_{\bar{k}} e^{ikx}\right),$$

so the equal-time commutators are

$$[\phi(t,\bar{x}), \phi(t,\bar{y})] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3q}{(2\pi)^3 2\omega_q} \times$$

$$\times \left([\hat{a}_k \hat{a}^+_{\bar{q}}] e^{+i\bar{k}\bar{x} - i\bar{q}\bar{y}} e^{-i\omega_{\bar{k}} t + i\omega_{\bar{q}} t} + \right.$$

$$+ [\hat{a}^{+}_{\bar{k}} \hat{a}_{\bar{q}}] e^{-i\bar{k}\bar{x} + i\bar{q}\bar{y}} e^{i\omega_k t - i\omega_q t} \Big)$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\omega_{\bar{k}}} \int \frac{d^3 q}{(2\pi)^3 2\omega_{\bar{q}}} \Big((2\pi)^3 2\omega_{\bar{k}}\, \delta^{(3)}(\bar{k}-\bar{q}) e^{\cdots}$$

$$- (2\pi)^3 2\omega_{\bar{k}}\, \delta^{(3)}(\bar{k}-\bar{q}) e^{\cdots} \Big) =$$

$$= \int \frac{d^3 k}{(2\pi)^3 2\omega_{\bar{k}}} \Big(e^{i\bar{k}(\bar{x}-\bar{y})} - e^{-i\bar{k}(\bar{x}-\bar{y})} \Big).$$

This is zero, since an odd function of $\bar{k}$ is integrated from $-\infty$ to ∞.

For $[\phi(t,\bar{x}), \pi(t,\bar{y})]$ commutator we find:

$$[\phi(t,\bar{x}), \pi(t,\bar{y})] = -\frac{i}{2} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \int \frac{d^3 q}{(2\pi)^3}$$

$$\times \Big(-[\hat{a}_{\bar{k}} \hat{a}^{+}_{\bar{q}}] e^{i\bar{k}\bar{x} - i\bar{q}\bar{y}} e^{-i\omega_{\bar{k}} t + i\omega_{\bar{q}} t} +$$

$$+ [\hat{a}^{+}_{\bar{k}} \hat{a}_{\bar{q}}] e^{-i\bar{k}\bar{x} + i\bar{q}\bar{y}} e^{i\omega_k t - i\omega_q t} \Big) =$$

$$= i \int \frac{d^3 k}{(2\pi)^3} e^{i\bar{k}(\bar{x}-\bar{y})} = i\, \delta^{(3)}(\bar{x}-\bar{y}).$$

• The function $\Delta(x_A - x_B) =$ $= [\phi(x_A), \phi(x_B)]$ is known as Pauli-Jordan function. Since $\phi(x)$ is linear in $\hat{a}, \hat{a}^+$, the commutator of $\phi(x_A)$ and $\phi(x_B)$ is a c-number and $\langle 0 | [\phi(x_A), \phi(x_B)] | 0 \rangle =$ $= [\phi(x_A), \phi(x_B)]$. We have then:

$$\Delta(x_A - x_B) = \int d^3\tilde{k}\, d^3\tilde{q} \Big([a_{\bar{k}} a^+_{\bar{q}}] e^{-ikx_A + iqx_B} +$$

$$+ [a^+_{\bar{k}} a_{\bar{q}}] e^{ikx_A - iqx_B} =$$

$$= \int d^3\tilde{k}\, d^3\tilde{q} \left(e^{-ikx_A + iqx_B} - e^{ikx_A - iqx_B} \right) \times$$
$$\times (2\pi)^3 2\omega_{\bar{k}}\, \delta(\bar{k} - \bar{q}) =$$

$$= \int d^3\tilde{k} \left(e^{-ik(x_A - x_B)} - e^{ik(x_A - x_B)} \right)$$

where the notation $\int d^3\tilde{k} \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\bar{k}}}$ was used. This expression can be written in a covariant form using

$$\delta(k^2-m^2) = \frac{1}{2\omega_{\bar{k}}}\left[\delta(k_0+\omega_{\bar{k}}) + \delta(k_0-\omega_{\bar{k}})\right]$$

$$\Rightarrow \quad \Delta(x_A - x_B) = \int \frac{d^4k}{(2\pi)^3}\, \delta(k^2-m^2)\, \varepsilon(k_0)\, e^{-ik(x_A-x_B)}$$

where $\varepsilon(k_0) = k_0/|k_0| = \operatorname{sgn} k_0$.

