

Collapse of Langmuir Waves

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It is shown that arbitrary Langmuir turbulence of sufficient intensity ($W/nT > (kr_D)^2$) is unstable. The instability leads to the development of significantly nonlinear phenomena, viz., the formation of low density regions (caverns) in the plasma which collapse during a finite time. Cavern collapse is an effective mechanism of energy dissipation of longwave Langmuir oscillations and plays an important role in the kinetics of Langmuir turbulency. The problem of the nonlinear stage of instability development of a monochromatic Langmuir wave is also considered.

INTRODUCTION

THE kinetics of Langmuir turbulence is characterized by a tendency to "condensation" of the waves in k -space. The main nonlinear processes, mainly the induced scattering by the electrons and ions, decay of the Langmuir waves with production of ion sound, and four-plasmon scattering, do not change the number of Langmuir quanta, and only decrease their wave vectors. As a result, the energy of the Langmuir oscillation becomes concentrated in the long-wave part of the spectrum, in which the linear damping (collision and Landau damping) is small. This poses the question of the mechanism whereby this energy is dissipated. It is shown in the present paper that this mechanism may be a unique three-dimensional focusing of the Langmuir waves, which leads to the appearance of local singularities of their amplitudes. Regions of decreased density (caverns) are produced thereby in the plasma, and serve as resonators for the Langmuir waves. After a finite time, the caverns "collapse" to a dimension at which the electron trajectories intersect and the oscillation energy is dissipated. This phenomenon, the collapse of the Langmuir waves, can be regarded as a nonlinear stage of the development of the instability, discovered by Vedenov and Rudakov^[1], of a cold Langmuir gas (see also^[2]).

Unlike the linear dissipation mechanisms, the collapse of Langmuir waves has an amplitude threshold that depends on the spectral distribution of the oscillations. The collapse is accompanied by the development of a strong plasma turbulence. Strongly-nonlinear processes such as collapse cannot be investigated by the random-phase approximation and by the kinetic equations for the waves. On the other hand, the dynamic description of the plasma by means of a system of kinetic equations is too complicated.

A simplified dynamic description of a plasma, based on averaging over the "fast time," is proposed here. The corresponding equations are derived in Sec. 1. The problem of instability of a monochromatic Langmuir wave of finite amplitude is solved in Sec. 2 within the framework of these equations, and the sufficient conditions for the instability of the turbulent spectra are obtained. The collapse proper (the development of the caverns and their collapse) is investigated in Sec. 3. Section 4 deals with the nonlinear stage of development of the instability of a monochromatic Langmuir wave, and Sec. 5 with the kinetics of four-plasmon processes

in a plasma in the presence of collapse.

1. FUNDAMENTAL EQUATIONS

We consider long-wave oscillations of a plasma ($kr_D \ll 1$, where r_D is the Debye radius). Two types of motion exist in the plasma, fast Langmuir oscillations with frequency $\omega \approx \omega_p$, and slow motion of the plasma as a whole. We assume that in the slow motions the plasma is quasineutral. We can then put for the ion density

$$n_i = n_0 + \delta n, \quad \delta n / n_0 \ll 1$$

and for the electron density

$$n_e = n_0 + \delta n + \delta n_e, \quad \delta n_e / n_0 \ll 1.$$

Here n_e is the variation of the electron density in the Langmuir oscillations. We use for the Langmuir oscillations a linearized system of hydrodynamic equations

$$\begin{aligned} \frac{\partial}{\partial t} \delta n_e + n_0 \operatorname{div} \mathbf{v}_e + \operatorname{div} \delta n \mathbf{v}_e &= 0, \\ \frac{\partial}{\partial t} \mathbf{v}_e + \frac{e}{m} \nabla \varphi_e + \frac{3}{2} \frac{T_e}{2mn_0} \nabla \delta n_e &= 0. \end{aligned} \quad (1.1)$$

Here φ_e is the high-frequency part of the electrostatic potential.

Equations (1.1) do not take into account the electronic nonlinearities, which begin to play a role if the characteristic time of the nonlinear process is $1/\tau \leq \omega_p (kr_D)^2 W/nT$ (see^[3]). We consider henceforth only the fast processes.

We introduced a slowly varying quantity ψ defined by

$$\varphi_e = 1/2 (\psi e^{-i\omega_p t} + \psi^* e^{i\omega_p t}).$$

Assuming that δn is independent of the time and neglecting the second derivative of ψ with respect to t , we obtain from (1.1):

$$\Delta \left(i \frac{\partial \psi}{\partial t} + \frac{3}{2} \omega_p r_D^2 \Delta \psi \right) = \frac{\omega_p}{2n_0} \operatorname{div} \delta n \nabla \psi. \quad (1.2)$$

Equation (1.2) describes the Langmuir oscillations after averaging over the "fast time" $1/\omega_p$, but does not presuppose averaging over the phase shift or over the wavelength. This equation has an integral of motion

$$I_1 = \frac{1}{8\pi} \int |\nabla \psi|^2 dr, \quad (1.3)$$

which equals, accurate to terms $(kr_D)^2$, the energy W of the Langmuir oscillations. More accurately speaking, $I_1 = 2W_0$, where W_0 is the averaged energy of the electrostatic field. We can treat the quantity $N = I_1/\omega_p$

as the conserved number of Langmuir quanta.

