Three regularizations as subgrid models

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ABSTRACT

As the implementation of numerical modeling of geophysical and astrophysical flows exceeds technological limits and since truncation of the omitted scales removes important physics, a closure (e.g.,subgrid (subfilter) modeling) is required. Regularization subgrid models for this closure problem have recently emerged. Unlike many Large Eddy Simulations (LES), these models have guarantees on the computability of their solutions, conserve energy, and recover the physical equations as the filter width vanishes. Three regularizations can be viewed as LES with successively more complex subgrid-stress terms: the Clark-α model, the Leray-α model, and the Lagrangian-averaged Navier-Stokes (LANS-α) model. Clark-α is associated with the first term of a Taylor expansion of the subfilter-scale stress. Leray-α and LANS-α represent advection by a smoothed velocity in an Eulerian and a Lagrangian sense, respectively. These regularizations are studied both for their sub-filter scale properties and for their potential as SGS models.

I) THE THREE REGULARIZATION MODELS

Navier-Stokes:

$$\partial_t v_i + \partial_j(v_j v_i) + \partial_i p = \nu \partial_{jj} v_i \quad \nabla\cdot\mathbf{v} = 0$$

Leray-α:

$$\partial_t v_i + \partial_j(u_j v_i) + \partial_i \mathcal{P} = \nu \partial_{jj} v_i$$

LANS-α:

$$\partial_t v_i + \partial_j(u_j v_i) + \partial_i \pi + v_j \partial_i u_j = \nu \partial_{jj} v_i$$

Clark-α :

$$\partial_t v_i + (1-\alpha^2\partial_{kk})\partial_j(u_j u_i) + \partial_i P + \alpha^2 \partial_j(\partial_k u_i \partial_k u_j) = \nu \partial_{jj} v_i$$

Common filter: $v_i = (1-\alpha^2\partial_{jj})u_i$

Energy conservation:

H1α(u) norm	L2(v) norm
Clark-α	Leray-α
α-model	
$\frac{dE_\alpha}{dt} = -2\nu\Omega_\alpha$	$\frac{dE}{dt} = -2\nu\Omega$
$E_\alpha = \frac{1}{2}\langle(\mathbf{u}-\alpha^2\nabla^2\mathbf{u})\cdot\mathbf{u}\rangle = \frac{1}{2}\langle\mathbf{v}\cdot\mathbf{u}\rangle$	$E = \frac{1}{2}\langle\lvert\mathbf{v}\rvert^2\rangle$
$\Omega_\alpha = \frac{1}{2}\langle\boldsymbol{\omega}\cdot\boldsymbol{\varpi}\rangle$	$\Omega = \frac{1}{2}\langle\lvert\boldsymbol{\omega}\rvert^2\rangle$

II) KÁRMÁN & HOWARTH SCALING

Clark-α

$$\partial_t \mathcal{Q}^C{}_{ij} = \frac{\partial}{\partial r^k}\left(\mathcal{T}^{Ck}_{ij} - \alpha^2 \mathcal{S}^{Ck}_{ij}\right)$$

$$\mathcal{Q}^C{}_{ij} = \langle v_i u'_j + v_j u'_i\rangle$$

$$\mathcal{T}^{Ck}_{ij} = \langle(v_i u'_j + v_j u'_i + v'_i u_j + v'_j u_i - u_i u'_j - u_j u'_i)u^k + (u_i u'_j + u_j u'_i)v^k\rangle$$

$$\mathcal{S}^{Ck}_{ij} = \left\langle(\partial_m u_i \partial_m u^k)u'_j + (\partial_m u_j \partial_m u^k)u'_i + g_\alpha * \tau^{C'k}_j v_i + g_\alpha * \tau^{C'k}_i v_j\right\rangle$$

Two possible (sub-filter scale) scalings result from the Kármán-Howarth equation for Clark-α:

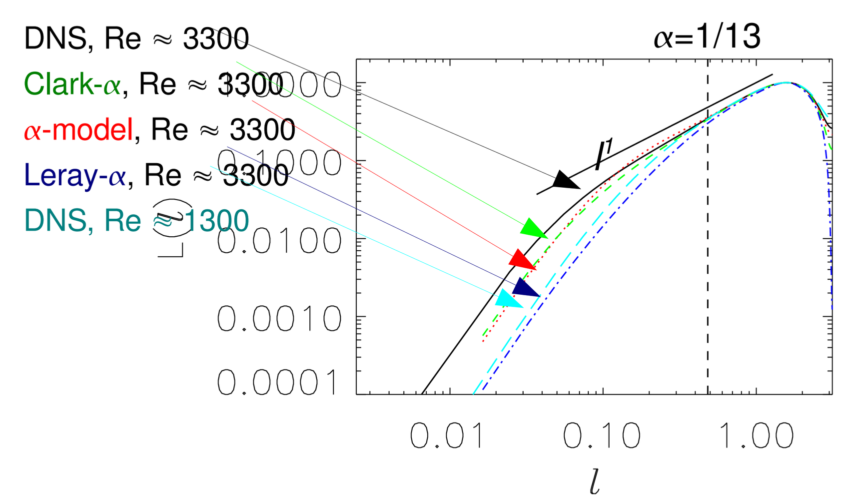
$$\varepsilon^C_\alpha \sim \frac{1}{l}\left(vu^2 + \frac{\alpha^2}{l^2}u^3 + u^3\right)$$

$l \ll \alpha$: $u = (1+\alpha^2/l^2)^{-1}v \sim l^2 v$

$$\left\langle(\delta u_\|(l))^2(\delta v_\|(l))\right\rangle \sim \varepsilon^C_\alpha l \qquad \left\langle(\delta u_\|(l))^3\right\rangle \sim \varepsilon^C_\alpha l$$

$$E^C_\alpha(k) \sim k^{-1} \qquad E^C_\alpha(k) \sim k^{1/3}$$

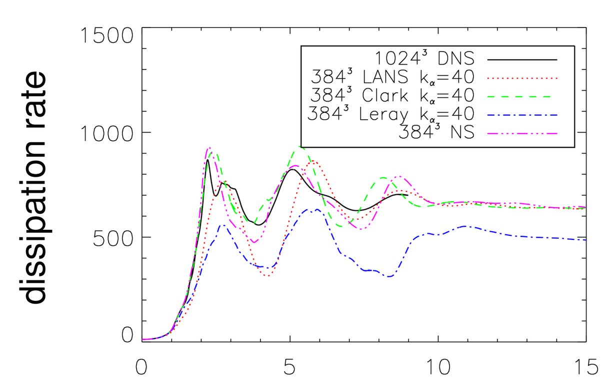
One possible Clark-α scaling dominates: $u^2 v \sim l$
(for forcing and boundary conditions employed)



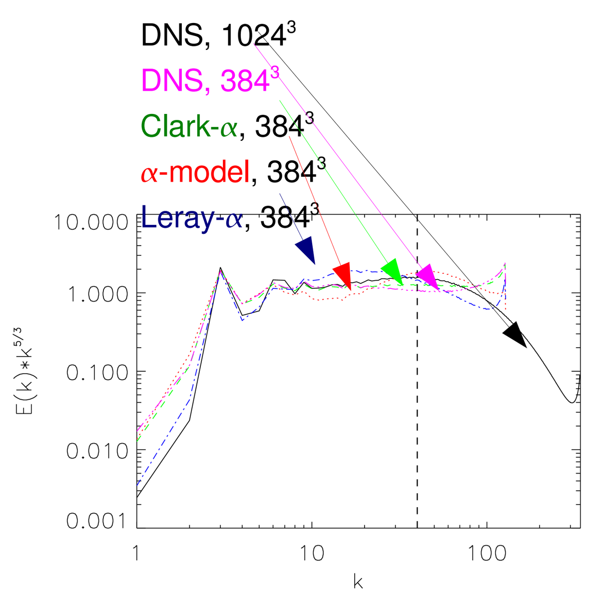
Therefore the predicted sub-filter scale spectrum for Clark-α is $E_\alpha^C(k) \sim k^1$.

IV) SGS PROPERTIES

(α=1/40, Re≈3300)



The leading effect of the Leray-α model is to lower the effective Reynolds number.