- For space-like separated x_A and x_B, we can just use the result for equal-time correlator,

$$[\phi(t,\bar{x}), \phi(t,\bar{y})] = \int \frac{d^3k}{(2\pi)^3 2\omega_{\bar{k}}} \left(e^{-ik(x-y)} - e^{ik(x-y)}\right) = 0,$$

and the fact that the measure $\frac{d^3k}{2\omega_{\bar{k}}}$ and $k(x-y)$ are Lor-invar. $\Rightarrow$ the Lor. boost connects the correlator at any space-like separated x_A, x_B to the equal-time corr. $\Rightarrow \langle 0|[\phi(x_A)\phi(x_B)]|0\rangle = 0$ for space-like separated x_A, x_B.

(c) For massless field $\phi(t, \bar{x})$, we have the expansion:

$$\phi(t,x) = \int \frac{d^3k}{2\omega_{\bar{k}}(2\pi)^3}\left(a_{\bar{k}} e^{-ikx} + a^*_{\bar{k}} e^{ikx}\right),$$

where $\omega_k = |\bar{k}|$.

If $\phi_{new}(\bar{x}, t=0) = \phi\left(\frac{\bar{x}}{\lambda}, t=0\right)$, then

$$\phi_{new}(\bar{x}, t=0) = \int \frac{d^3k}{2\omega_{\bar{k}}(2\pi)^3}\left(a_{\bar{k}} e^{i\bar{k}\bar{x}/\lambda} + a^*_{\bar{k}} e^{-i\bar{k}\bar{x}/\lambda}\right)$$

Changing var. to $k_i' = k_i/\lambda$, we get

$$\phi_{new}(\bar{x}, t=0) = \lambda^3 \int \frac{d^3k'}{2\omega_{\bar{k}'}} \frac{\omega_{\bar{k}'}}{\omega_{\bar{k}}} \frac{1}{(2\pi)^3}\left(a_{\bar{k}} e^{i\bar{k}'\bar{x}} + a^*_{\bar{k}} e^{-i\bar{k}'\bar{x}}\right)$$

So, $a_{new}(\bar{k}') = \lambda^3 \frac{\omega_{\bar{k}'}}{\omega_{\bar{k}}} a(\bar{k}) =$

$= \lambda^2 a(\lambda k')$

Thus, $a_{new}(\bar{k}') = \lambda^2 a(\lambda k')$.

The Hamiltonian transforms as

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, a^*(k)\, a(k) =$$

$$= \frac{1}{2}\int \frac{\lambda^3 d^3k'}{(2\pi)^3}\,\frac{1}{\lambda^4}\, a^*_{new}(k')\, a_{new}(k') =$$

$$= \frac{1}{\lambda} H_{new}. \quad \text{Thus,} \quad E_{new} = \lambda E.$$

(d) The quantum Hamiltonian for a free massless scalar field is

$$:\hat{H}: = \int d^3\tilde{k}\ \omega_{\bar{k}}\, \hat{a}^+_{\bar{k}} \hat{a}_{\bar{k}},$$

where $\omega_{\bar{k}} = |\bar{k}|$ in massless case.

Acting on a single-particle state $|k\rangle$, it gives $:\hat{H}: |k\rangle = \omega_{\bar{k}} |k\rangle$. Multi-particle states $|k_1 \ldots k_n\rangle$ are eigenstates of $:\hat{H}:$ with eigenvalue $\omega_{\bar{k}_1} + \ldots \omega_{\bar{k}_n}$.

Under $k \to k/\lambda$: $\omega_{\bar{k}} \to \omega_{\bar{k}}/\lambda$, i.e. the energy of massless modes decreases by a factor of λ, with excitation number unchanged.

- For massive modes, $\omega_{\bar{k}}^2 = \bar{k}^2 + m^2$, so in rel. case $(|\bar{k}| \gg m)$ $\omega_{\bar{k}} \to \omega_{\bar{k}}/\lambda$, in non-rel $(|\bar{k}| \ll m)$: $\omega_{\bar{k}} \to \omega_{\bar{k}}$.