To close Eq. (1.2), we include the action of the Langmuir oscillations on the slow motions of the plasma and regard it as the action of a high-frequency force with a potential $U = e^2 |\nabla\psi|^2 / 4m\omega_p^2$ (cf., e.g., [4,5]). We note further that the averaged electron density (as well as the ion density) represents a Boltzmann distribution in a field with an effective potential $U_{\text{eff}} = -e\varphi + U$. Here φ is the low-frequency component of the electrostatic potential. Thus,

$$\delta n = n_0 \left[\exp\left(\frac{e\varphi - U}{T}\right) - 1 \right]. \quad (1.4)$$

The ions are described by the kinetic equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} - \frac{e}{m} \frac{\partial f}{\partial v} \nabla\varphi = 0. \quad (1.5)$$

Equations (1.2)–(1.5) together with the relation

$$\delta n = \int f dv - n_0 \quad (1.6)$$

constitute a complete system. We consider two methods of simplifying the resultant system of equations. Assume that the following inequalities are satisfied:

$$w \ll \mu T_i / T_e, \quad (kr_D)^2 \ll \mu,$$

where

$$w = W / nT, \quad \mu = m / M.$$

In this case the characteristic time of all the processes

$$1/\tau \sim \omega_p (kr_D)^2 \ll kv_{Ti}$$

and the characteristic velocities of the slow motions are small compared with the thermal velocity of the ions. It can be assumed that the ions have a Boltzmann distribution in the low-frequency electrostatic field, so that $\delta n = n_0 [\exp(e\varphi/T_i) - 1]$. Comparing with (1.4), we get

$$\delta n = n_0 \left[\exp\left(-\frac{U}{T_i + T_e}\right) - 1 \right] \approx \frac{n_0 U}{T_i + T_e}. \quad (1.7)$$

After substituting in (1.2) we obtain

$$\Delta \left(i\psi_i + \frac{3}{2} \omega_p r_D^2 \Delta\psi \right) = -\frac{e^2 q}{4m\omega_p T_e} \text{div} |\nabla\psi|^2 \nabla\psi, \quad (1.8)$$

$$q = T_e / (T_e + T_i).$$

This approximation can be called static.

Another method of simplifying the initial system is to go over to a hydrodynamic description of the ions. The condition for the applicability of the hydrodynamics, $1/\tau \gg kv_{Ti}$, is the opposite of the condition for the applicability of the static approximation. The linearized hydrodynamic description is by means of the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_i^2 \Delta \right) \delta n = \frac{1}{16\pi M} \Delta |\nabla\psi|^2, \quad (1.9)$$

$$c_i^2 = \frac{T_e + 3/2 T_i}{M},$$

which completes Eq. (1.2) in this case.

For long-wave $((kr_D)^2 \ll \mu)$ plasma motions, the condition for the applicability of the hydrodynamic description yields

$$w \gg \mu T_i / T_e.$$

If $T_i \ll T_e$, then, for the amplitudes satisfying the condition

$$\mu T_i / T_e \ll w \ll \mu,$$

we can neglect in (1.9) the term $\partial^2 \delta n / \partial t^2$. We then arrive at Eq. (1.8), but now $q = T_e / (T_e + 3/2 T_i)$.

2. INSTABILITY OF MONOCHROMATIC LANGMUIR WAVE

The stability of Langmuir spectra that are narrow in k -space can be assessed in the simplest cases by examining the stability of a monochromatic Langmuir wave having the corresponding k_0 . In particular, the stability of an intensive isotropic spectrum can be assessed from the stability of a wave with $k_0 = 0$.

A stationary monochromatic wave corresponds to a solution of a fundamental system of equations in the form

$$\psi = (A/k_0) \exp\{-i\omega_k t + ik_0 r\}, \quad \omega_k = 3/2 \omega_p k_0 r_D^2, \quad \delta n = 0.$$

We linearize the fundamental system against the background of this solution and put

$$\delta f \sim \exp\{i\Omega t + i\kappa r\}, \quad \delta\psi \sim \exp\{i\omega_k t + i\Omega t + i\kappa r\},$$

$$o\psi^* \sim \exp\{-i\omega_k t + i\Omega t + i\kappa r\}.$$

We obtain for Ω the dispersion equation

$$1 + \frac{\omega_p}{4} \frac{W}{n_0^2} G_{a,\kappa} \left[\frac{(k_0, k_0 + \kappa)^2}{k_0^2 |k_0 + \kappa|^2} \frac{1}{-\Omega + \omega_{k_0 - \kappa} - \omega_{k_0}} + \frac{(k_0, k_0 - \kappa)^2}{k_0^2 |k_0 - \kappa|^2} \frac{1}{\Omega + \omega_{k_0 - \kappa} - \omega_{k_0}} \right] = 0, \quad (2.1)$$

$$W = |A|^2 / 8\pi.$$

Here

$$G_{a,\kappa} = \frac{\delta n_{a,\kappa}}{U_{a,\kappa}} = L_{a,\kappa} / M \left(1 - \frac{4\pi e^2}{M k_D^2} L_{a,\kappa} \right),$$

$$L_{a,\kappa} = \int \frac{1}{-\Omega + \kappa v} \kappa \frac{\partial f}{\partial v} dv.$$

In the hydrodynamic approximation we get

$$L_{a,\kappa} = \frac{n_0 \kappa^2}{\Omega^2}, \quad G_{a,\kappa} = \frac{n_0}{M} \frac{\kappa^2}{\Omega^2 - c_i^2 \kappa^2}. \quad (2.2)$$

In the static approximation

$$G_{a,\kappa} = -n_0 / (T_i + T_e). \quad (2.3)$$

We present the results of investigations of (2.1) in different cases. The character of the instability depends essentially on the value of k_0 . If $(k_0 r_D)^2 > \mu/3$, then the character of the instability depends also on the ratio of the electron and ion temperatures. At $T_i \ll T_e$ and at sufficiently small amplitudes ($w \ll \mu^{1/2} kr_D$) we get the usual decay instability with excitation of ion sound^[6]. The wave vector κ of the perturbation lies, for this instability, on the surface

$$\omega_{k_0} = \omega_{k_0 - \kappa} + \Omega_\kappa,$$

and the increment is equal to

$$\gamma = \frac{\omega_p}{2} \mu^{1/4} w^{1/2} (\kappa r_D)^{1/2} \frac{|(k_0, k_0 - \kappa)|}{|k_0| |k_0 - \kappa|}. \quad (2.4)$$

