A fundamental feature of any advanced (or even not so advanced) technological civilization is measuring things.

- Distance from Oxford to London ~ 80 km
- Speed of a car ~ 60 miles/hour
- Speed of light ~ 300,000 km/sec
- Volume of a bottle ~ 0.75 litre
- Mass of water in a glass ~ 200 gram
- Radius of the Earth ~ 6400 km etc.

Acceleration of gravity $g = 9.8 \text{ m/s}^2$

What do all these statements mean?

Usually, they mean that we have chosen some units in which to measure a quantity, and they can compare it with some standard (e.g., standard of a meter kept in the Bureau of Weights and Measures in Paris).

Or we can define a way to measure some quantity in terms of several units: e.g., speed in units of length per unit time.

Thus, there are independent and dependent (derived) units.

For any given class of physical phenomenon (e.g., mechanics), we can choose a fundamental set of units: length, time, mass — and fix that we can express everything else in terms of these.
E.g. velocity = \frac{\text{length}}{\text{time}}, \text{ acceleration} = \frac{\text{length}}{\text{time}^2}, \text{ etc.}

\text{SI system: meter, kg, sec} \rightarrow \text{ velocity} = \frac{\text{m}}{\text{sec}} \text{ etc.}

\text{What constitutes an adequate system of units depends on the range of physical phenomena we are interested in.}

\text{E.g. Geometric [site of objects]: just length}

\text{Kinematics [moving objects]: length \& time}

\text{Dynamics [objects moving \& subject to forces]: length, time, \text{ man}}

\text{E\&M: have to add a unit of charge (in SI, Coulomb)}}

\text{Note: The choice of a system of units is not unique.}

\text{E.g. we could use velocity \& time instead of length \& time. Then length becomes a dependent unit, expressed as speed \times time (e.g., distances measured in light years).} 1 \text{ knot} = 2 \text{ km/hr}

\text{What if I change the units themselves (rather than what they measure)?}

\text{E.g. use km, tonne, hour (truck driver's units).}

\text{length \& man \& time}
Then all new quantities previously expressed in m, kg, sec must be multiplied or divided by some conversion factors:

\[
\begin{align*}
\text{Length} & \rightarrow \text{Length} / L \\
\text{Time} & \rightarrow \text{Time} / T \\
\text{Mass} & \rightarrow \text{Mass} / M \\
\text{Velocity} & \rightarrow \text{Velocity} / (L/T) \\
\text{Acceleration} & \rightarrow \text{Acceleration} / (L/T^2)
\end{align*}
\]

\[
\begin{align*}
L &= 10^3 \text{ (m in km)} \\
T &= 3600 \text{ (s in hr)} \\
M &= 10^3 \text{ (kg in tonne)} \\
\text{Density} &\rightarrow \frac{\text{Density}}{M/L^3}
\end{align*}
\]

This allows us to introduce the concept of dimension of a physical quantity: it is the factor that determines the factor by which a physical quantity changes if we change units of measurement:

\[
\begin{align*}
[l] &= L \\
[t] &= T \\
[m] &= M \\
[v] &= LT^{-1} \\
[a] &= LT^{-2} \\
[p] &= ML^{-3} \\
\end{align*}
\]

What if we used a different system, say TVM instead of LMT?

\[
\begin{align*}
[t] &= T \\
[v] &= V \\
[m] &= M \\
[l] &= VT \\
[a] &= VT^{-1} \\
[p] &= MV^{-3}T^{-3} \\
\end{align*}
\]

\[
\frac{\text{kg}}{\text{knot}^3 \text{ sec}^3}
\]
So: units are independent if we cannot derive their dimensions from each other.

Two important exercises:

- What are the dimensions of force?
  \[ f = ma \]
  so \[ [f] = MLT^{-2} \]

  (Newton's 2nd law)

  Physical laws are independent of the units of measurement.

  and so both sides of equations that express them must have the same dimensions.

  This is the key principle, which will allow us to discover some amazing things shortly.