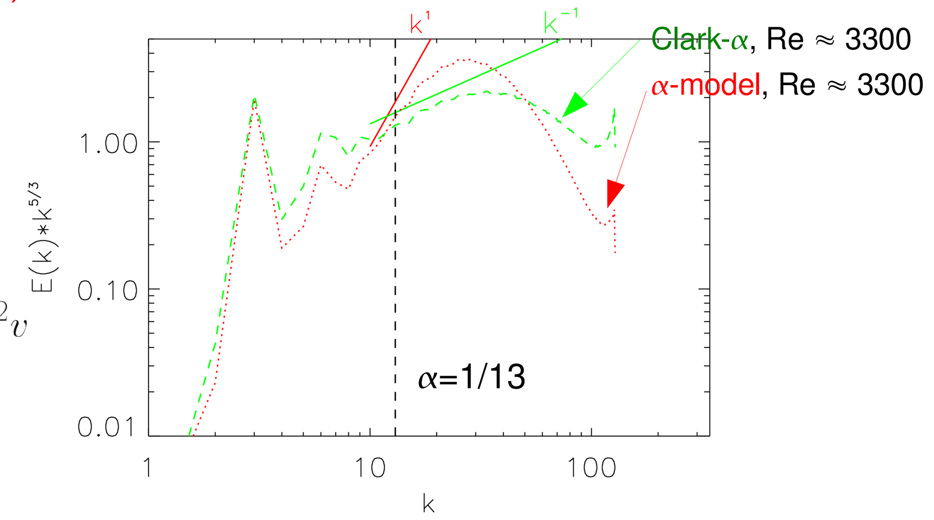


Clark-α is the only model that demonstrates a significant improvement in the energy spectrum prediction over under-resolving the Navier-Stokes equations.

III) SUB-FILTER SCALE PROPERTIES

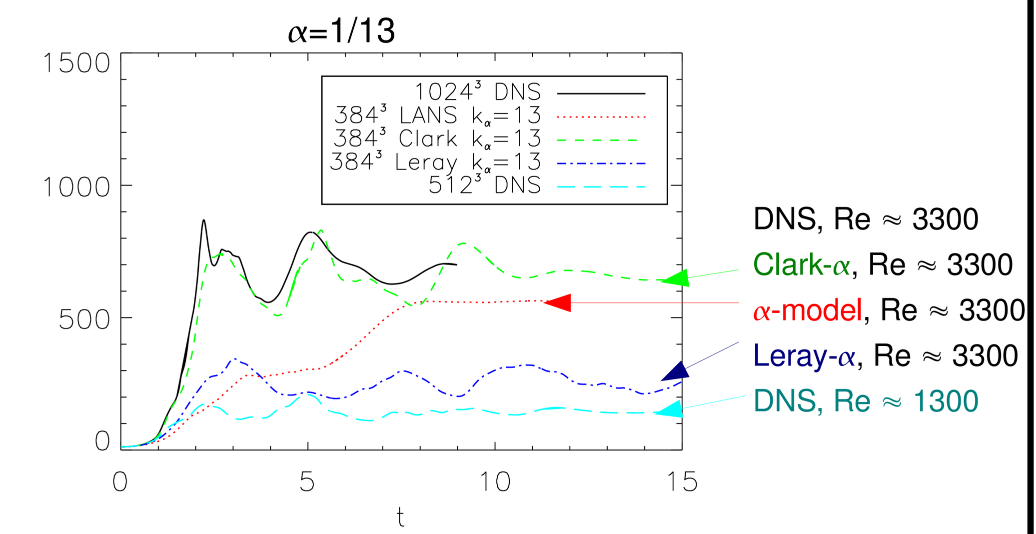
A) SPECTRUM

The Kármán-Howarth equation for LANS-α also predicts $E_\alpha(k) \sim k^1$.