The increment is maximal at $\kappa \sim 2k_0$; then

$$\gamma \approx 1/2 \omega_p \mu^{1/4} (2k_0 r_D)^{1/2} w^{1/2}.$$

Formula (2.4) does not hold for sufficiently small κ , for if $\kappa < \kappa_{CR}$, where $\kappa_{CR} r_D \sim w \mu^{1/2}$, then the instability becomes more complicated (see^[7]). $\kappa_{CR} \rightarrow k_0$ with increasing amplitude, and when $w \sim \mu^{1/2} kr_D$ the decay

instability is completely realigned. At $w \gg \mu^{1/2} \kappa r_D$ we have a modified decay instability with an increment (at $\kappa \approx 2k_0$)

$$\gamma \approx \omega_p (W_0 k^2 / M n_0 \omega_p^2)^{1/2}. \quad (2.5)$$

It develops near the surface $\omega_{k_0 - \kappa} = \omega_{k_0}$.

We note that formula (2.5) does not contain the plasma temperature and is valid also at higher energy density of the Langmuir oscillations ($w \gtrsim 1$).

At $\kappa r_D > \mu^{1/4}$ the modified decay instability retains its character up to amplitudes $w \sim \mu^{1/2} (\kappa r_D)^{-2}$, above which it becomes necessary to take into account the electronic nonlinearity. If $\mu^{1/4} > \kappa r_D > \mu^{1/2}$, then the modified decay instability takes place up to intensities $w \sim (\kappa r_D)^2 / \mu$. When $(\kappa r_D)^2 / \mu < w < \mu^{1/2} (\kappa r_D)^{-2}$ we get an instability with an increment whose order of magnitude is given by (2.5). This instability, however, is not localized near the surface $\omega_{k_0 - \kappa} = \omega_{k_0}$; its increment is approximately constant in a wide region near $\kappa \approx k_0$.

In an isothermal plasma, waves of low amplitude ($w < \mu$) experience an instability connected with induced scattering by the ions. The maximum increment $\gamma \sim \omega_p w$ of this instability occurs at $\kappa \approx 2k_0$. When $w > \mu$, at small wave numbers of the perturbation, $\kappa r_D \sim w \mu^{-1/2}$, a modified decay instability sets in with a maximum increment $\gamma \sim \omega_p w$. For these waves, therefore, the instability increment has two maxima of the same order of magnitude, at $\kappa \approx 2k_0$ and at $\kappa r_D \sim w \mu^{-1/2}$. With increasing amplitude, the second maximum comes closer to the first, and the two coalesce $w \sim \mu^{1/2} / k_0 r_D$. Starting with these amplitudes, the ion temperature has no qualitative effect on the character of the instability.

The instability of long Langmuir waves ($(\kappa r_D)^2 < \mu/3$) depends little on the electron and ion temperature ratio. At low amplitudes ($w < (\kappa r_D)^2$) an instability of the automodulation type^[8] sets in, with an increment

$$\gamma = [^{3/4} q \omega_p^2 \kappa^2 r_D^2 w - ^{1/4} \omega_p^2 \kappa^4 r_D^4]^{1/2}. \quad (2.6)$$

Formula (2.6) is valid in the static $w \ll \mu T_i / T_e$, $q = T_e / (T_e + T_i)$ and in the hydrodynamic ($k^2 r_D^2 \gg w T_i / T_e$, $q \approx 1$) approximations and is not valid when $w \sim \mu T_i / T_e$.

If $w \gg (\kappa_0 r_D)^2$, then we can put $k_0 = 0$. In this important case the dispersion equation reduces to

$$\Omega^2 - \omega_{\kappa}^2 = ^{1/2} \omega_p W n_0^{-2} \omega_{\kappa} G(\Omega, \kappa) \cos^2 \alpha. \quad (2.7)$$

Here α is the angle between κ and the direction of the electric vector in the initial oscillation. In the static approximation we have

$$\gamma = \omega_p [^{3/4} q \kappa^2 r_D^2 w \cos^2 \alpha - ^{1/4} \kappa^4 r_D^4]^{1/2}. \quad (2.8)$$

The maximum increment

$$\gamma_{max} = ^{1/4} q \omega_p w$$

is reached at $\cos^2 \alpha = 1$, $(\kappa r_D)^2 = ^{1/6} w$.

In the hydrodynamic limit ($w \gg \mu T_i / T_e$), Eq. (2.7) takes the simpler form

$$(\Omega^2 - c^2 \kappa^2) (-\Omega^2 + ^{9/4} \omega_p^2 r_D^4 \kappa^4) + ^{3/4} \omega_p^4 (\kappa r_D)^2 \mu w \cos^2 \alpha = 0. \quad (2.9)$$

Equation (2.9) without the term $(^{9/4} \omega_p^2 \kappa^4 r_D^4)$ was obtained earlier by Rudakov and Vedenov^[1]. The term disregarded by them, however, is quite important, for

in its absence we have $\Omega \sim \kappa$ and the maximum instability increment turns out to be infinite (as $\kappa \rightarrow \infty$). At $\mu T_i / T_e < w < \mu$ the instability is again described by (2.8). For large amplitude waves ($w \gg \mu$), Eq. (2.9) takes the form

$$\Omega^2 (-\Omega^2 + ^{9/4} \omega_p^2 r_D^4 \kappa^4) + ^{9/4} \omega_p^4 (\kappa r_D)^2 \mu w \cos^2 \alpha = 0; \quad (2.10)$$

at small κ we have

$$\gamma \approx \omega_p \kappa r_D (^{3/4} \mu w \cos^2 \alpha)^{1/4}.$$

The maximum increment

$$\gamma_{max} = 3^{-1/2} \omega_p (\mu w)^{1/2}$$

is reached when $\kappa r_D \sim (\mu w)^{1/2}$. Equations (2.9) and (2.10) are valid when $w \ll 1$. In the opposite case, the electronic nonlinearities must be taken into account.

