

§17. Degenerate Fermi Gas

Let us summarise the general programme that must be followed in dealing with a quantum gas:

1) Calculate $\mu(n, T)$ from [see eq. (28) p. 175]:

$$N = \int_0^{\infty} \frac{dE g(E)}{e^{\beta(E-\mu)} + 1} \quad (1)$$

↑
for Fermi

✓ NB: does not depend on T

where $g(E) = \frac{(2s+1) V m^{3/2}}{\sqrt{2} \pi^2 \hbar^3} \sqrt{E}$ for non-relativistic ^{3D} gas.

2) Given μ , calculate mean energy [see eq. (30) p. 176]:

$$U = \int_0^{\infty} \frac{dE g(E) E}{e^{\beta(E-\mu)} + 1}, \quad (2)$$

✓ for non-relativistic ^{3D} gas

which is also $U = -\frac{3}{2} \Phi = \frac{3}{2} P V$ ($\frac{1}{3}$ for ultrarelat. 1 for 2D) (3)

and from this we can infer

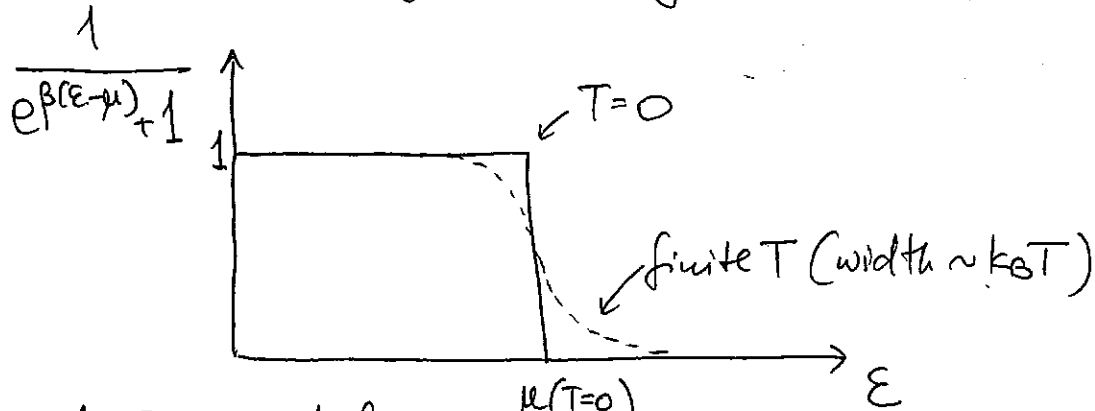
the equation of state and the rest of thermodynamics,

e.g. entropy $S = \frac{5}{3} \frac{U - \mu N}{T}$ (4)

heat capacity $C_V = \left(\frac{\partial U}{\partial T} \right)_V$ [or use S to get C at const any temp else]

So, the bottom line is we want the two integrals (1) & (2).

Consider Fermi gas at very low T , so $\beta \rightarrow \infty$. Then



So, at $T=0$, particles stack up to a certain fixed energy $\rightarrow \boxed{E_F = \mu(T=0)}$ Fermi energy

17.1 Fermi Energy and Fermi Gas at $T=0$

Calculations become very easy:

$$1) \text{ Eq. (1): } N = \int_0^{E_F} dE g(E) = \frac{(2S+1)V m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^{E_F} dE \sqrt{E} \quad (5)$$

$$\text{So } \boxed{E_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{2S+1} \right)^{2/3}} = \frac{\hbar^2 k_F^2}{2m} \quad (6)$$

$$k_F = \left(\frac{6\pi^2 n}{2S+1} \right)^{1/3}$$

Fermi wave # (max. momentum for the particles)

This tells us

- chemical potential $\mu(0) = E_F$
- max energy per particle E_F (~~all~~ ^{all} simple-particle levels are occupied up to E_F)
- criterion for the $T=0$ approximation:

$$k_B T \ll E_F \sim \hbar^2 n^{2/3} / m \quad \text{precisely the degeneration temperature}$$

\nearrow width of the "step"

$$(T_F = \frac{E_F}{k_B} \sim 10^4 \text{ K for electrons in metals})$$

Note (again) that "low T " means $T \ll T_F$ and T_F can be very high for systems with large density and low fermion mass.

E.g., for electrons in white dwarves (see PS-7),
 $E_F \sim \text{MeV}$ so $T_F \sim 10^{10} \text{ K} \sim m_e c^2$ so actually
 they are relativistic — and all our calculations
 must be redone for $\epsilon(k) = \hbar k c$ [Ex.]

