Revision Lecture : TT-2014

① KINETIC THEORY

Classical ideal gas: in equilibrium, particles have Maxwellian distribution

$$f(\vec{v}) = \frac{e^{-v^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}, \quad \text{where} \quad v_{th} = \sqrt{\frac{2k_B T}{m}}$$

[see p.19 of my Notes]

(in general, this is true locally in space)

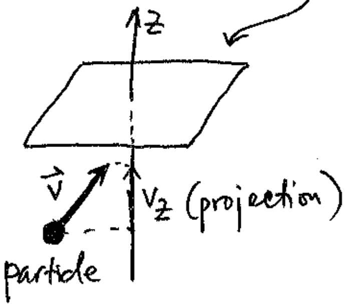
[See p.33 of my Notes]

Key step in deriving nearly all the relevant results (and solving exam questions):

The # of particles with velocities in the "cube"

$$[\vec{v}, \vec{v}+d^3\vec{v}] = [v_x, v_x+dv_x] \times [v_y, v_y+dv_y] \times [v_z, v_z+dv_z]$$

that hit a unit area perpendicular to z axis per unit time



$$d\Phi(\vec{v}) = \underset{\substack{\uparrow \\ \text{velocity at which they are moving towards the area of interest}}}{v_z} \underbrace{n f(\vec{v})\, d^3\vec{v}}_{\substack{\# \text{ of particles per unit volume, with } \vec{v}\in[\vec{v},\vec{v}+d^3\vec{v}]}}$$

[pp. 11-12 of Notes]

Derivation (not required unless specifically asked for)

volume where a particle must be to hit area A over time t

density of particles

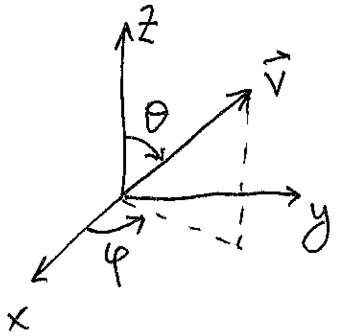
$$d\Phi(\vec{v}) = \frac{\boxed{A\cdot v_z t}\cdot\boxed{n}\cdot\boxed{f(\vec{v})\,d^3\vec{v}}}{A\cdot t}$$

← fraction of particles with velocities in $[\vec{v}, \vec{v}+d^3\vec{v}]$

per unit area per unit time

Same quantity for $f = f(\overset{\swarrow\,\text{isotropic}}{v})$, in polar coordinates:

$$d\Phi(\vec v) = v\cos\theta\, n f(v)\, v^2 \sin\theta\, dv\, d\theta\, d\varphi =$$



$$= \underbrace{n v^3 f(v) dv}_{\substack{\text{distribution}\\ \text{of}\\ \text{speeds}}}\ \underbrace{\cos\theta \sin\theta\, d\theta\, d\varphi}_{\substack{\text{distribution}\\ \text{of}\\ \text{angles}}}$$

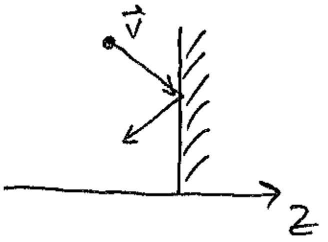
$\theta \in [0,\pi]$

$\varphi \in [0, 2\pi]$

$v \in [0,\infty]$

Now we can calculate everything:

1) Pressure = momentum flux onto wall



$$p = \int d\Phi(\vec v) \cdot 2 m v_z =$$

↑ momentum deposited by bouncing particles

↗ NB integral over $\theta \in [0, \frac{\pi}{2}]$ half the interval of θ's because particles must be moving towards the wall

equivalently, over $v_z > 0$ →

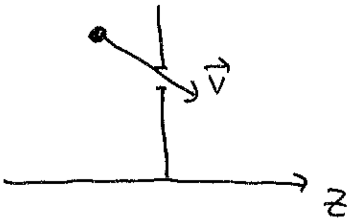
$$= \int_{(v_z>0)} 2 m v_z \cdot v_z \cdot n f(v)\, d^3\vec v = mn\langle v_z^2\rangle \overset{\text{isotropic}}{=} \frac{1}{3} mn \langle v^2 \rangle$$

$$\overset{\text{Maxwellian}}{=} n k_B T$$

↑ can drop 2 and integrate over $v_z \in [-\infty, +\infty]$ because $f(v_z) = f(-v_z)$

[p.13,20 of Notes]