Q6. (a) In the Schrödinger picture, states are time-dep. and obey

$$i\hbar \frac{\partial}{\partial t} |\varphi\rangle_s = \hat{H} |\varphi\rangle_s .$$

In the Heisenberg picture, operators are time-dep and obey

$$\frac{d}{dt}\hat{O}_H = \frac{i}{\hbar}[\hat{H}, \hat{O}_H]$$

Since $|\varphi(\bar{x},t)\rangle_s = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}|\varphi(\bar{x},t_0)\rangle_s$

$$\equiv \hat{U}(t,t_0)|\varphi(\bar{x},t_0\rangle_s =$$

$$= \hat{U}(t,t_0)|\varphi(t_0)\rangle_H .$$

and we must have

$$\langle\varphi(\bar{x},t)|\hat{O}_s|\varphi(\bar{x},t)\rangle_s = \langle\varphi(t_0)|\hat{O}_H|\varphi(t_0)\rangle ,$$

$$\hat{O}_H = \hat{U}^+(t)\,\hat{O}_s\,\hat{U}(t).$$

Interaction picture is convenient for Hamiltonians of the type

$$\hat{H} = \hat{H}_0 + \hat{H}_I ,$$

where $\hat{H}_0$ is the Hamiltonian of a free theory. Introduce

$$\hat{\mathcal{O}}_{int.p.} = \hat{U}_0^+ \hat{\mathcal{O}}_s \hat{U}_0 \, , \text{ where}$$

$$\hat{U}_0 = e^{-\frac{i}{\hbar}\hat{H}_0 (t-t_0)} \, .$$ Then

$$\dot{\mathcal{O}}_{int.p.} = \frac{i}{\hbar}\left[\hat{H}_0, \hat{\mathcal{O}}_{int.p.}\right]$$

i.e. e.o.m. for $\hat{\mathcal{O}}_{int.p.}$ are the same as Heis. e.o.m. for free fields.

Again, phys. quantities (expect. values of operators) must be the same in any picture $\Rightarrow$

$$|\varphi(t)\rangle_s = U_0(t)\, |\varphi(t)\rangle_{int.p.}$$

Since

$$i\hbar \frac{\partial}{\partial t} |\varphi(t)\rangle_s = \left(\hat{H}_0 + \hat{H}_I\right) |\varphi(t)\rangle_s \, ,$$

for $|\varphi(t)\rangle_{int.p.}$ we find the eq.

$$i\hbar \frac{\partial}{\partial t} |\varphi(t)\rangle_{int.p.} = \hat{H}_I^{int.p.} |\varphi(t)\rangle_{int.p.} \, ,$$

where $\hat{H}_I^{i.p.} \equiv \hat{U}_0^{+} \hat{H}_I \hat{U}_0$.

(b) A formal solution to the Schrödinger eq in the int. picture can be written as $|\varphi(t)\rangle_{i.p.} = \hat{U}_I(t,t_0)|\varphi(t_0)\rangle_{i.p.}$, where $\hat{U}_I(t,t_0) = e^{-\frac{i}{\hbar}\hat{H}_I^{ip}(t-t_0)}$

This evolution operator itself obeys the same eq.

$$i\hbar \frac{\partial}{\partial t} \hat{U}_I(t,t_0) = \hat{H}_I^{i.p.} \hat{U}_I(t,t_0)$$

which can be re-written as an integral eq

$$\hat{U}_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{H}_I^{ip}(t') \hat{U}_I(t',t_0).$$

The integral eq. can be solved by iterations in powers of $\hat{H}_I^{ip}$ (more precisely, a small parameter such as the inter. constant contained in $\hat{H}_I^{ip}$) - this produces a <u>formal</u> series

0^{th} order : $\hat{U}_I(t, t_0) = \mathbb{1}$

1^{st} order : $\hat{U}_I(t, t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{H}_I^{ip}(t')$

2^{nd} order : $\hat{U}_I(t, t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{H}_I^{ip}(t') \times$

$$\times \left(\mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t'} dt'' \hat{H}_I^{ip}(t'') \right) =$$

$$= \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt' \hat{H}_I^{ip}(t') + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \hat{H}_I^{ip}(t') \times$$