We consider now an arbitrary distribution of the Langmuir waves. Assume that the characteristic wave number for this distribution is k . This means that there exist spatial regions with dimension $L \lesssim 1/k$, inside which the electron oscillations relative to the ions can be regarded as homogeneous. Let the intensity of this oscillation equal W , and let us examine its instability. If the dimension over which this instability develops is smaller than L , then the spectral distribution in question is unstable. It is easy to estimate the instability conditions. Using the results of the present section, we find that the distribution is unstable if

$$\begin{aligned} (\kappa r_D)^2 < w & \text{ for } w < \mu, \\ (\kappa r_D)^2 < (\mu w)^{1/2} & \text{ for } w > \mu. \end{aligned} \quad (2.11)$$

In some cases (see^[8]) the kinetics of the Langmuir turbulence proceeds in such a way that it is possible to separate in the spectrum two components, namely an intense long-wave "core" and an extensive short-wave "halo." In this case we must assume that the parameters to be substituted in (3.9) are those of the "core," since the "halo" leads only to a negligible change of the effective thermal velocity of the electrons.

3. CAVERNS AND THEIR COLLAPSE

Let us examine the nonlinear stage of instability development of a monochromatic wave. For short waves ($(\kappa r_D)^2 > \mu/3$) the instability leads primarily to the appearance of Langmuir waves with $k \approx -k_0$ and to a gradual filling of the vicinity of the sphere $\omega_k = \omega_{k_0}$. The instability of long ($(\kappa r_D)^2 < \mu/3$) monochromatic waves of low amplitude ($w < (\kappa r_D)^2$) will be considered in Sec. 4. In the present section we consider the development of the instability of a homogeneous Langmuir oscillation. In fact, with an aim at applications to the stability of turbulent spectra, we should examine the development of the instability of a localized Langmuir oscillation of sufficiently high intensity. We examine this question under the assumption that the oscillation is spherically symmetrical.

Introducing the quantity $E = \partial \psi / \partial r$, we obtain from (1.2) the equation

$$i \frac{\partial E}{\partial t} + \frac{3}{2} \omega_p r_D^2 \frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 E = \frac{\omega_p}{2 n_0} \delta n E \quad (3.1)$$

with the obvious boundary condition $E(0) = 0$. In the hydrodynamic limit, (3.1) is completed with the equation

$$\frac{\partial^2}{\partial t^2} \delta n - c_s^2 \Delta_r \delta n = \frac{1}{16\pi M} \Delta_r r |E|^2, \quad \Delta_r r = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \quad (3.2)$$

In the static limit we have

$$i \frac{\partial E}{\partial t} + \frac{3}{2} \omega_p r_D^2 \frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} r^2 E + \frac{e^2 q}{4m\omega_p T_e} |E|^2 E = 0. \quad (3.3)$$

Making the change of variables

$$r = 3 \sqrt{\frac{q}{2\mu}} r_D \rho, \quad t = \frac{3}{\mu} q \frac{\tau}{\omega_p}, \\ \delta n = \frac{2}{3q} n_0 \mu V, \quad E = 4 \left[\frac{2\pi n_0 T_e}{q} \mu \right]^{1/2} \varphi,$$

we reduce (3.1)–(3.3) to the form

$$i \frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial \rho} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \varphi = V \varphi, \quad (3.4)$$

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{\rho\rho} \right) V = \Delta_{\rho\rho} |\varphi|^2, \quad (3.5)$$

$$i \frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial \rho} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \varphi + |\varphi|^2 \varphi = 0. \quad (3.6)$$

Let us investigate (3.6). It has the integrals of motion

$$I_1 = \int_0^\infty \rho^2 |\varphi|^2 d\rho, \\ I_2 = \int_0^\infty [|(\rho\varphi)_\rho|^2 + 2|\varphi|^2 - 1/2 \rho^2 |\varphi|^4] d\rho.$$

We introduce the quantity

$$A = \int_0^\infty \rho^4 |\varphi|^2 d\rho.$$

We can easily obtain directly from (3.6) the relation

$$\frac{d^2 A}{dt^2} = 6I_2 - 2 \int_0^\infty |(\rho\varphi)_\rho|^2 d\rho - 4 \int_0^\infty \rho^2 |\varphi|^4 d\rho < 6I_2,$$

which leads to the inequality

$$A < 3I_2 t^2 + C_1 t + C_2, \quad (3.7)$$

where C_1 and C_2 are constants. If $I_2 < 0$ then, by virtue of the positiveness of A , the inequality (3.7) can be satisfied only for not too high values of t . This means that the solution of the initial problem with $I_2 < 0$ exists only for a finite time and should lead to a singularity at a certain $t = t_0$ (cf. [9, 10]). The condition $I_2 < 0$ means $|\varphi|^2 l^3 \gtrsim l^2$, which agrees in order of magnitude with the instability condition ($w > (krD)^2$). To determine the character of the singularity that results from the instability, we note that Eq. (3.6) admits of the self-similar substitution

$$\varphi = \exp \left\{ i \lambda^2 \int \frac{dt}{f(t)} \right\} \frac{1}{\lambda f(t)} R \left(\frac{\rho}{\lambda f(t)} \right), \\ f(t) = \beta \sqrt{t_0 - t}, \quad \rho / \lambda f(t) = \xi. \quad (3.8)$$

for $R(\xi)$ we have

$$\Delta_{\xi\xi} R - \frac{2}{\xi^2} R + |R|^2 R = i\alpha \frac{\partial}{\partial \xi} \xi R.$$

