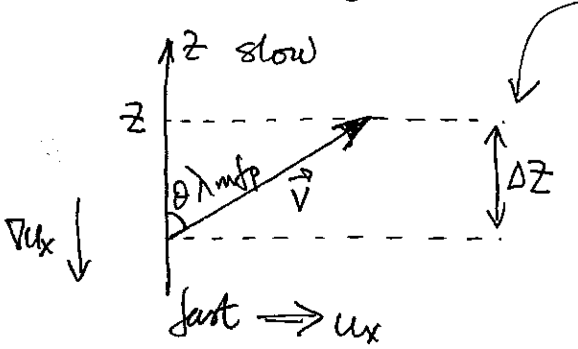
§6. Kinetic Calculation of Transport Coefficients.

We now turn to the task of deriving the transport equations and the expressions for viscosity and thermal conductivity from kinetic theory.

6.1 A popular, simple but dodgy derivation

Continue assuming $\vec{u} = u_x(z)\hat{x}$, $T = T(z)$, $n = \text{const}$

Viscosity.



# of molecules that hit this plane per unit time per unit area (see p.24, eq.(1))

$$d\Phi(\vec{v}) = n v^3 f(v)\, dv \cos\theta \sin\theta\, d\theta\, d\varphi \tag{1}$$

These molecules have travelled distance λ_{mfp} since their last collision — i.e., since they last "communicated" with the gas collectively.

Where in z was this? At $z - \Delta z$, $\Delta z = \lambda_{mfp}\cos\theta$.

But the mean momentum at z and at $z-\Delta z$ is different, so the particle that last collided at $z - \Delta z$ brings to z some extra momentum:

$$\Delta p = -m\frac{\partial u_x}{\partial z}\Delta z = -m\frac{\partial u_x}{\partial z}\lambda_{mfp}\cos\theta \tag{2}$$

[because $u_x(z-\Delta z) - u_x(z) \approx \frac{\partial u_x}{\partial z}(-\Delta z)$]

Thus, the flux of momentum is

$$\Pi_{zx} = \int_{\substack{\text{all}\\ \text{velocities}}} d\Phi(\vec{v})\cdot\Delta p = -mn\frac{\partial u_x}{\partial z}\lambda_{mfp}\int_0^\infty dv\, v^3 f(v)\underbrace{\int_0^\pi d\theta\cos^2\theta\sin\theta}_{=2/3}\underbrace{\int_0^{2\pi} d\varphi}_{=2\pi}$$

$$= -\frac{1}{3}mn\lambda_{mfp}\frac{\partial u_x}{\partial z}\underbrace{4\pi\int dv\, v^3 f(v)}_{=\langle v\rangle}$$

Thus,

$$\boxed{\Pi_{zx} = \underbrace{-\frac{1}{3} mn\lambda_{mfp}\langle v\rangle}_{= \eta \text{ viscosity}} \frac{\partial u_x}{\partial z}} \qquad (3)$$

Since $\langle v\rangle = \frac{2}{\sqrt{\pi}} v_{th}$, $\eta = \frac{2}{3\sqrt{\pi}} mn\lambda_{mfp} v_{th}$, so we have the expression we expected [eq.(23) p.42] with a particular numerical coefficient (which is wrong anyway as you'll realise later).

<u>Thermal conductivity</u>: precisely the same story, viz., particles that arrive at z after having collided at $z-\Delta z$ bring to z some extra energy because $T(z) \neq T(z-\Delta z)$:

$$\Delta E = -\underbrace{\frac{3}{2}k_B}_{= c_v \text{ per particle}} \frac{\partial T}{\partial z}\Delta z = -\underbrace{\frac{3}{2}k_B}\frac{\partial T}{\partial z}\lambda_{mfp}\cos\theta \qquad (4)$$

Therefore the flux of energy (heat flux) is

$$\boxed{J_z = \int d\Phi(\vec{v})\,\Delta E = \underbrace{-\frac{1}{3}\cdot n c_v \lambda_{mfp}\langle v\rangle}_{= \varkappa \text{ heat capacity}} \frac{\partial T}{\partial z}} \qquad (5)$$

and so $\varkappa = \frac{2}{3\sqrt{\pi}} n c_v \lambda_{mfp} v_{th} = \frac{n k_B}{\sqrt{\pi}} \lambda_{mfp} v_{th}$.