2) Effusion flux = flux of particles through a hole



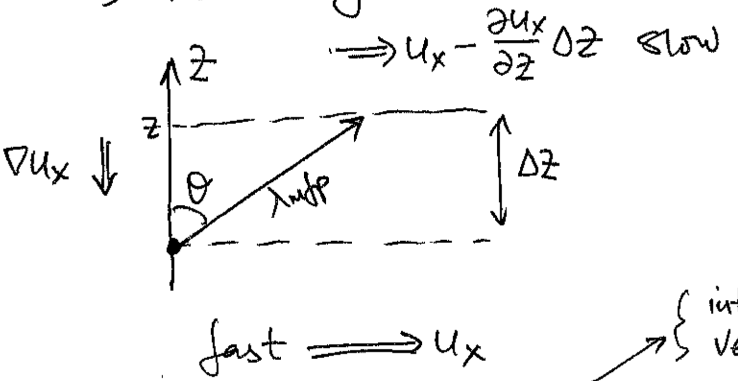
$$\Phi = \int_{v_z>0} d\Phi(\vec v) =$$

$$= n \int_0^\infty dv\, v^3 f(v) \underbrace{\int_0^{\pi/2} d\theta \cos\theta \sin\theta}_{=\frac{1}{2}} \underbrace{\int_0^{2\pi} d\varphi}_{= 2\pi} =$$

$$= n \cdot \frac{1}{4} \cdot \int_0^\infty dv \underbrace{4\pi v^2 f(v)}_{\substack{\uparrow \\ \text{distribution of speeds} \\ \text{[p.14 of Notes]}}} \cdot v = \frac{1}{4} n \langle v \rangle \overset{\substack{\text{for Maxwellian} \\ \swarrow}}{=} \frac{p}{\sqrt{2\pi m k_B T}}$$

$(p = n k_B T)$
[p.24 of Notes]

3) Viscosity



Momentum flux through z:

$$\Pi_{zx} = \int d\Phi(\vec{v}) \cdot \Delta p =$$

{ integral over all velocities

including ones with $v_z<0$ because those are the ones coming from $z+\Delta z$

{ mean extra momentum brought to z by particles that had last collision at $z-\Delta z$

$$\boxed{\begin{aligned} \Delta p &= -m\Delta u_x = -m\frac{\partial u_x}{\partial z}\Delta z \\ \Delta z &= \lambda_{mfp}\cos\theta \end{aligned}}$$

$$= \cancel{m} \int n v^3 f(v)\, dv \cos\theta \sin\theta\, d\theta\, d\varphi \cdot \left(-m\frac{\partial u_x}{\partial z}\lambda_{mfp}\cos\theta\right)$$

$$= -mn\frac{\partial u_x}{\partial z}\lambda_{mfp}\int_0^\infty dv\, v^3 f(v) \underbrace{\int_0^\pi d\theta \cos^2\theta \sin\theta}_{\overset{\shortparallel}{2/3}} \underbrace{\int_0^{2\pi} d\varphi}_{\overset{\shortparallel}{2\pi}}$$

$$= -\frac{1}{3} mn\lambda_{mfp}\frac{\partial u_x}{\partial z} \underbrace{\int_0^\infty dv\, 4\pi v^3 f(v)}_{\overset{\shortparallel}{\langle v \rangle}}$$

$$= \underbrace{-\frac{1}{3} mn\lambda_{mfp}\langle v \rangle}_{\overset{\shortparallel}{\eta} \text{ viscosity}} \frac{\partial u_x}{\partial z} = \frac{F}{A} \text{ force per unit area}$$

[p.49 of Notes]

4) Thermal conductivity

Same story, but for flux of energy (heat flux):

$$J_z = \int d\Phi(\vec{v})\,\underbrace{\Delta E}_{} = \dots = \underbrace{-\frac{1}{3} n c_v \lambda_{mfp} \langle v \rangle}_{} \frac{\partial T}{\partial z}$$

excess energy brought by particles $\quad \Delta E = -\underbrace{\frac{3}{2}k_B}_{c_v \text{ per particle}} \frac{\partial T}{\partial z}\Delta z$

$= \varkappa$ heat conductivity

[p. 50 of Notes]