The substitution $R = F/\xi$ reduces this equation to

$$F'' - 2F/\xi^2 + |F|^2 F - F = i\alpha \xi F'_\xi, \quad \alpha = \beta / \lambda^2. \quad (3.9)$$

Let us examine the asymptotic behavior of F as $\xi \rightarrow \infty$. At large ξ we can neglect the nonlinearity as well as the term $-2F/\xi^2$. The asymptotic form of $F(\xi)$ is

$$F(\xi) \rightarrow c_1 \xi^{i/\alpha} + c_2 \int_0^\infty \exp \left\{ \frac{i\alpha \xi^2}{2} \right\} d\xi. \quad (3.10)$$

We see from (3.9) that the integral I_1 for the self-similar solution is infinite for all c_1 and c_2 . The integral I_2 is finite if $c_2 = 0$. In this case $I_2 \equiv 0$, since it follows from the self-similar substitution that $I_2(t) = f(t)I_2(0) = I_2(0)$. The requirement $c_2 = 0$ is equivalent to the requirement $d(\xi R)/d\xi \rightarrow 0$ as $\xi \rightarrow \infty$. This condition together with the condition $R(0) = 0$ specifies the boundary-value problem that determines the eigenvalues α for (3.9). The quantity $|c_1|$ can be expressed in terms of α . As $t \rightarrow t_0$ we get $\xi \rightarrow \infty$ and $\varphi(\rho)$ tends to its limiting value, $\varphi \rightarrow c_1 \rho^{1/\alpha-2}$, in the entire region of ρ . From the condition $I_2 = 0$ we get $1/2 |c_1|^2 = \alpha^{-2} + 3$.

Thus, the development of the instability can be represented as the formation of a region in which the plasma has a reduced density—a cavern. The electrons inside the cavern execute intense radial oscillations and are expelled from the cavern by the high-frequency pressure. After a finite time, the cavern collapses. When the instant of collapse is approached, the density distribution established everywhere in the cavern, with the exception of the central region, is

$$\delta n = -3n_0 |c_1|^2 r_D^2 / r^2.$$

This formula is valid for $r \gg r_0(t)$, and the cavern radius $r_0(t)$ tends to zero like $\sqrt{t_0 - t}$. The density at the center of the cavern remains unperturbed.

Actually, however, Eq. (3.6), together with the self-similar solution, is valid only for not too large variation of the density ($\delta n/n_0 < \mu$). In addition, if $T_i \ll T_e$, then the solution obtained does not hold for $\delta n/n \approx w \sim \mu T_i/T_e$, since it is necessary to take into account the Landau damping by the ions. Since the invariant (1.3) (the total number of the Langmuir quanta) is conserved, we can state that these regions of the parameters are “traversed” without an appreciable energy loss.

When $w \gg \mu$ we can describe the collapse by using the hydrodynamic approximation. We can then simplify (3.4) and (3.5) by putting

$$\varphi \approx \exp \left\{ i \int E(t) dt \right\} \Phi(r, t),$$

where Φ is real, and neglecting the term $\Delta_{\rho\rho} V$. The resultant system

$$\Delta_{\rho\rho} \Phi - \frac{2}{\rho^2} \Phi - V \Phi = E \Phi, \quad \frac{\partial^2}{\partial t^2} V = \Delta_{\rho\rho} |\Phi|^2 \quad (3.11)$$

describes the quasistationary eigenvalue problem for a Schrödinger equation with a potential $-2/\rho^2 - V$ that depends in self-consistent fashion on the wave function Φ . The system (3.11) admits of the self-similar substitution

$$\Phi = \frac{\lambda A^{1/2}}{f^{1/2}} R \left(\frac{\lambda \rho}{f} \right), \quad V = \frac{\lambda^2}{\rho^2} V_0 \left(\frac{\lambda \rho}{f} \right) \\ E = \lambda^2 / f^2(t), \quad f(t) = A(t_0 - t)^{1/2}. \quad (3.12)$$

For R and V we have

$$\Delta_{\xi\xi} R - \frac{2}{\xi^2} R - V_0 R = R, \\ \frac{2}{9} \left(\frac{5}{\xi} \frac{\partial}{\partial \xi} \xi^2 V_0 + 2 \frac{\partial^2}{\partial \xi^2} \xi^2 V_0 \right) = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \xi^2 \frac{\partial}{\partial \xi} R^2, \quad (3.13) \\ I_1 = \int \rho^2 |\varphi|^2 d\rho.$$

It follows from the self-similar substitution (3.12) that the integral

$$I_1 = \int \rho^2 |\varphi|^2 d\rho.$$

is finite and is conserved. This means that the cavern is a resonator in which a Langmuir oscillation is "trapped." However, the connection between the wave function of the oscillations and the density variation is now different and the course of the density variation remains qualitatively the same as before. We can state that once the density variation reaches a value $\delta n \approx -n_0 \mu$ the oscillation contained in the central zone of the cavern "breaks away" and collapses independently of anything else. The behavior of the density outside the central zone of the cavern is determined by the solution of (3.13), in which we substitute $R^2 = 0$. From this we get $\delta n \sim r^{-2}$. The dimension of the central region decreases like $(t_0 - t)^{2/3}$. This process begins when the wave amplitude in the cavern reaches the value $w \sim \mu$. The characteristic dimension of the central zone is in this case $L \sim r_D \mu^{-1/2}$. This allows us to estimate the amount of energy entering in the collapse:

$$\Delta \varepsilon \sim WL^3 \approx n T r_D^3 \mu^{-3/2}. \quad (3.14)$$

When the intensity of the oscillations at the center of the cavern increases to a value $w \sim 1$ (in which case the density variation $\delta n/n_0$ becomes of the order of $\mu^{1/2}$), it becomes necessary to take into account the electronic nonlinearities. Qualitatively, however, the situation does not change. The cavern will continue to collapse, since the plasma is expelled from it by the high-frequency pressure. This process continues until the amplitude of the oscillations increases enough for an intersection of the trajectories of the electrons moving toward the center. Following this intersection, the electrons, having an ordered velocity greatly exceeding the thermal velocity, leave the cavern and carry away with them the energy of the Langmuir oscillations. The depleted cavern is "collapsed" by the ion-acoustic shock wave that converges towards the center.