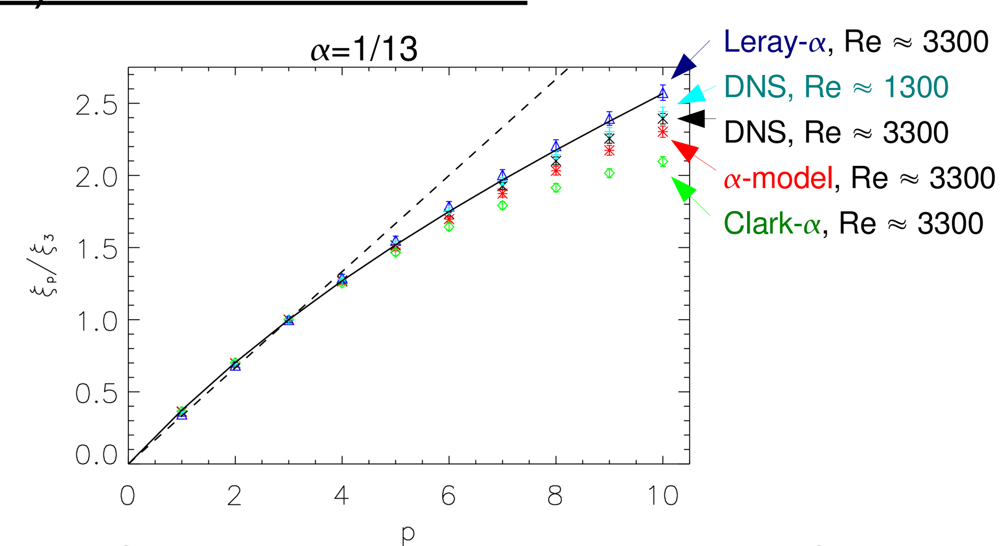
Clark-α possesses the predicted scaling, but LANS-α's spectrum is k^{+1}.

B) DISSIPATION RATE



LANS-α reduces sub-filter scale dissipation and Leray-α drastically so.

C) INTERMITTENCY



Clark-α is more intermittent than Navier-Stokes at sub-filter scales.

Acknowledgments

Computer time was provided by NCAR and by the National Science Foundation Terascale Computing System at the Pittsburgh Supercomputing Center. The NSF Grant No. CMG-0327888 at NCAR supported this work in part and is gratefully acknowledged. The National Center for Atmospheric Research is sponsored by the National Science Foundation.

V) "RIGID BODIES" IN LANS-α

We hypothesize that LANS-α introduces "rigid bodies" or "polymerized portions" of the flow for three reasons:

- LANS-α describes an incompressible second-grade non-Newtonian fluid (under a modified dissipation)
- Taylor's frozen in turbulence hypothesis drastically reduces spectrally local small-scale interactions
- From a spectral initial value problem standpoint LANS-α results in the heat equation advected by a uniform velocity:

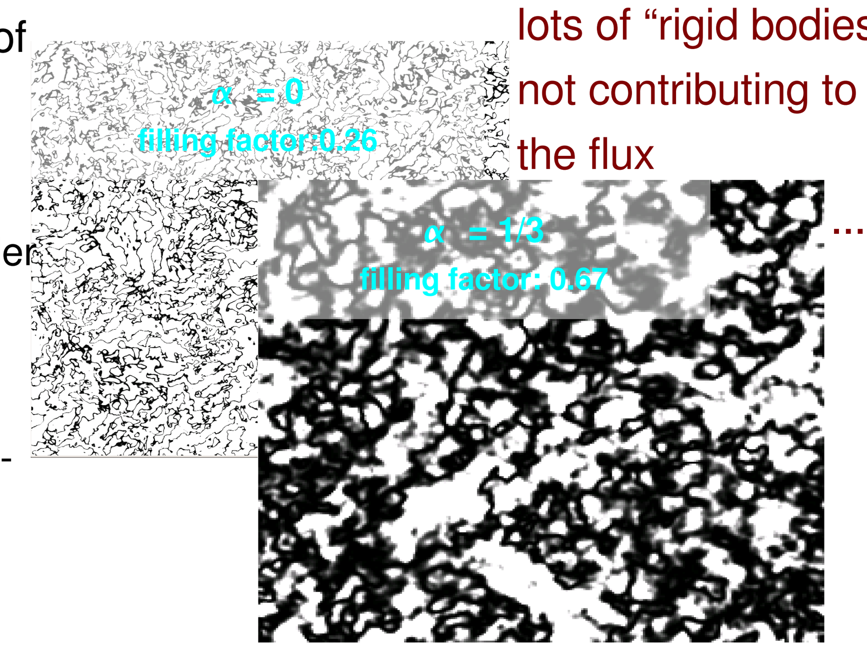
$$\partial_t\hat{\mathbf{v}}(\mathbf{k}) + \mathfrak{F}\left[\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T\right] - i\mathbf{k}\hat{\pi}(\mathbf{k}) = -\nu k^2\hat{\mathbf{v}}(\mathbf{k})$$