5) Particle diffusivity

Flux of labelled particles through z:

$$\Phi_z^* = \int d\Phi(\vec{v}) \cdot \frac{1}{n} \cdot \Delta n^*$$

Δn^* ← excess density of labelled particles coming from $z - \Delta z$

$\frac{1}{n}$ ← to get rid of n dependence in $d\Phi(\vec{v})$

$$\Delta n^* = -\frac{\partial n^*}{\partial z}\lambda_{mfp}\cos\theta$$

$$= \underbrace{-\frac{1}{3}\lambda_{mfp}\langle v \rangle}_{} \frac{\partial n^*}{\partial z}$$

$= D$ particle diffusivity

Note: Easy way to check your derivations is by verifying that the result has correct dimensions:

$$\frac{\partial}{\partial t}\frac{3}{2} n k_B T = -\frac{\partial}{\partial z} J_z = \varkappa \frac{\partial^2 T}{\partial z^2} \Rightarrow \text{Dimension of } \frac{\varkappa}{\frac{3}{2} n k_B} \text{ is } \frac{L^2}{t}$$

energy density $\qquad$ flux

So $\varkappa \sim n k_B v_{th} \lambda_{mfp}$

$$\frac{\partial}{\partial t} m n u_x = -\frac{\partial}{\partial z}\Pi_{zx} = \eta \frac{\partial^2 u_x}{\partial z^2} \Rightarrow \text{Dimension of } \frac{\eta}{mn} \text{ is } \frac{L^2}{t}$$

momentum density $\qquad$ flux

So $\eta \sim m n v_{th} \lambda_{mfp}$

$$\frac{\partial}{\partial t} n^* = -\frac{\partial}{\partial z}\Phi_z^* = D \frac{\partial^2 n^*}{\partial z^2} \Rightarrow \text{Dimension of } D \text{ is } \frac{L^2}{t}$$

So $D \sim v_{th}\lambda_{mfp}$

[p. 42 of Notes]

② QUANTUM GASES

Starting point:

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} \pm 1}$$

⊕ Fermi-Dirac

⊖ Bose-Einstein

[derivation p. 171 of Notes]

mean occupation # of single-particle microstate i

ε_i is energy of that microstate

This discussion is inspired by Q10 of the 2012 paper

NB: $i = (\vec{p}, s_z$, any other relevant quantum #'s)

momentum $\vec{p} = \hbar\vec{k}$

spin

Density of states: $g(k)dk$ = # of i's in the interval $[k, k+dk]$

$g(\varepsilon)d\varepsilon$ = # of i's in the interval $[\varepsilon, \varepsilon+d\varepsilon]$

$g(\omega)d\omega$ = # of i's in the interval $[\omega, \omega+d\omega]$ (for photons usually, $\varepsilon = \hbar\omega$).

Calculating g allows us to convert

$$\sum_i = \int dk\, g(k) = \int d\varepsilon\, g(\varepsilon) \quad \text{etc.}$$

$$g(k)\,dk = \frac{4\pi k^2 dk}{(2\pi)^3/V} \cdot (2s+1)$$

"volume" of the shell $[k, k+dk]$ in $\vec{k}$ space

# of spins

"volume" in $\vec{k}$ space corresponding to each discrete value of $\vec{k}$

$$= \frac{(2s+1)}{2\pi^2} V k^2 dk \quad \text{in 3D}$$

$$g(k)\,dk = \frac{2\pi k\, dk}{(2\pi)^2/A} \cdot (2s+1) = \frac{(2s+1) A k\, dk}{2\pi} \quad \text{in 2D}$$

} NB

Conversion from $g(k)$ to $g(\varepsilon)$ depends on the relationship between ε and k.

Non-relativistic gas: $\varepsilon = \dfrac{\hbar^2 k^2}{2m}$ (ultrarelativistic gas or photons: $\varepsilon = \hbar k c$)

Change variables:

$$\underset{\underset{\frac{\hbar^2 k\,dk}{m}}{\|}}{g(\varepsilon)d\varepsilon} = g(k)dk \quad \Rightarrow \quad g(\varepsilon) = \frac{g(k)m}{k\hbar^2} \underset{\boxed{3D}}{=} \frac{(2s+1)Vmk}{2\pi^2\hbar^2}$$

$$= \frac{(2s+1)Vm}{2\pi^2\hbar^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}} = \frac{(2s+1)Vm^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}\sqrt{\varepsilon}$$

[p175 of Notes]

Similarly in 2D:

$$g(\varepsilon) = \frac{(2s+1)Am}{2\pi\hbar^2} = \text{const}$$

(v. convenient)

From this, usually calculate

$$\bar{N} = \sum_i \bar{n}_i = \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