We can obtain an upper bound for the energy of the electrons leaving the cavern. In any case, the dimension of the cavern cannot be smaller than the Debye radius. It follows from (3.14) that the energy of the outgoing electrons satisfies the inequality $\varepsilon < T_e \mu^{-1/2}$.

The formation and collapse of the caverns leads to a Langmuir-wave dissipation characterized by a certain damping decrement. To calculate this decrement, we note that the development of the caverns is connected with a local increase of the wave number. Since the development of the caverns is the fastest of all the nonlinear processes, the average wave number will actually always lie at the boundary of the instability excitation, so that we have

$$(kr_D)^2 \sim w \text{ for } w < \mu, \\ (kr_D)^2 \sim (\mu w)^{1/2} \text{ for } w > \mu$$

(as before, $w = W/nT$, $\mu = m/M$).

We consider first the case $w \ll \mu$. The characteristic cavern development time is then given by $1/\tau$

$\sim \omega_p w$. Assuming that at a given instant of time one cavern is located on the average in a region having a volume on the order of k^{-3} , and that an energy $\Delta \varepsilon \sim n T r_D^3 \mu^{-1/2}$ is dissipated in the cavern, we obtain the effective damping decrement

$$\gamma_{\text{eff}} \sim \omega_p \mu^{-1/2} w^{3/2} \text{ for } w \ll \mu.$$

Let now $w \gg \mu$. The characteristic cavern development time is then given by $1/\tau \sim \omega_p (\mu w)^{1/2}$. At intensities of this order, the entire energy entering the cavern is dissipated, so that we can assume that the energy dissipated in a time τ is of the order of the total energy of the Langmuir waves. This yields the estimate

$$\gamma_{\text{eff}} \sim \omega_p (\mu w)^{1/2} \text{ for } w \gg \mu.$$

4. AUTOMODULATION OF A MONOCHROMATIC LANGMUIR WAVE

A monochromatic Langmuir wave of small amplitude ($w \ll (kr_D)^2 \ll \mu$) is unstable (see Sec. 2) against the excitation of a growing modulation with characteristic dimension $L \sim r_D w^{-1/2}$. We shall show that the nonlinear evolution of this instability leads to the development of caverns and to absorption of the wave energy as a result of Langmuir collapse. We start from Eq. (1.8). Assuming

$$\psi = \exp(ikr - i\omega t) G,$$

where G is the complex envelope of the wave, we obtain

$$i \left(\frac{\partial}{\partial t} + v \nabla \right) G + \frac{3}{2} \omega_p r_D^2 \Delta G + \frac{e^2 q k_0^2}{4m\omega_p T_e} |G|^2 G = 0. \quad (4.1)$$

Changing over to the co-moving reference frame and to the dimensionless variables

$$\mathbf{r} - \mathbf{v}t = \sqrt{3/2} r_D \boldsymbol{\rho}, \quad t = \tau / \omega_p,$$

$$G = \frac{1}{k_0} \left(\frac{e^2 q}{4m\omega_p T_e} \right)^{1/4} E,$$

we obtain for E

$$iE_t + \Delta E + |E|^2 E = 0. \quad (4.2)$$

Equation (4.1) has the integrals of motion

$$I_1 = \int |E|^2 dr, \quad I_2 = \int \left(i|\nabla E|^2 - \frac{1}{2}|E|^4 \right) dr.$$

Introducing

$$A = \int r^2 |E|^2 dr,$$

we obtain from (4.1)

$$\frac{d^2 A}{dt^2} = 6I_2 - 4 \int_0^\infty |E|^4 dr < 6I_2,$$

from which follows the inequality (3.7). Reasoning as in Sec. 3, we conclude that the solution of the initial-value problem for Eq. (4.1) should terminate in a singularity if $I_2 < 0$. This means that the development of automodulation of a monochromatic wave (for which $I_2 = -\infty$) produces, in the regions where the wave intensities are high, "foci" that move at the group velocity and whose amplitude becomes formally infinite after a finite time. The plasma density is lower near these foci.

The development of the focus is similar in many respects to the development of a plasma cavern. We assume that the wave envelope near the focus is spherically symmetrical and make the self-similar substitution (3.8). As a result we obtain a boundary-value problem for the eigenvalues

$$\Delta_{\xi} R + |R|^2 R - R = i\alpha \frac{\partial}{\partial \xi} \xi R$$

with boundary conditions

$$R_{\xi}'|_{\xi=0} = 0, \quad (\xi R)_{\xi}'|_{\xi \rightarrow \infty} \rightarrow 0$$

and with properties analogous to those of the boundary-value problem (3.9). In particular, the asymptotic expansion (3.10) and the identity $I_2 = 0$ hold for this problem. The coefficient c_1 and the eigenvalue α are connected by the relation $(\frac{1}{2})|c_1|^2 = \alpha^{-2} + 1$.

It follows from the self-similar substitution that the amplitude of the Langmuir wave near the focus becomes infinite like $G \sim 1/\sqrt{t_0 - t}$. Actually, Eq. (4.2) ceases to hold already at $|G^2|/nT \sim (kr_D)^2$. The amplitude of the wave in the region of the focus reaches a value such that its collapse becomes possible, namely, the caverns come into being and collapse over a dimension on the order of the wavelength. This leads to a nonlinear damping of the wave energy, and this indeed limits the growth of the amplitude at the focus.

The development of foci leads also to a strong broadening of the wave spectrum.