These new expressions for η and $\varkappa$ <u>look</u> more quantitative, but are in fact not!

<u>Ex.</u> Do the same derivation for the flux of an admixture, i.e., given $\frac{\partial n^*}{\partial z}$, calculate Φ_z^*.

So, what makes this derivation dodgy?

1) I included in the derivation of fluxes the fact that particles had an angle distribution, but at the same time assumed that they all travelled exactly the same distance λ_{mfp} between collisions and all carried exactly the same momentum and energy - but surely all of this depends on particle velocity?!

e.g., if I had said instead that they all travelled for the same time τ_c between collisions, I would have had to replace $\lambda_{mfp} \to v\tau_c$ and got $\frac{\eta}{mn} = \frac{\varkappa}{nc_V} = \frac{1}{3}\langle v^2 \rangle \tau_c =$ (eq. (7) p.30)
$= \frac{1}{3}\frac{\langle v^2\rangle}{\langle v\rangle}\lambda_{mfp} = \sqrt{\frac{\pi}{24}}\lambda_{mfp} v_{th}$ different coefficient!

2) I used homogeneous (isotropic) Maxwellian distributions $f(\vec{v})$ even though we now only have a local Maxwellian, which depends on z via $T(z)$ and $u_x(z)$.
This might have been OK because the scale of inhomogeneities is long ($\gg \lambda_{mfp}$) and $u_x \ll v_{th}$, but I did not set up this approximation systematically!

So, clearly, if I am to claim that I am doing a better job than just the dimensional guess on p. 42, I must provide a more systematic derivation!

6.2 Kinetic expressions for fluxes

Let us go back to basics. Suppose we know the particle distribution $F(\vec{r}, \vec{v})$. What are the fluxes of momentum and energy?
$\vec{r}$: just z!

Momentum: $$\Pi_{zx}^{(z)} = \int d^3\vec{v}\, \underbrace{m v_x}_{\text{particle momentum}} \cdot \underbrace{v_z}_{\text{velocity of particles along } z} \cdot \underbrace{F(z, \vec{v})}_{\#\text{ of particles}} \qquad (6)$$

Energy (heat) (assume $\vec{u}=0$):

$$J_z(z) = \int d^3\vec{v}\, \underbrace{\frac{mv^2}{2}}_{\text{particle energy}} \cdot \underbrace{v_z}_{\text{velocity of particle}} \cdot \underbrace{F(z, \vec{v})}_{\#\text{ of particles}} \qquad (7)$$

But if $$F(z, \vec{v}) = \frac{n}{[\pi v_{th}^2(z)]^{3/2}} e^{-\frac{[v_x - u_x(z)]^2 + v_y^2 + v_z^2}{v_{th}^2(z)}}$$ local Maxwellian

$$= \frac{n}{(\pi v_{th}^2)^{3/2}} e^{-w^2/v_{th}^2} \qquad v_{th}(z) = \sqrt{\frac{2k_B T(z)}{m}}$$

Then $\Pi_{zx} = 0$ and $J_z = 0$ because they ~~are~~ both have v_z under the integral and F is even in v_z!

So this means that fluxes come from the fact that the distribution function is not <u>exactly</u> a local Maxwellian:

and we must find this bit!

$$F = F_M + \delta F(z, \vec{v}) \qquad (8)$$

To do this, we will need ~~to~~ an evolution <u>equation</u> for $F(t, \vec{r}, \vec{v})$.