- if $\bar{N}$ is fixed, this is the equation for $\mu(\underset{\underset{N/V}{\uparrow}}{n}, T)$
- if μ is known (e.g. $\mu = 0$ for photons), this tells you how many particles there are (equivalently, their equilibrium density)

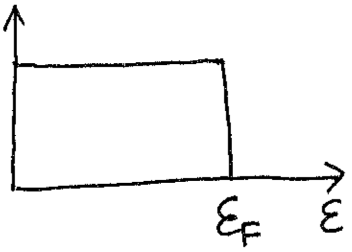
Energy:

$$U = \sum_i \varepsilon_i \bar{n}_i = \int_0^\infty \frac{d\varepsilon\, \varepsilon\, g(\varepsilon)}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

Pressure: $P = \underset{\underset{\text{in 3D, for non-relat. gas}}{\uparrow}}{\boxed{\frac{2}{3}}} \dfrac{U}{V}$ calculated via grand potential (p.177 of notes)

For Fermi gas at T=0,

$$\frac{1}{e^{\beta(\epsilon-\mu)}+1} =$$



$\uparrow$
$\varepsilon_F = \mu(T=0)$

and all the integrals become very simple:

$$N = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon) \;\Rightarrow\; \text{Calculate } \varepsilon_F \text{ from this}$$

3D: $\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{2s+1}\right)^{2/3}$

2D: $\varepsilon_F = \frac{\hbar^2}{2m}\,\frac{4\pi}{2s+1}\,\frac{N}{A}$

$$U = \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon\, g(\varepsilon) \;\Rightarrow$$ Can calculate mean energy per particle, for which you only need to know that $g \propto \sqrt{\varepsilon}$ (in 3D!)

$$\frac{U}{N} = \frac{\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon\sqrt{\varepsilon}}{\int_0^{\varepsilon_F} d\varepsilon\sqrt{\varepsilon}} = \frac{2/5}{2/3}\,\varepsilon_F = \frac{3}{5}\varepsilon_F$$

in 2D: $$\frac{U}{N} = \frac{\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon}{\int_0^{\varepsilon_F} d\varepsilon} = \frac{1/2}{1} = \frac{1}{2}\varepsilon_F$$

Heat capacity requires knowledge of T dependence of energy U, i.e.

$$U(T) = \underset{\frac{3}{5}\varepsilon_F N}{\underset{\shortparallel}{U(T=0)}} + \delta U(T)$$

small correction calculated in the limit $k_B T \ll \varepsilon_F$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial \delta U}{\partial T}\right)_V$$

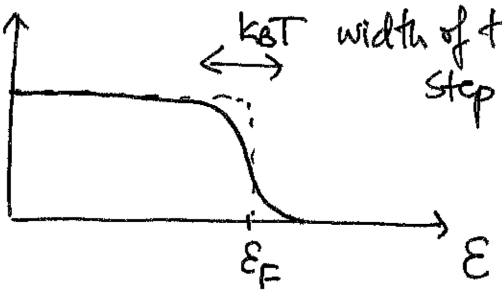
In my lectures (p. 190), I calculated C_V systematically — I do not expect that such a calculation could be required in an exam. What may be required (and is useful anyway) is a qualitative understanding of why it turns out that

$$C_V \approx \underbrace{Nk_B}_{\substack{C_V \text{ of}\\ \text{classical}\\ \text{gas}}} \cdot \underbrace{\frac{k_BT}{\varepsilon_F}}_{\substack{\text{small}\\ \text{number.}}} \cdot \text{const}$$

↑ this happens to be $\pi^2/2$, but you can't show that qualitatively.

At finite T s.t. $k_BT \ll \varepsilon_F$,

$$\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \approx$$



I.e., a small # of fermions with energies of order $\sim \varepsilon_F$ can be kicked out of the ground state to energies $> \varepsilon_F$. What is this #?

$$\Delta N_{\text{excited}} \sim \underbrace{g(\varepsilon_F)}_{\substack{\text{density of states around}\\ \varepsilon \approx \varepsilon_F}} \underbrace{\Delta\varepsilon}_{\substack{\text{energy spread}\\ \Delta\varepsilon \sim k_BT}} \sim g(\varepsilon_F)\, k_BT$$

Each of these ~~electrons~~ fermions will have on the order of $\sim \Delta\varepsilon \sim k_BT$ more energy than it would have done at $T=0$