2) $E_{q. (2)} : \underline{\text{energy}}$
 $c^2 E_F$

2) Eq. (2): energy

$$U = \int_0^{\epsilon_F} d\epsilon g(\epsilon) \epsilon = \frac{N}{\frac{2}{3} \epsilon_F^{3/2}} \underbrace{\int_0^{\epsilon_F} d\epsilon \epsilon^{3/2}}_{= \frac{2}{5} \epsilon_F^{5/2}} = \frac{3}{5} N \epsilon_F \quad (7)$$

Thus, mean energy per particle is

$$\frac{U}{N} = \frac{3}{5} \epsilon_F \quad (8)$$

and the equation of state is

$$P = \frac{2}{3} \frac{U}{V} = \frac{2}{5} n \epsilon_F \stackrel{\text{eq. (6)}}{=} \frac{\hbar^2}{5m} \left(\frac{6\pi^2}{2s+1} \right)^{2/3} n^{5/3} \quad (9)$$

This is of course independent of T (indeed, $T=0$)
and so the gas might be said to behave as
a "pure mechanism" (no entropy / no heat involved)



Note that (9) $\Leftrightarrow PV^{5/3} = \text{const}$ - the general adiabatic law [eq. (35) p. 177], but it is only the eq. of state at $T=0$ (when $S=0=\text{const!}$.)

17.2 Finite-T Corrections: Heat Capacity

We are not done yet because we have not calculated the heat capacity at low T :
indeed knowing $U(T=0)$ does not help us calculate

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V \text{ for } T \ll E_F$$

Also, by the way, we have not really proven that $S=0$ at $T=0$: indeed, from (4),

$$S = \frac{\frac{5}{3} U - \mu N}{T} \rightarrow 0 \quad \text{but what is the ratio?}$$

So, the goal now is to calculate finite- T corrections to our $T=0$ results, i.e., to expand the integrals (1) & (2) in $k_B T / E_F \ll 1$.

This task will require a little bit of maths:

we must learn how to calculate integrals of the form

$$I = \int_0^\infty \frac{d\varepsilon f(\varepsilon)}{e^{\beta(\varepsilon-\mu)} + 1}, \quad \text{where } f(\varepsilon) = g(\varepsilon) \propto \sqrt{\varepsilon} \text{ in (1)}$$

$$= \varepsilon g(\varepsilon) \propto \varepsilon^{3/2} \text{ in (2)}$$

(or other powers for other limits,
e.g. ultrarelativistic or 2D)

Change variables:

$$\beta(\varepsilon-\mu) = x \Rightarrow \varepsilon = \mu + k_B T x$$

$$\text{Then } I = k_B T \int_{-\mu/k_B T}^\infty \frac{dx f(\mu + k_B T x)}{e^x + 1} =$$

$$= k_B T \int_0^{\infty} \frac{dx f(\mu + k_B T x)}{e^x + 1} + k_B T \int_0^{\mu/k_B T} \frac{dx f(\mu - k_B T x)}{e^{-x} + 1} =$$

here we changed $x \rightarrow -x$

use

$$\frac{1}{e^{-x} + 1} = 1 - \frac{1}{e^x + 1}$$

$$= k_B T \int_0^{\mu/k_B T} dx f(\mu - k_B T x) +$$

because $\mu \gg k_B T$,
pick exp'lly small error $\rightarrow \infty$

$$+ k_B T \left[\int_0^{\infty} \frac{dx f(\mu + k_B T x)}{e^x + 1} - \int_0^{\mu/k_B T} \frac{dx f(\mu - k_B T x)}{e^x + 1} \right]$$

$$\approx \int_0^{\mu} d\epsilon f(\epsilon) + k_B T \int_0^{\infty} \frac{dx}{e^x + 1} \underbrace{\left[f(\mu + k_B T x) - f(\mu - k_B T x) \right]}$$

SS

$$2 k_B T x f'(\mu)$$

$$= \int_0^{\mu} d\epsilon f(\epsilon) + 2 (k_B T)^2 f'(\mu) \underbrace{\int_0^{\infty} \frac{dx x}{e^x + 1}}_{\substack{\parallel \\ \frac{\pi^2}{12}}} + \dots \quad \text{higher-order terms if needed}$$

Thus, we have the foll.

useful formula: if $k_B T \ll \mu$, then

see, e.g. Landau & Lifshitz §58

$$I = \int_0^{\infty} \frac{d\epsilon f(\epsilon)}{e^{\beta(\epsilon - \mu)} + 1} \approx \int_0^{\mu} d\epsilon f(\epsilon) + \frac{\pi^2}{6} f'(\mu) (k_B T)^2 + \dots \quad (10)$$

We can now calculate finite-T corrections

to anything we like by substituting the appropriate form of $f(\epsilon)$.