6.3 Kinetic Equation

Simplest derivation: for some small Δt,

$$F(t+\Delta t, \vec{r}, \vec{v}) = F(t, \vec{r}-\vec{v}\Delta t, \vec{v}) + \Delta F_c$$

Particles found at $\vec{r}$ with velocity $\vec{v}$ at $t+\Delta t$ were at $\vec{r}-\vec{v}\Delta t$ at time t

Since some collisions occurred during Δt, F changed

$$\approx F(t,\vec{r},\vec{v}) - \vec{v}\cdot\nabla F\,\Delta t + \Delta F_c$$

can Taylor-expand as long as $v\Delta t \ll [\nabla \ln F]^{-1}$ (small enough Δt)

Thus,

$$\boxed{\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F = \left(\frac{\partial F}{\partial t}\right)_c} \qquad (9)$$

Kinetic Equation

NB: We assume no forces act on particles other than collisions

Ex. What if particles have an acceleration $\vec{a}$? (e.g. gravity)

LHS: conservation of particle density in phase space

$$\left[\text{indeed, } \frac{\partial F}{\partial t} = -\nabla\cdot(\vec{v}F) + \dots\right]$$

flux of particles in phase space

RHS: $\left(\frac{\partial F}{\partial t}\right)_c$ "collision operator".

For ideal gas of hard spheres it was derived by Boltzmann. I will not present this derivation (see, e.g. Chapman & Cowling) but instead use some very simple criteria that it must satisfy to produce an extremely simple model for it (not quantitatively correct, but good enough for our purposes)

6.4 Collision Operator

1) If collisions are elastic, they should not change the total #, momentum or energy of particles (at a point).

So, $\int d^3\vec{v}\,\Delta F_c = 0 \Rightarrow \int d^3\vec{v}\left(\frac{\partial F}{\partial t}\right)_c = 0$ particle #

Similarly, $\int d^3\vec{v}\, m\vec{v}\left(\frac{\partial F}{\partial t}\right)_c = 0$ momentum

$\int d^3\vec{v}\,\frac{mv^2}{2}\left(\frac{\partial F}{\partial t}\right)_c = 0$ energy (10)

2) The effect of collisions should be to drive the distribution of particles locally towards a Maxwellian

$$F_M = \frac{n(t,\vec{r})\, m^{3/2}}{[2\pi k_B T(t,\vec{r})]^{3/2}}\, e^{-\frac{m|\vec{v}-\vec{u}(t,\vec{r})|^2}{2k_B T(t,\vec{r})}}$$ [see eq.(7) p.33]

Once this distribution is achieved, collisions should not change it:

$$\left(\frac{\partial F_M}{\partial t}\right)_c = 0 \qquad (11)$$

NB: In fact, if one derives the explicit form of the collision operator from considering the microphysics of particle collisions (as Boltzmann did), one can then prove that $\left(\frac{\partial F}{\partial t}\right)_c = 0$ implies $F = F_M$

and also that collisions always drive the distribution to F_M (never away from it). This is associated with the so called Boltzmann's H-theorem, which is the law of entropy increase for kinetic systems. This is outside the syllabus but you should look it up!

3) The distribution should <u>relax to local Maxwellian equilibrium on the collisional time scale</u> $\tau_c = \frac{1}{\nu_c}$, $\nu_c = \sigma n v_{th}$
[NB: depends on n and T, so can be function of $\vec{r}$; in ~~reality~~ reality, it can also be function of $\vec{v}$]

To satisfy these three criteria, we postulate

$$\boxed{\left(\frac{\partial F}{\partial t}\right)_c = -\nu_c (F - F_M) = -\nu_c \delta F} \quad (12)$$

"Krook operator" (or BGK = Bhatnagar-Gross-Krook)

where, to ensure conservation laws, we demand

$$\int d^3\vec{v}\, \delta F = 0, \quad \int d^3\vec{v}\, m\vec{v}\, \delta F = 0 \quad \text{and} \quad \int d^3\vec{v}\, \frac{mv^2}{2}\, \delta F = 0,$$

i.e., the deviation from Maxwellian cannot contain any perturbation of n, $\vec{u}$, T (we can always do this because any changes in n, $\vec{u}$, T can be absorbed into the Maxwellian)

NB: The Krook operator is <u>very</u> simplified and inadequate for many kinetic calculations, especially when the collision rate is low. However, it'll be fine for our calculation of ~~η~~ η and κ.