- How many independent quantities are there in this set:
  \[ \rho, \ p, \ v \]

  pressure, density, velocity?

  \[ [v] = LT^{-1} \]
  \[ [p] = ML^{-3} \]
  \[ [\rho] = \left[ \frac{f}{L^2} \right] = ML^{-1}T^{-2} \]

  So \[ [\frac{p}{\rho}] = \frac{L^2}{T^2} = [v^2] \]

  has units of velocity

  What is this velocity?

  \[ c_s = \sqrt{\frac{p}{\rho}} \]

  is speed of sound

  \[ 14 \text{ in air at room temp.} \]

  \[ \text{constant cannot be determined from dim. analysis.} \]

  So, just by considering the dimensions, we have been able to discover that a fluid or a gas has a special speed associated with it!
This was the first example of dimensional analysis. I could have asked the question so:

**What is the speed of sound in any given medium?** Clearly it must depend on \( p \) and \( p \).

**What can the speed be equal to if it depends on \( p \) and \( p \)?** It must be proportional to \( \sqrt{p/p} \), so

\[
C_s = \sqrt{\frac{p}{p}} \cdot \text{constant} \]

To get this we only need one good measurement!

(you can be confident of this because the relationship between \( C_s \) and \( p \) is a physical law and it cannot change if we change units — so, scaling units on the RHS must produce the same scaling factor on the LHS etc.)

Note that we did not have to solve any equations (of motion, wave propagation etc.) to get this result. And since it has no choice but to hold, we can even check the dimensions of the results of our calculations.
A systematic example:

The Pendulum.

What is the period of small oscillations of a pendulum? Let us find it without recourse to solving any equations. What can $t_p$ depend on?

$l, m, g$

$[l] = L \quad [m] = M \quad [g] = LT^{-2} \quad [t_p] = T$

$\Pi = \frac{t_p}{\sqrt{2l/g}}$ is dimensionless (i.e., if I change units, $\Pi$ will not change).

In principle, it may be that $\Pi = \Pi(l, m, g)$ - but is it?

Change unit of mass: $m \rightarrow m/M$, but $\Pi$ is unchanged. So independent of mass.

Similarly with $l$ and $g$.

So $\frac{t_p}{\sqrt{2l/g}} = \Pi = \text{constant} \Rightarrow t_p = \text{constant} \sqrt{\frac{l}{g}}$

We solved an interesting physics problem from nothing! (just by analyzing dimensions.)
But things are not always quite so simple...

**Drag force on a moving body**

Consider a body (say, a sphere) moving through a gas at high speed (constant).

What is the drag it feels? In other words, how much power does it need to move?

Let's not worry about friction (we will worry about that later on in these lectures)—so the force will be all due to inertia of the gas as it is kept further apart by the body.

**Parameters that matter:**

- \( [P] = \frac{M}{L^2} \)  
  - Density of gas
- \( [p] = \frac{M}{LT^2} \)  
  - Pressure
- \( [U] = \frac{L}{T} \)  
  - Velocity of body
- \( [d] = L \)  
  - Diameter of body

- \( [F] = \frac{MLU}{T^2} \)  
  - Drag force
- \( \frac{F}{\rho U^2 d^2} = \Pi (p, p, U, d) = \Pi (p, U, d, Ma) \)
  - There are independent!
  \( \rho \sim U^2 \) speed of sound, so we have another Sim-len combination: \( Ma = \frac{U}{C_s} \) Mach Number

By the same argument as before, \( \Pi \) cannot depend on \( p, U, d \) but it can (and does!) depend on \( Ma \).
So we have learned that

\[ f = \rho U^2 d^2 \Pi (Ma) \]

unknown function, which we cannot fix from dim. analysis.

This is less conclusive than with Froudean, but still very useful: a priori, we have \( f = f(\rho, \rho, U, d) \) function of 4 parameters. We have now reduced the problem to find just one function of one dim-len parameter: \( \Pi (Ma) \) [we have also figured out what matters physically].