5. COLLAPSE AND KINETICS OF WEAK TURBULENCE

The condition for the collapse of Langmuir waves ($w \gtrsim (kr_D)^2$) is the inverse of the condition for the applicability of weak turbulence ($w \ll (kr_D)^2$). Nonetheless, collapse and weak turbulence can coexist in different regions of the spectrum. The collapse, by effecting self-consistent damping of long waves, exerts an appreciable influence on the kinetics of the weak turbulence in all of space.

The equations describing the weak turbulence can be derived from the system (1.2), (1.4)–(1.6). This is particularly simple for long-wave turbulence, when $(kr_D)^2 < \mu$. (Such a turbulence can be excited, for example by an electron beam having a velocity $v_n > v_{Te} \mu^{-1/2}$.) In the first approximation we can describe such a turbulence by starting with the dynamic equation (1.8). Taking its Fourier transform with respect to the coordinates and changing over to the variable $c_k = k\psi_k/\sqrt{8\pi\omega_D}$, we obtain

$$i \frac{\partial c_k}{\partial t} - \omega_k c_k = \int \mathcal{W}_{kk_1 k_2 k_3} c_{k_1} c_{k_2} \delta(k + k_1 - k_2 - k_3) dk_1 dk_2 dk_3, \quad (5.1)$$

$$\mathcal{W}_{kk_1 k_2 k_3} = -\frac{1}{8} \frac{q\omega_p}{n_0 T_e} \frac{(kk_2)(k_1 k_3) + (kk_3)(k_1 k_2)}{kk_1 k_2 k_3}. \quad (5.2)$$

Averaging (5.1) over the phases, we obtain a kinetic equation for the quantity defined by $n_k \delta(k - k')$ = $\langle c_k c_{k'}^* \rangle$:

$$\frac{\partial n_k}{\partial t} = 2\pi \int |\mathcal{W}_{kk_1 k_2 k_3}|^2 \delta(k + k_1 - k_2 - k_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times (n_{k_1} n_{k_2} n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3}) dk_1 dk_2 dk_3. \quad (5.3)$$

Equation (5.3) has integrals of motion

$$N = \int n_k dk, \quad \varepsilon = \int \omega_k n_k dk.$$

The expression (5.2) for the kernel of the four-plasmon equation coincides with the one calculated earlier by Tsytovich and Pikel'ner by another method^[11].

If necessary, the induced scattering by the ions can be taken into account by refining Eq. (5.1). We note for this purpose that since we assume on going over to averaging over the phases that the nonlinearity is small, we can assume in the expression

$$\delta n_{k,\omega} = G_{\omega,k} U_{k,\omega} = \frac{e^2}{4m\omega_p} \int G_{\omega,k}(k_1 k_2) \psi_{k_1, \omega_1} \psi_{k_2, \omega_2} \delta(k + k_1 - k_2) \times \delta(\omega + \omega_1 - \omega_2) d\omega_1 d\omega_2 dk_1 dk_2$$

that $\psi_{k,\omega} \sim \psi_K \delta(\omega - \omega_K)$ is small. Then

$$\delta n_{k,t} = \frac{e^2}{4m\omega_p} \int G_{\omega, k_2 - \omega, k_1} \delta(k + k_1 - k_2) (k_1 k_2) \psi_{k_1, \omega_1} \psi_{k_2, \omega_2} dk_1 dk_2 d\omega_1 d\omega_2. \quad (5.4)$$

Substituting (5.14) in (1.2) and changing over to the variables c_k , we obtain a more precise version of (5.1), in which the kernel \mathcal{W} is replaced by $\tilde{\mathcal{W}}$, where

$$\tilde{\mathcal{W}}_{kk_1 k_2 k_3} = -\frac{1}{8} \frac{\omega_p}{n_0^2} \frac{1}{kk_1 k_2 k_3} [(kk_2)(k_1 k_3) G_{\omega, k_2 - \omega, k_1} + (kk_3)(k_1 k_2) G_{\omega, k_2 - \omega, k_1}].$$

Since $G_{\omega} \approx n_0/(T_i + T_e)$, the real parts of the kernels \tilde{W} and \tilde{W} can be assumed to be equal, with good accuracy. The kinetic equation (5.3), however, acquires a term that describes the induced scattering:

$$\left(\frac{\partial n}{\partial t} \right)_i = 2n_k \int \tilde{\mathcal{W}}_{kk_1 n_{k_1}} dk_1,$$

$$\tilde{\mathcal{W}}_{kk_1} = \text{Im} \tilde{\mathcal{W}}_{kk_1, kk_1} = -\frac{1}{8} \frac{\omega_p}{n_0^2} \frac{(kk_1)^2}{k^2 k_1^2} \text{Im} G_{\omega, k_2 - \omega, k_1}.$$

Analogously, the decay kinetic equations can be derived from the initial system for $(kr_D)^2 > \mu$.

When $(kr_D)^2 < \mu$ we can neglect the induced scattering if $w \gg (kr_D)^2 T_i/T_e$. We shall consider just this case, although the collapse plays an important role in the establishment of the stationary spectrum also at small wave amplitudes.

We average (5.3) over the angles in each of the k -spaces and change over to the variable $\omega = k^2$. We obtain

$$\bar{\omega} \frac{\partial n_{\omega}}{\partial t} = \iint S_{\omega, \omega_2 + \omega_3 - \omega, \omega_2, \omega_3} (n_{\omega_2 + \omega_3 - \omega} n_{\omega_2} n_{\omega_3} + n_{\omega} n_{\omega_2} n_{\omega_3} - n_{\omega} n_{\omega_2 + \omega_3 - \omega} n_{\omega_2} - n_{\omega} n_{\omega_2 + \omega_3 - \omega} n_{\omega_3}) d\omega_2 d\omega_3. \quad (5.5)$$

The integration is now over the region shown in the figure. Then

- in region I: $S_{\omega\omega_1, \omega_2\omega_3} = T_{\omega\omega_1, \omega_2\omega_3} > 0$;
- in region II: $S_{\omega\omega_1, \omega_2\omega_3} = T_{\omega_1\omega, \omega_2\omega_3}$;
- in region III: $S_{\omega\omega_1, \omega_2\omega_3} = T_{\omega_3\omega_2, \omega\omega_1}$;
- in region IV: $S_{\omega\omega_1, \omega_2\omega_3} = T_{\omega_2\omega_3, \omega\omega_1}$.