The "enlightened guessing" procedure we followed to introduce it is also very instructive as a general illustration of how one might devise a simple physically sensible model where the precise nature of the underlying process (collisions) is unknown but it is clear what criteria must be satisfied a priori.

6.5 Local Conservation Equations Rederived

Let us now again consider a simplified case where the local equilibrium only depends on z and mean velocity is along x and density is constant:

$$\left\{ \begin{array}{l} n = \int d^3\vec{v}\, F(t,\vec{r},\vec{v}) = \text{const} \\ \vec{u} = \int d^3\vec{v}\, \vec{v}\, F(t,\vec{r},\vec{v}) = n u_x(z)\hat{x} \\ F(t,\vec{r},\vec{v}) = F(t,z,\vec{v}) \end{array} \right. \qquad (13)$$

Let us take moments of the kinetic equation (9):

<u>Density</u>

$$\frac{\partial}{\partial t} n = \int d^3\vec{v}\, \frac{\partial F}{\partial t} = \int d^3\vec{v} \left[-\underbrace{\vec{v}\cdot\nabla F}_{= v_z \frac{\partial F}{\partial z}} + \left(\frac{\partial F}{\partial t}\right)_c \right] =$$

$$= -\frac{\partial}{\partial z} \underbrace{\int d^3\vec{v}\, v_z F}_{\substack{=0 \\ \text{because } u_z = 0}} + \underbrace{\int d^3\vec{v} \left(\frac{\partial F}{\partial t}\right)_c}_{\substack{=0 \text{ because collisions conserve} \\ \text{particles.}}} = 0 \qquad (14)$$

So, $\frac{\partial n}{\partial t} = 0$ if we start at $n = \text{const}$, it will stay const.

<u>Momentum</u>

$$\frac{\partial}{\partial t} m n u_x = \int d^3\vec{v}\, m v_x \frac{\partial F}{\partial t} = \int d^3\vec{v}\, m v_x \left[-v_z \frac{\partial F}{\partial z} + \cancel{\left(\frac{\partial F}{\partial t}\right)_c} \right]$$

0 because collisions conserve momentum

$$= -\frac{\partial}{\partial z} \underbrace{\int d^3\vec{v}\, m v_x v_z F}_{\substack{= \Pi_{zx} \text{ as in (6) (p.52)} \\ \text{momentum flux}}} = -\frac{\partial \Pi_{zx}}{\partial z} \qquad (15)$$

[same as eq. (14) p. 38]

Since Π_{zx} vanishes for $F = F_M$ (~~odd~~ integrand odd in v_z), $\boxed{\Pi_{zx} = m\int d^3\vec{v}\, v_x v_z\, \delta F}$ (16)

Energy

ordered energy

$\mathcal{E} = \frac{3}{2} n k_B T$ internal energy [see eq.(6) p.32

$$\frac{\partial}{\partial t}\int d^3\vec{v}\,\frac{mv^2}{2}F = \underbrace{\frac{\partial}{\partial t}\left(\frac{mnu_x^2}{2}\right.}_{\text{from eq.(15)}} + \left.\left\langle\frac{mw^2}{2}\right\rangle n\right)$$

$$mnu_x\frac{\partial u_x}{\partial t} = -u_x\frac{\partial \Pi_{zx}}{\partial z} \qquad (17)$$

On the other hand,

$$\frac{\partial}{\partial t}\int d^3\vec{v}\,\frac{mv^2}{2}F = \int d^3\vec{v}\,\frac{mv^2}{2}\frac{\partial F}{\partial t} = \int d^3\vec{v}\,\frac{mv^2}{2}\left[-v_z\frac{\partial F}{\partial z} + \cancel{\left(\frac{\partial F}{\partial t}\right)_c}\right]$$

0 because collisions conserve energy

$$= -\frac{\partial}{\partial z}\underbrace{\int d^3\vec{v}\,\frac{mv^2}{2}v_z F}$$

this looks like heat flux, eq.(7) on p.52, but we have to be careful as heat is only the random part of the motion, while $\vec{v}$ now also contains mean $\vec{u} = u_x\hat{x}$.