It is often possible to solve such problems completely in some limits. E.g., consider supersonic motion \( Ma \gg 1 \). If \( \Pi (Ma) \to \text{finite limit as } Ma \to \infty \), we get

\[ f = \text{const} \cdot \rho U^2 d^2 \quad \text{as} \quad U \gg C_s \]

In this case, it works: experimentally,

\[ \Pi \sim \frac{\rho U^2 d^2}{C_s} \]

Note: Power needed to move body:

\[ \Phi \sim f \cdot U \sim \rho U^3 d^2 \]

Quite a strong scallop.
So, general recipe:

1) Find parameters on which quantity of interest depends. [Here one needs to have some physical insight into what is relevant and what is not.]
   (E.g., we neglected friction)

2) Find parameters with independent dimensions

3) Find dimensionless combinations. Then:

\[ \text{Dimensionless combination involving quantity of interest} = \text{function of all other dimension combinations}. \]

\[ \text{lecture ended here} \]

3 \textbf{The 17 Theorem.}

What has been above been shown to work by example can be formally generalized. Here are the steps (no proof).

1) The dimension function is always a power-law monomial; i.e., the dimension of any physical quantity \( a \) is

\[ [a] = L^x M^y T^z R \ldots \text{ (and other units if appropriate, e.g., charge Q)} \]

2) Recall that quantities \( a_1, a_2, \ldots, a_k \) have independent dimensions if none of them can be expressed as product of dimensions of others. If we have a system of \( k \) independent (fundamental) units (e.g., \( k = 3 \) for LMT) and \( k \) quantities \( a_1 \ldots a_k \)
with independent dimensions, it is always possible to change to a system of units that have the same dimensions as \( q_1 \ldots q_k \) — and so, we can then always change units so that any one of the \( q_i \)'s changes by some specified factor, while all other \( q_i \)'s remain unchanged.

Eg. LMT, in the drag force problem

\[
\begin{align*}
q_1 &= \rho \\
q_2 &= U \\
q_3 &= d
\end{align*}
\]

were indep.

So we could measure everything in units of density, velocity and length — and could scale these units independently.

(3) Now consider any given physical problem. It always reduces to finding some relationship(s) of the form

\[
a = f \left( \frac{q_1 \cdots q_k}{b_1 \cdots b_m} \right)
\]

desired quantity (e.g. drag force)

\[
\begin{align*}
\text{e.g.:} & \quad \rho, U, d \\
\text{e.g.} & \quad p
\end{align*}
\]

Can always express

\[
\begin{align*}
[b_1] &= [q_1]^d [q_2]^p_1 \\
& \vdots \\
[b_m] &= [q_1]^d_m [q_2]^{p_m}
\end{align*}
\]

and \([a] = [q_1]^d [q_2]^p \) ...
How to find the exponents? Just by solving a system of simultaneous linear equations:

\[ [f] = M^4 L^4 T^{-2} = [e]^x [U]^y [d]^z = \]

\[ = (ML^{-3})^x (LT^{-1})^y L^z = \]

\[ = M^x L^{-3x + y + z} T^{-y} \]

So, \( x = 1 \)

\[-3x + y + z = 1 \quad \Rightarrow \quad y = 1 + 3 - 2 = 2 \quad \Rightarrow \quad [f] = [p U^2 d^2] \]

\[ [p] = M^4 L^{-1} T^{-2} = [p]^y [U]^z [d]^w \]

\[ = (ML^{-3})^y (LT^{-1})^z L^w = \]

\[ = M^y L^{-3x + y + z} T^{-z} \]

\( z_1 = 1 \)

\[-3x_1 + y_1 + z_1 = -1 \quad \Rightarrow \quad y_1 = -1 + 3 - 2 = 0 \quad \Rightarrow \quad [p] = [p U^2] \]

\[-y_1 = -2 \quad \Rightarrow \quad x_1 = 2 \]