Therefore, the excess mean energy compared to T=0 will be

$$\delta U(T) \sim \underbrace{g(\varepsilon_F) k_B T}_{\substack{\#\text{ of excited} \\ \text{fermions}}} \cdot \underbrace{k_B T}_{\substack{\text{excess energy} \\ \text{per fermions}}} = g(\varepsilon_F)(k_B T)^2$$

So,

$$U(T) = U(T=0) + \text{const} \cdot g(\varepsilon_F)(k_B T)^2$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \text{const } g(\varepsilon_F) k_B^2 T \sim$$

$$\sim k_B \underbrace{g(\varepsilon_F)\varepsilon_F}_{\substack{S \\ N = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)}} \cdot \frac{k_B T}{\varepsilon_F} \sim k_B N \frac{k_B T}{\varepsilon_F} \quad \text{q.e.d.}$$

For Bose gas at $T \to 0$, the physics is quite different because bosons all want to be in the $\varepsilon = 0$ state, but $g(\varepsilon) \propto \sqrt{\varepsilon}$ in 3D and so ~~[illegible]~~ the integral

$$N = \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)}{e^{\beta(\varepsilon - \mu)} - 1}$$

cannot "see" a finite number of particles accumulating at $\varepsilon = 0$.

In contrast, in 2D, $g(\varepsilon) = \text{const}$ and ~~[illegible]~~ small-ε particles are adequately included in the ~~[illegible]~~ continuous approximation.

(This discussion is inspired by Q10 of the 2013 paper)

In dealing with this problem mathematically, you should not worry too much about remembering how to do the ζ-function integrals.

Basic idea is as follows.

- As $T \to 0$, bosons all fall into the ground state, so

$$\bar{n}_0 = \frac{1}{e^{-\beta\mu}-1} \to N \quad \Rightarrow \quad \beta\mu \to -\ln\left(1+\frac{1}{N}\right) \approx -\frac{1}{N} \to 0$$

$$N = \underbrace{\int_0^\infty \frac{d\varepsilon \cdot \sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)}-1}} \cdot \frac{(2s+1)Vm^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}$$

|| change variable $x = \beta\varepsilon$

$$\underbrace{\int_0^\infty \frac{dx\sqrt{x}}{e^{x-\beta\mu}-1}} \cdot \frac{1}{\beta^{3/2}} = \text{const} \cdot \frac{1}{\beta^{3/2}}$$

$$\simeq \int_0^\infty \frac{dx\sqrt{x}}{e^x-1}$$

This is just a number, it does not really matter what it is (obviously order unity). The important ~~pointers~~ detail is that the integral converges: indeed, as $x \ll 1$, $\frac{\sqrt{x}}{e^x-1} \sim \frac{1}{\sqrt{x}}$ and this is integrable.

Thus,

$$N = \text{const} \cdot \frac{(2s+1)\,Vm^{3/2}}{\hbar^3}\,(k_B T)^{3/2} \qquad (*)$$

So, when T is below this value, there is condensation

[p. 194 of Notes]

This gives $T_c = \text{const}\cdot\frac{\hbar^2}{mk_B}\left[\frac{N}{V(2s+1)}\right]^{2/3} \sim \frac{\hbar^2 n^{2/3}}{mk_B}$

NB : $k_BT_c \sim \varepsilon_F \sim k_B T_{deg.}$

↑ degeneracy temperature.

You can actually guess this dimensionally, by asking what energy scale can be cooked up from the only available parameters: $\hbar$, m and n.

For $T<T_c$, (*) is just the formula for the # of bosons in the excited state. Normalising,

$$\frac{N_{exc}}{N} = \left(\frac{T}{T_c}\right)^{3/2} \qquad \left(\text{because } N_{exc}=N \text{ for } T=T_c\right)$$

<u>The energy</u> integral is handled similarly:

$$U = \int_0 \frac{d\varepsilon\, \overset{\text{extra factor} = x\cdot k_BT}{\boxed{\varepsilon}}\, g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1} = \text{const}\cdot N_{exc}\cdot\boxed{k_BT} \quad \text{dimensionally obvious!}$$

$$= \text{const}\cdot Nk_BT_c\left(\frac{T}{T_c}\right)^{5/2} \qquad [\text{p. 197 of Notes}]$$

<u>Heat capacity</u> $C_V = \left(\frac{\partial U}{\partial T}\right)_V = \text{const}\, Nk_B\left(\frac{T}{T_c}\right)^{3/2}$

<u>Pressure</u> $P = \frac{2}{3}\frac{U}{V}$ etc.