Let $\vec{v} = \vec{u} + \vec{w}$. Then $d^3\vec{v} = d^3\vec{w}$ and

$$\int d^3\vec{v}\,\frac{mv^2}{2}v_z F = \int d^3\vec{w}\left(\cancel{\frac{mu_x^2}{2}} + mu_x w_x + \frac{mw^2}{2}\right)w_z F =$$

Vanishes because $\int d^3\vec{w}\, w_z F = 0$ by definition of $\vec{w}$

$$= \int d^3\vec{w}\,\frac{mw^2}{2}w_z F + u_x\underbrace{\int d^3\vec{w}\,m w_x w_z F} \qquad (18)$$

This is J_z heat flux

↳ precisely Π_{zx} because, from eq.(16),

$$\Pi_{zx} = m\int d^3\vec{w}\,(w_x + \cancel{u_x})w_z F = m\int d^3\vec{w}\, w_x w_z F$$

Vanishes because $\int d^3\vec{w}\, w_z F = 0$ by def of $\vec{w}$

So, we have

$$\frac{\partial}{\partial t}\int d^3\vec{v}\,\frac{mv^2}{2}F \overset{\text{eq.(17)}}{=} \frac{\partial}{\partial t}\frac{3}{2}nk_BT - u_x\cancel{\frac{\partial \Pi_{zx}}{\partial z}}$$ Cancellation

$$\overset{\text{eq.(18)}}{=} -\frac{\partial J_z}{\partial z} - \frac{\partial}{\partial z}u_x\Pi_{zx} = -\frac{\partial J_z}{\partial z} - \Pi_{zx}\frac{\partial u_x}{\partial z} - u_x\cancel{\frac{\partial \Pi_{zx}}{\partial z}}$$

So,

$$\boxed{\frac{\partial}{\partial t}\frac{3}{2}nk_BT = -\frac{\partial J_z}{\partial z} - \Pi_{zx}\frac{\partial u_x}{\partial z}} \qquad (19)$$

same as eq.(13) p.37 (for $u_x=0$)

where $J_z = \int d^3\vec{w}\,\frac{mw^2}{2}w_zF$ (20)

<u>heat flux</u>

$$= \int d^3\vec{w}\,\frac{mw^2}{2}w_z\,\delta F$$

because it is zero for $F=F_M$

$$\left(F_M = \frac{n}{(\pi v_{th}^2)^{3/2}}e^{-w^2/v_{th}^2}\right)$$

What is this?!

Ex. Figure out what this is!

Hint. Look at what happens to total energy $\frac{mnu_x^2}{2} + \frac{3}{2}nk_BT$

from eq.(15) from eq.(19)

OK, so we have rederived the local conservation laws and have expressions for fluxes in terms of δF. It remains to figure out what is δF — we will do this from the kinetic equation (9)

NB: Since $\Pi_{zx} = -\eta\frac{\partial u_x}{\partial z}$, eq.(19) becomes

$$\frac{\partial}{\partial t}\frac{3}{2}nk_BT = -\frac{\partial J_z}{\partial z} + \underbrace{\eta\left(\frac{\partial u_x}{\partial z}\right)^2}_{\text{positive! So heating term associated with viscosity}}$$

Show that total energy $\int d^3\vec{r}\left(\frac{mnu_x^2}{2} + \frac{3}{2}nk_BT\right)$ is conserved!

6.6 Solution of Kinetic Equation

The kinetic equation with the Krook operator is

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F = -\nu_c \delta F \quad , \quad F = F_M + \delta F \qquad (21)$$

Suppose $\delta F \ll F$, i.e. the distribution is close to the local Maxwellian equilibrium.

Suppose also that spatiotemporal evolution of δF happens on the same ~~scales~~ scales as for F_M (i.e. for T and u_x).

We will confirm these assumption shortly.