So, this means we can introduce \( m+1 \) dim-less combinations:

\[ \Pi = \frac{a}{a_1^{x_1} a_2^{y_1} ...} \]

\[ \Pi_1 = \frac{\theta_1}{a_1^{x_1} a_2^{y_1} ...} \]

\[ \Pi_m = \frac{\theta_m}{a_1^{x_m} a_2^{y_m} ...} \]

\[ \Pi = \frac{\theta}{p u^2 d^2} \]

\[ \Pi_1 = \frac{\Pi}{p u^2} \]

\[ \frac{1}{M a^2} \]

and we can use these physical relationships as

\[ \Pi = \frac{f(a_1, a_2, \theta, \ldots \theta_m)}{a_1^{x_1} a_2^{y_1} ...} = \frac{f(a_1, a_2, \theta, \Pi_1, a_1^{x_m} a_2^{y_m} \ldots, \Pi_m, a_1^{x_1} a_2^{y_1} ...)}{a_1^{x_1} a_2^{y_1} ...} \]

\[ = f(a_1, \ldots, a_k, \Pi_1, \ldots, \Pi_m) \]
But now, since both sides are dimensionally similar, any of the parameters $\alpha_i$ by an arbitrary factor is equivalent to simply changing units, so should not change values of $\Pi, \Pi_1, \ldots, \Pi_m$ because they are dimensionless or values of the rest of $\alpha_i$'s because they are independent. Therefore, $F$ is independent of $\alpha_1, \ldots, \alpha_k$ and we obtain the Hamiltonian:

$$\Pi = F(\Pi_1, \ldots, \Pi_m)$$

or

$$a = a_1 a_2 \ldots = F(\Pi_1, \ldots, \Pi_m)$$

E.g., $f = p v^2 d^2 F(M_0)$.

So, if we have $k$ independent units in our fundamental system of units ($k=3$ for LMT) and $n$ governing parameters in the problem under scrutiny, we expect to be able to express the answer to any undetermined function of $m = n - k$ dimensionless combinations.
Ex. 5  Rising Bubbles (and related stories)

How fast do bubbles rise depend on their size?

Find \( U \) as a function of \( d \) velocity

Let us first try a "quick and dirty" solution.

\[ [U] = \left[ \frac{L}{T} \right], \text{ velocity depends on } L, T \]

If we had two governing parameters with independent times involving only \( L \) and \( T \), we'd know what to do. OK, there are

\[ [d] = L \text{ and } [g] = \frac{L}{T^2} \text{ acceleration of gravity} \]

So, immediately, \[ U = \text{const.} \sqrt{gd} \]

(Bubble 4 times the size rises twice as fast)

Does this make sense? Well, it's just force balance:

Archimedes force \( \sim \rho \pi d^3 g \) = drag force

\( (\text{buoyancy}) \)

So, indeed, \( U^2 \sim gd \).

But recall that the drag force result involved assumption "high speed" - we did not quantify this...
assumption, which was necessary to neglect viscosity of the fluid. But surely this was a dodgy assumption? – especially for bubbles, which in our experience rise rather slowly and at quite different speeds in fluids of varying viscosity.

So, we need to include the effect of viscosity – which means that we need to introduce some quantity that characterizes the viscosity of a fluid, a quantity that could be measured for any given fluid.

Viscosity is basically a measure of how difficult it is to move fluid differentially with itself.

\[
\frac{A}{d} \quad \text{plate moving at velocity } v
\]

\[
\text{fluid} \quad \text{fluid at rest} \quad \text{because fluid "sticks" to two plates.}
\]

It is known empirically that force on moving plate \( f \) is \( \propto \frac{V}{d} A \) and the (dimensionless!) constant of proportionality is, within some class of fluids and ambient conditions, approximately independent of \( V \) or \( d \). So let

\[
[f] = \frac{MLT^{-1}}{L^2} \quad [\mu] = \frac{MLT^{-1}}{T^2 L^2}
\]

Note: This is one of the ways physics moves forward: empirical laws are parameterized, then understood on a deeper level (e.g. viscosity from him-then)
Physically, viscosity is relevant to the determination of the drag force on a moving object because there are two sets of forces opposing motion:

1) Inertial forces - object pushes the medium apart as it moves.

2) Viscous forces - fluid in the immediate vicinity of the object has to move at the speed of the object, while fluid far away must be at rest, so the object sets up a differential flow.