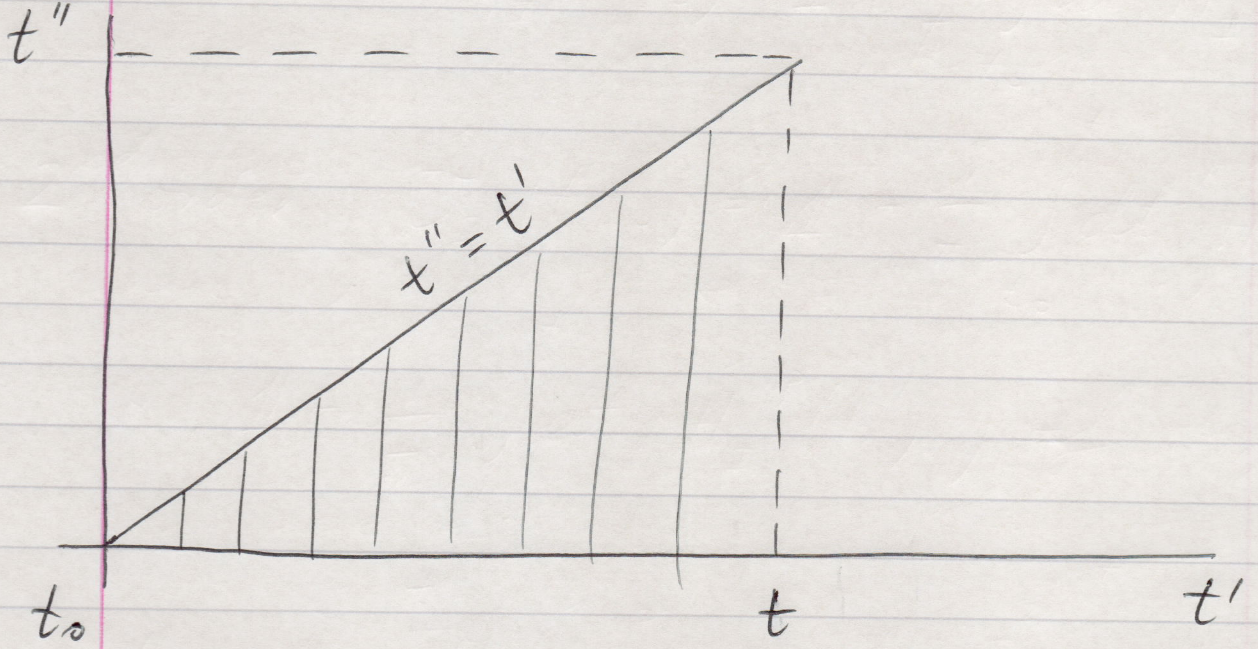
$$\times \int_{t_0}^{t'} dt'' \hat{H}_I^{ip}(t'') .$$

The last term can be written as

$$\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \hat{H}_I^{ip}(t') \hat{H}_I^{ip}(t'') =$$

$$= \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt'' \int_{t''}^{t} dt' \hat{H}_I^{ip}(t') \hat{H}_I^{ip}(t'')$$

It is helpful to consider the integration region and limits in Fig:



Changing the dummy var. $t'' \to t'$, $t' \to t''$, we have

$$\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t'}^{t} dt'' \, \hat{H}_I^{ip}(t'') \hat{H}_I^{ip}(t')$$

Thus, $\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, \hat{H}_I^{ip}(t') \hat{H}_I^{ip}(t'')$

$$= \frac{(-i/\hbar)^2}{2} \int_{t_0}^{t} dt' \left[\int_{t_0}^{t'} dt'' \, \hat{H}_I^{ip}(t') \hat{H}_I^{ip}(t'') + \int_{t'}^{t} dt'' \, \hat{H}_I^{ip}(t'') \times \hat{H}_I^{ip}(t') \right] =$$

$$= \frac{(-i/\hbar)^2}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, T\left[\hat{H}_I^{ip}(t') \hat{H}_I^{ip}(t'')\right],$$

where $T\left[\hat{\Theta}(t')\hat{\Theta}(t'')\right] = \begin{cases} \hat{O}(t')\hat{O}(t''), & t' > t'' \\ \hat{\Theta}(t'')\hat{\Theta}(t'), & t' < t''. \end{cases}$

Therefore,

$$\hat{U}_I(t, t_0) = \mathbb{1} - \frac{i}{\hbar} \int_{t_0}^{t} dt' \, \hat{H}_I^{ip}(t') +$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \frac{1}{2} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \, T\left[\hat{H}_I^{ip}(t') \hat{H}_I^{ip}(t'')\right]$$

$$+ \ldots$$

Finally,

$$\hat{U}_I(t, t_0) = T\left\{ \exp\left[-\frac{i}{\hbar} \int_{t_0}^{t} dt' \, \hat{H}_I^{ip}(t') \right] \right\}.$$

For scattering, $t \to +\infty$, $t_0 \to -\infty$,

$$\hat{S} = \hat{U}_I(\infty, -\infty) = T \exp\left[-\frac{i}{\hbar} \int d^4x \, \hat{\mathcal{H}}_I^{ip}(x) \right],$$

where $\mathcal{H}$ is the Hamiltonian density.