Then, in the LHS of (21), we can approximate $F \approx F_M$ and so

$$\delta F \approx -\frac{1}{\nu_c}\Big(\frac{\partial F_M}{\partial t} + \underbrace{\vec{v}\cdot\nabla F_M}_{\displaystyle = v_z \frac{\partial F_M}{\partial z} \sim \frac{v_{th}}{\ell} F_M}\Big) \quad , \quad F_M = \frac{n m^{3/2}}{(2\pi k_B T)^{3/2}} e^{-\frac{m|\vec{v}-u_x\hat{x}|^2}{2k_BT}}$$

ℓ ← spatial scale of T, u_x

Thus,

$$\frac{1}{\nu_c} v_z \frac{\partial F_M}{\partial z} \sim \frac{v_{th}}{\ell \nu_c} F_M \sim \frac{\lambda_{mfp}}{\ell} F_M \ll F_M \text{ as we assumed.}$$

Also

$$\frac{1}{\nu_c}\frac{\partial F_M}{\partial t} \sim \left(\frac{\lambda_{mfp}}{\ell}\right)^2 F_M \ll F_M \text{ and also} \ll \text{than} \quad \text{so } \frac{\partial F_M}{\partial t}$$

should be neglected completely!

see p.45 eq.(26)

$$\frac{\partial}{\partial t} \sim D\frac{\partial^2}{\partial z^2} \sim \frac{\nu_c \lambda_{mfp}^2}{\ell^2}$$

← you might worry that we don't technically speaking know this yet - but we can anticipate it and then confirm that our solution satisfies it!

Thus, $\delta F \approx -\frac{1}{\nu_c}\vec{v}\cdot\nabla F_M = -\frac{v_z}{\nu_c}\frac{\partial}{\partial z}F_M =$

$$= -\frac{v_z}{\nu_c}F_M\left[\frac{m|\vec{v}-u_x\hat{x}|^2}{2k_BT^2}\frac{\partial T}{\partial z} - \frac{3}{2}\frac{1}{T}\frac{\partial T}{\partial z} + \frac{m}{2k_BT}2(\vec{v}-u_x\hat{x})\cdot\hat{x}\frac{\partial u_x}{\partial z}\right]$$

$$= -\frac{1}{\nu_c}F_M(w)\,w_z\left[\left(\frac{w^2}{v_{th}^2}-\frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial z} + \frac{2w_x}{v_{th}^2}\frac{\partial u_x}{\partial z}\right] \quad (22)$$

where, as before, $\vec{w} = \vec{v}-\vec{u}$, $v_{th}^2 = \frac{2k_BT}{m}$ and $F_M = \frac{n e^{-w^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}$

So we have solved the kinetic equation.

NB: If the collision operator had been more realistic, the solution would have involved inverting it, which can be quite complicated [see Chapman & Cowling or Landau-Lifshitz - vol 10]

<u>Ex.</u> Check that eq. (22) satisfies the conservation of particles, momentum and energy (see p. 55)

6.7 Calculation of Fluxes

<u>Momentum flux</u>: $\Pi_{zx} = \int d^3\vec{w}\, m w_x w_z \delta F =$ sub. from eq. (22)

$$= -\frac{m}{\nu_c}\int d^3\vec{w}\, w_x w_z^2 F_M(w)\left[\cancel{\left(\frac{w^2}{v_{th}^2}-\frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial z}} + \frac{2w_x}{v_{th}^2}\frac{\partial u_x}{\partial z}\right]$$

vanishes because integrand is odd in w_x

$$= -\frac{2m}{\nu_c v_{th}^2}\int d^3\vec{w}\, w_x^2 w_z^2 F_M(w)\frac{\partial u_x}{\partial z} \equiv -\eta\frac{\partial u_x}{\partial z}$$