So, let's repeat our drag force calculation.

\[ [F] = \frac{ML}{T^2}, \quad [p] = \frac{M}{L^3}, \quad [p] = \frac{M}{LT^2}, \quad [U] = \frac{1}{L}, \quad [d] = L \]

\[ \mu = \frac{M}{LT} \]

\[ \nu = 5 \text{gov. parameters} \]

\[ k = 3 \text{ independent, say } p, U, d, \text{ as before} \]

\[ m = 2 \text{ - we'll have 2 dimensionless combinations!} \]

Find them:

\[ [p] = [e^{U^2}] \text{ we already know } (p, U), \text{ so} \]

\[ [\mu] = [e]^{\chi^2} [U]^{a} [d]^{b} [d]^{c} = M^x L^{-3x+\beta+\gamma} T^{-\beta} = \frac{M}{LT} \]

So \( x = 1, \beta = 1, \gamma = 1 \)

\[ Re = \frac{p Ud}{\mu} \]
From the \( \Pi \)-theorem, some function of 2 dimensionless numbers:

\[
f = \rho U^2 d^2 \mathcal{F}(Ma, Re)
\]

So we now see what it meant to move "fast". We needed \( Re \gg 1 \) and implicitly assumed that \( \mathcal{F}(Ma, \infty) \) was finite (NB: this sort of thing is not always vindicated).

Clearly, for subcritical \( Ma \ll 1 \) \( (U \ll c_s) \).

So let's assume that \( \mathcal{F}(0, Re) \) is finite and we only need to figure out the \( Re \) dependence.

1) Familiar limit is \( Re \gg 1 \): \( f \approx \text{const} \rho U^2 d^2 \)

It turns out that this is a situation in which the fluid behind the bubble becomes turbulent, so viscous forces no longer matter ("turbulent drag").

2) Opposite limit: \( Re \ll 1 \)

It's clear \( \mathcal{F}(0, 0) \) cannot be finite because then we'd get the same answer independent of viscosity. So we need a little physical insight to guess what \( \mathcal{F}(Re) \) looks like at small \( Re \).

[This is the only way to deal with things that do not follow from slip analysis formally]
If viscosity is large, let's argue that drag force should not depend on density — because inertia of the fluid is no longer important.

Since \( f = pU^2d^2F(\text{Re}) \) and \( \text{Re} = \frac{pUd}{\mu} \), the only way to arrange for this is

\[
F(\text{Re}) \propto \frac{\text{const}}{\text{Re}}, \quad \text{so} \quad f = \text{const} \cdot \mu U d \]  

Stokes' formula

(in fact, already Aristotle thought for \( \text{Re} \approx 1 \) — before Newton knew better)

For \( \text{Re} \ll 1 \) (so for slow velocities, large viscosities, low densities or small bubbles)

Balance with Archimedes force:

\[
pd^3 g \sim \mu U d \quad \Rightarrow \quad U = \text{const} \frac{pg}{\mu} d^2
\]

\[
\text{So speed of risers likely increases quite fast with their size if they are small.}
\]

NB: \( \text{Re} \sim \frac{p^2gd^3}{\mu^2} \sim 1 \) when \( d_c \sim \frac{\mu^{1/3}}{\rho^{1/3}g^{1/3}} \) (transition around these values) \( \sim C \sim (\rho g/\rho)^{1/3} \)
Pythagoras Theorem (This is quite amateur)

Area of a right triangle is completely determined by its hypotenuse $c$ and one (let's say the smaller) of its acute angles $\phi$.

Similarly, $A = c^2 f(\phi)$.

Divide it into 2 triangles $\Box$ as $\Box$,

$A_1 = a^2 f(\phi), A_2 = b^2 f(\phi)$

But $A = A_1 + A_2$, so

$c^2 f(\phi) = a^2 f(\phi) + b^2 f(\phi)$

$c^2 = a^2 + b^2 \quad \text{q.e.d.}$

Note that this is based on operatip in flat space. If we were in curved space — e.g. a triangle on the surface of a sphere, there would be another parameter — $r$, radius of the sphere, so we would have $A = c^2 f(\phi, \frac{c}{r})$.

So we'd have

$c^2 f(\phi, \frac{c}{r}) = a^2 f(\phi, \frac{a}{r}) + b^2 f(\phi, \frac{b}{r})$

Can't cancel $f$! — unless $\frac{c}{r}, \frac{a}{r}, \frac{b}{r} \leq 1$, so we take the limit $f(\phi, 0)$ and recover previous result.