$$\hat{\mathbf{u}}(\mathbf{k}) = \frac{\hat{\mathbf{v}}(\mathbf{k})}{1-\alpha^2k^2}$$

$$\lim_{\alpha\to\infty}\hat{\mathbf{u}}(\mathbf{k}\neq 0) = 0$$

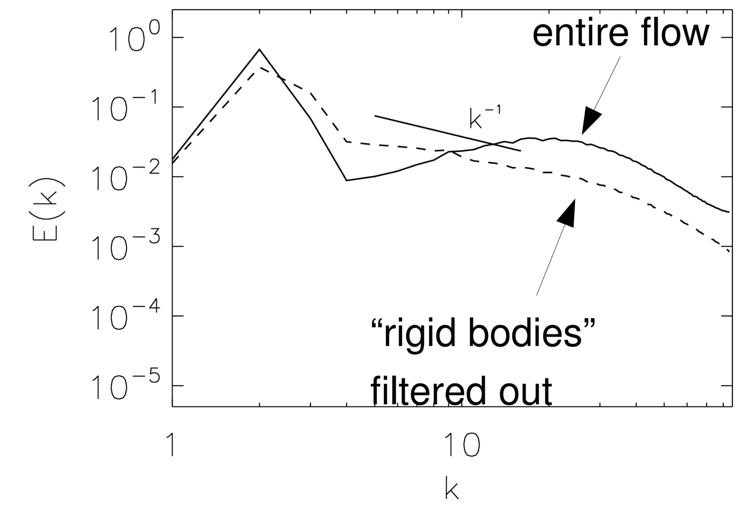
i.e., ***u*** has no spatial dependence

We observe lots of "rigid bodies" not contributing to the flux ...

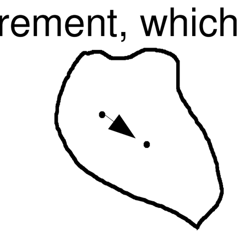


$(\delta u_\|(\mathbf{r}))^3 < 10^{-2}$ for black pixels

... and the predicted energy spectrum in the regions between the "rigid bodies."



Rigid bodies give no contribution to the turbulent cascade because their associated velocity has no longitudinal increment, which would give stretching or contraction:



$$\delta\mathbf{u}(\mathbf{l}) = \boldsymbol{\Omega}\times\mathbf{l}$$

$$\delta u_\|(l) = \delta\mathbf{u}(\mathbf{l})\cdot\mathbf{l}/l = 0$$

$$<(\delta u_\|(l))^3> = 0$$

Yet, *they agree with the observed energy spectrum!* $u^2 \sim l^0$

$$u = (1+\alpha^2/l^2)^{-1}v \sim l^2 v$$

$$E_\alpha(k)k \sim uv \sim k^2$$

$$\boldsymbol{E_\alpha(k) \sim k^1}$$

We also find that α must be tied to the dissipation scale to adequately model the energy spectrum: $\alpha \approx 4\eta_K$.

Therefore, computational savings are modest and not a function of *Re*: $dof_\alpha = dof_{NS}/12$.

CONCLUSIONS

- Leray-α results in an excessive reduction of the nonlinearity of the flow.
- One of two possible scalings resulting from the Karman-Howarth equation for Clark-α as well as its associated k^1 energy spectrum is confirmed.
- For LANS-α, the predicted k^1 scaling is conjectured to coexist with a k^1 scaling in different spatial portions of the flow.
- This latter spectrum is consistent with the absence of stretching in the sub-filter scales due to the Taylor frozen-in hypothesis employed as a closure in the derivation of LANS-α. The $E(k)\sim k^1$ scaling is subdominant to k^1 in the energy spectrum, but is responsible for the direct energy cascade, as no cascade results from the motions with no internal degrees of freedom.
- ***Clark-α is found to be the best approximation for reproducing the super-filter scale energy spectrum and the total dissipation rate, whereas intermittency properties for larger values of α are best reproduced by LANS-α.***

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