1) Eq. (1): $f(\epsilon) = g(\epsilon) = \frac{N}{\frac{2}{3} \epsilon_F^{3/2}} \sqrt{\epsilon}$ and so

$$N = \frac{N}{\frac{2}{3} \epsilon_F^{3/2}} \left[\frac{2}{3} \mu^{3/2} + \frac{\pi^2}{6} \frac{1}{2\sqrt{\mu}} (k_B T)^2 + \dots \right]$$

$\mu = \epsilon_F + \delta\mu$

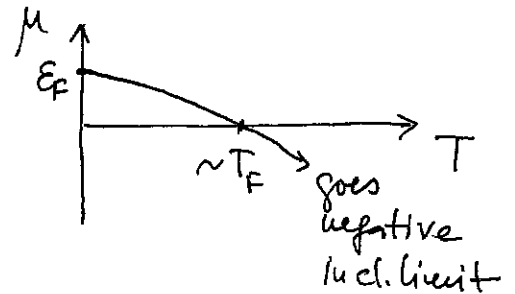
$\mu \approx \epsilon_F$ (can neglect $\delta\mu$ here because this is already inside small term)

$$1 = \left(1 + \frac{\delta\mu}{\epsilon_F}\right)^{3/2} + \frac{\pi^2}{8} \left(\frac{k_B T}{\epsilon_F}\right)^2 + \dots$$

So,

$$\frac{\mu}{\epsilon_F} = 1 + \frac{\delta\mu}{\epsilon_F} = \left[1 - \frac{\pi^2}{8} \left(\frac{k_B T}{\epsilon_F}\right)^2 + \dots\right]^{2/3} \approx 1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F}\right)^2$$

$$\boxed{\mu \approx \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F}\right)^2\right]} \quad (11)$$



2) Eq. (2): now calculate energy:

$$f(\epsilon) = g(\epsilon) \epsilon = \frac{N}{\frac{2}{3} \epsilon_F^{3/2}} \epsilon^{3/2} \text{ and so}$$

$$U = \frac{N}{\frac{2}{3} \epsilon_F^{3/2}} \left[\frac{2}{5} \mu^{5/2} + \frac{\pi^2}{6} \frac{3}{2} \sqrt{\mu} (k_B T)^2 + \dots \right]$$

$\mu \xrightarrow{\text{eq. (11)}} \epsilon_F$

$$\approx N \left\{ \frac{3}{5} \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F}\right)^2\right]^{5/2} + \frac{3\pi^2}{4} \frac{(k_B T)^2}{\epsilon_F} + \dots \right\}$$

$$= \frac{3}{5} N \epsilon_F \left[1 - \frac{5}{2} \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 + \frac{5}{8} \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$

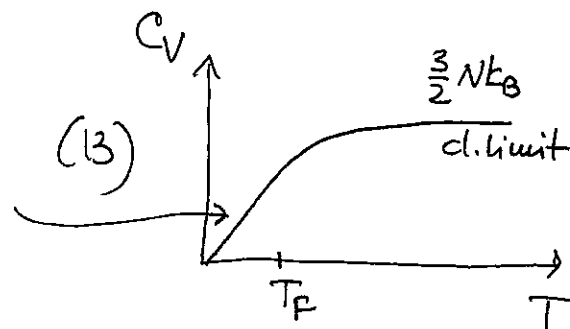
$$= \frac{3}{5} N \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] \quad (12)$$

↑ this is the finite-T correction we have been after

Heat capacity:

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V = N k_B \frac{\pi^2}{2} \frac{k_B T}{\epsilon_F} \quad (13)$$

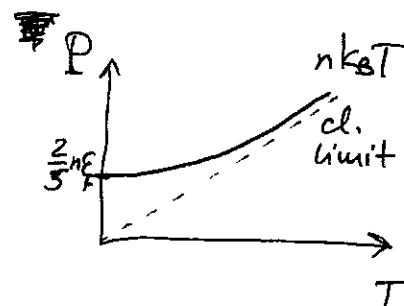
So it is linear in T as $T \rightarrow 0$



For metals, this heat capacity due to electrons (which are degenerate) is the dominant part - because the heat capacity due to lattice vibrations behaves as $\propto T^3$ at low T, so \ll the linear electron contribution

Equation of state:

$$P = \frac{2}{3} \frac{U}{V} \underset{\text{eq. (12)}}{=} \frac{2}{5} n \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$



Entropy:

eq. (11) & (12)

$$S = \frac{1}{T} \left[\frac{5}{3} U - \mu N \right] \underset{\checkmark}{=} \frac{N}{T} \left[\epsilon_F \left(1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right) - \epsilon_F \left(1 + \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right) \right]$$

$$= N \frac{k_B \pi^2}{2} \frac{k_B T}{\epsilon_F} \rightarrow 0 \text{ as } T \rightarrow 0 \quad (3^{\text{rd}} \text{ law holds})$$

Ex. Work out $\frac{C_P}{C_V}$ and show that it is not $\frac{5}{3}$ as $T \rightarrow 0$.