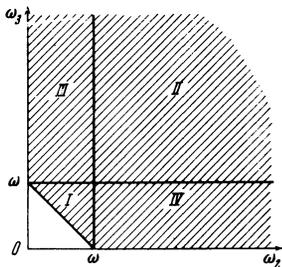
The function T was calculated in^[11] and is a homogeneous function of degree $\frac{1}{2}$. For $\omega_1 \ll \omega$ we have

$$T_{\omega\omega_1, \omega_2\omega_3} \approx \bar{\omega}_{\omega_1}. \quad (5.6)$$

Let us examine the stationary solutions of (5.5). To this end, we equate its right-hand side to zero. We seek the stationary solutions in the form $n_{\omega} = \omega^X$. Using the linear-fractional transformations (see^[3, 12])

$$\omega_2 \rightarrow \frac{\omega\omega_3}{\omega_2 + \omega_3 - \omega}, \quad \omega_3 \rightarrow \frac{\omega\omega_2}{\omega_2 + \omega_3 - \omega};$$

in region II,



$$\omega_2 \rightarrow \frac{(\omega_2 + \omega_3 - \omega)\omega}{\omega_3}, \quad \omega_3 \rightarrow \frac{\omega^2}{\omega_3}$$

in region III, and

$$\omega_2 \rightarrow \frac{\omega^2}{\omega_2}, \quad \omega_3 \rightarrow \frac{(\omega_2 + \omega_3 - \omega)\omega}{\omega_2}$$

in region IV, we obtain

$$\iint \frac{T_{\omega, \omega_2 + \omega_3 - \omega, \omega_2, \omega_3}}{\omega_2^x \omega_3^x (\omega_2 + \omega_3 - \omega)^x} \left[1 + \left(\frac{\omega}{\omega_2 + \omega_3 - \omega} \right)^x - \left(\frac{\omega}{\omega_2} \right)^x - \left(\frac{\omega}{\omega_3} \right)^x \right] \times \left[1 + \left(\frac{\omega}{\omega_2 + \omega_3 - \omega} \right)^{1/2 + 3x} - \left(\frac{\omega}{\omega_2} \right)^{1/2 + 3x} - \left(\frac{\omega}{\omega_3} \right)^{1/2 + 3x} \right] d\omega_2 d\omega_3 = 0,$$

from which we get the possible values of x:

$$x_1 = 0, \quad x_2 = -1, \quad x_3 = -3/2, \quad x_4 = -7/6.$$

The first two distributions, $n_\omega = \text{const}$ and $n_\omega \sim T/\omega$ are thermodynamically in equilibrium and have no physical meaning when applied to the turbulence problem. The third solution $n \sim \omega^{-3/2} \sim k^{-3}$ has the meaning of a Kolmogorov spectrum corresponding to a constant flux of the integral ϵ in the region of large k . This solution, however, cannot be realized because the integrals diverge in the collision term at small k .

The only physically realistic solution is

$$n_\omega \sim Q^{1/6} / \omega^{7/6} \sim Q^{1/3} / k^{7/3};$$

here Q is the constant flux of the number of particles (of the integral N) in the region of small wave numbers. It is easy to verify (in analogy with^[3]) that the integrals in the collision term converge for this solution at both large and small wave numbers¹⁾.

The stationary spectrum $n_k \sim Q^{1/3} / k^{7/3}$ presupposes the presence of wave absorption at small wave numbers. Langmuir collapse is such a mechanism. To verify the existence of collapse, we rewrite the spectrum in the form

$$n_k \approx \frac{W}{4\pi k_0 \omega_p} \left(\frac{k_0}{k} \right)^{7/6}.$$

Here k_0 is the characteristic wave number of the energy-containing turbulence region.

We consider the quantity

$$W_k = 4\pi \int_0^k \omega_p k^2 n_k dk,$$

which has the meaning of the energy density in a sphere of radius k . As $k \rightarrow 0$, the quantity $W_k \sim k^{2/3}$ decreases more slowly than $\omega_k \sim \omega_p (kr_D)^2$, thus indicating the existence of collapse. The boundary k_S of the "collapse zone" can be obtained from the condition $W_k / nT \sim (k_S r_D)^2$, which yields

$$(k_S r_D)^2 \sim \frac{1}{k_0 r_D} \left(\frac{W}{nT} \right)^{3/4};$$

when $k > k_S$ we get weak turbulence, and when $k \lesssim k_S$ we get strong turbulence due to collapse.

If instability exists at $k \sim k_0$ (say, two-stream instability), with a characteristic increment γ , then four-plasmon scattering limits the instability to the level

$$W / nT \sim k_0 r_D (\gamma / \omega_p)^{1/2}.$$

This yields for the boundary of the strong turbulence

$$(k_S r_D)^2 \sim (k_0 r_D)^{1/2} (\gamma / \omega_p)^{1/4}.$$

When $\gamma / \omega_p \gtrsim (k_0 r_D)^2$, there is no region of weak turbulence at all, and the characteristic dimension of the collapsing cavern is of the order of $L \sim 1/k_D$. The foregoing example shows that it is important to take into account the collapse effect in plasma-turbulence problems.

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Translated by J. G. Adashko
201

¹⁾The spectrum $k^2 n_k \sim 1/k^{2.84}$ obtained by Pikel'ner and Tsytovich^[1] is a result of a computation error in formula (2.16) of their article.