Thus, viscosity is

$$\eta = \frac{2m}{\nu_c v_{th}^2} \underbrace{\int_0^{2\pi} d\varphi \cos^2\varphi}_{=\pi} \underbrace{\int_0^{\pi} d\theta \sin^3\theta \cos^2\theta}_{=\frac{4}{15}} \underbrace{\int_0^{\infty} dw\, w^6 \frac{n e^{-w^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}}_{=\frac{n v_{th}^4}{\pi^{3/2}} \frac{\sqrt{\pi}}{2} \frac{5\cdot 3\cdot 1}{2^3} = \frac{15}{16\pi} n v_{th}^4}$$

$$= \frac{m n v_{th}^2}{2\nu_c} = \frac{n k_B T}{\nu_c} = \frac{P}{\nu_c} \qquad (23)$$

Cf. eq. (3) (p. 50): $\eta = \frac{1}{3} m n \lambda_{mfp} \langle v \rangle = \frac{1}{3} m n \frac{\langle v \rangle^2}{\nu_c} = \frac{4}{3\pi} \frac{m n v_{th}^2}{\nu_c}$

So we have ended up with a different numerical coefficient (as expected!) but still ~~without a~~ in line with our a priori dimensional guess [p. 42 eq. (23)]:

$$\eta \sim m n v_{th} \lambda_{mfp} \sim \frac{m v_{th}}{\sigma}$$

The numerical coefficient depends on the precise form of the collision operator.

The point is that if we know what this form is (depends on particle collision microphysics) then we can compute viscosity precisely!

Nothing fundamentally dodgy about the above calculation anymore.

<u>Heat flux</u>: $J_z = \int d^3\vec{w}\, \frac{mw^2}{2} w_z \delta F =$

$$= -\frac{m}{2\nu_c}\int d^3\vec{w}\; w^2 w_z^2 F_M(w)\left[\left(\frac{w^2}{v_{th}^2}-\frac{3}{2}\right)\frac{1}{T}\frac{\partial T}{\partial z} + \cancel{\frac{2w_x}{v_{th}^2}\frac{\partial u_x}{\partial z}}\right]$$

vanishes because integrand odd in w_x

$$= -\frac{m}{2\nu_c T}\cdot 2\pi \underbrace{\int_0^\pi d\theta \sin\theta\cos^2\theta}_{=\frac{2}{3}} \underbrace{\int_0^\infty dw\, w^6 \left(\frac{w^2}{v_{th}^2}-\frac{3}{2}\right)\frac{n e^{-w^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}}_{=\frac{n v_{th}^4}{\pi^{3/2}}\frac{\sqrt{\pi}}{2}\left(\frac{7\cdot5\cdot3\cdot1}{2^4}-\frac{3}{2}\cdot\frac{5\cdot3\cdot1}{2^3}\right)=\frac{15}{8\pi}n v_{th}^4} \frac{\partial T}{\partial z}$$

$$= -\varkappa \frac{\partial T}{\partial z}$$

<u>Ex.</u> Check this calculation.

where thermal conductivity is

$$\varkappa = \frac{5}{4}\frac{mnv_{th}^4}{\nu_c T} = \frac{5}{2} n k_B \frac{v_{th}^2}{\nu_c} = \frac{5}{3} n c_V \frac{v_{th}^2}{\nu_c} \qquad (24)$$

Cf. eq. (5) (p. 50): $\varkappa = \frac{1}{3} n c_V \lambda_{mfp} \langle v\rangle = \frac{4}{3\pi}\frac{n c_V v_{th}^2}{\nu_c}$

Same story with numerical coefficients.

As before, $\varkappa \sim n c_V v_{th}\lambda_{mfp} \sim \frac{c_V v_{th}}{\sigma}$

so the a priori dimensional guess was right.

So the key insight is that relaxation of gradients (local → global equilibrium) happens via fluxes that are due to small departures of the pdf from Maxwellian

Further reading.

- My lecture notes for MT-2011 – there I derived transport eqns somewhat more generally (3D mean flow, 3D spatial dependence of T and $\vec{u}$ and n)
- Kardar's book – derivation of transport eqns using Krook operator
- Landau & Lifshitz vol. 10 / Chapman & Cowling } kinetic theory in full detail (including real collision operators and their inversion)