(c) Wick's theorem expresses time-ordered products of fields in terms of normal-ordered product and contractions:

$$T\left[\phi_{ip}(x_1)\cdots\phi_{ip}(x_n)\right] = :\phi_{ip}(x_1)\cdots\phi_{ip}(x_n):$$

$$+ :\text{all possible contractions}:,$$

where the contraction is defined as

$$\overbrace{\phi_{ip}(x)\,\phi_{ip}}(y) = D_F(x-y), \text{ with}$$

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{-ik(x-y)}.$$

• To compute $\langle 0|T\,\phi(x_1)\phi(x_2)\phi(x_3)\phi^3(x_4)|0\rangle$, first recall that vac. expectation value of normal-ordered uncontracted terms vanishes, since they are of the form $\langle 0|a^\dagger\ldots a|0\rangle = 0$. Thus, we need to list all possible contractions only:

$$\langle 0 | T\phi(x_1)\phi(x_2)\phi(x_3)\phi^3(x_4)|0\rangle =$$

$$= 3\mathcal{D}_F(x_1-x_2)\mathcal{D}_F(x_3-x_4)\mathcal{D}_F(0) +$$

$$+ 3\mathcal{D}_F(x_1-x_3)\mathcal{D}_F(x_2-x_4)\mathcal{D}_F(0) +$$

$$+ 3\mathcal{D}(x_1-x_4)\mathcal{D}(x_2-x_3)\mathcal{D}_F(0) +$$

$$+ 6\mathcal{D}_F(x_1-x_4)\mathcal{D}_F(x_2-x_4)\mathcal{D}_F(x_3-x_4).$$

This expression is divergent since $\mathcal{D}_F(0)$ is divergent.

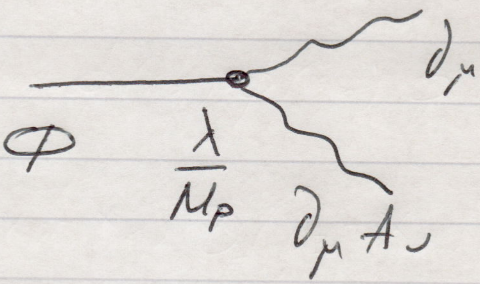
(d) $\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{int}$,

$$\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu a\partial^\mu a$$

$$\mathcal{L}_{int} = \frac{1}{2}m^2\phi^2 + \frac{1}{2}m_a^2 a^2 + \frac{\lambda\phi}{M_P}F_{\mu\nu}F^{\mu\nu} +$$

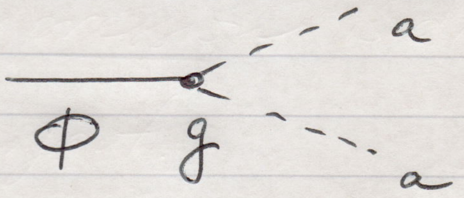
$$+ g\phi a^2$$

The field ϕ couples to A_μ via the (gauge-invar.) term $\phi F_{\mu\nu}F^{\mu\nu}$ and to a via ϕa^2. The two vertices are



[Note that this is the derivative coupling, i.e. in momentum space the vertex involves the two momenta k_1^μ and k_2^μ (in a gauge-invar. combination $\eta^{\mu\nu} k_1 \cdot k_2 - k_1^\mu k_2^\mu$).]

and



So we have 2 channels of decay for Φ.

Total decay width

$$\Gamma = \frac{1}{2\omega_\Phi} \sum_{n=1,2} \int d\Pi_n \left| \mathcal{M}_{fi}^{(n)} \right|^2 = \Gamma_1 + \Gamma_2$$

and the life-time $\tau = 1/\Gamma$.

In the rest frame of Φ, $\omega_\Phi = m$.

Also, $\mathcal{M}_{fi}^{(1)} \sim \frac{\lambda}{M_P} \left(\eta^{\mu\nu} k_1 k_2 - k_1^\mu k_2^\nu \right)$

$\mathcal{M}_{fi}^{(2)} \sim g$

Kinematically, channel 1 is available $\forall\, m$ (since photons are massless), and $\mathcal{M}_{fi}^{(1)} \sim \dfrac{\lambda}{M_p}\,\mathcal{E}_\gamma^2$, with $\mathcal{E}_\gamma \sim m/2$.

Thus, $\Gamma_1 \sim \dfrac{\lambda^2}{M_p^2}\,\dfrac{1}{m}\,m^4 \sim \dfrac{\lambda^2 m^3}{M_p^2}$.

Channel 2 is only available for $m > 2m_a$, and $\Gamma_2 \sim g^2/m$.

Therefore, for $0 < m < 2m_a$, we have $\tau \sim \tau_1 = M_p^2/\lambda^2 m^3$. For $m > 2m_a$, the second channel gives $\tau_2 \sim m/g^2$. With $g^{-1}M_p \gg 1$, $M_p/m \gg 1$, we have $\Gamma_1 \ll \Gamma_2$ (when channel 2 is available).

In summary: