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§16. Quantum Ideal Gases

So far, in all our calculations of partition functions for ideal gas, we have stayed within the classical limit, where the key assumption was that the # of single-particle states available ≫ # particles and so the probability of more than one particle occupying any given single-particle state ≪ 1.

The time has now come to *relax this assumption*, but let us first explain what are those *quantum correlations*, which we have so far tried so assiduously to avoid having to take into account.

16.1 Bosons and Fermions

- Consider a 2-particle wave function

$$\psi(\underset{\substack{\nearrow\\\text{first particle}\\\text{in state 1}}}{1},\underset{\substack{\nwarrow\\\text{in state 2}}}{2})\ \longleftarrow\ \text{second particle}$$

- Swap the particles: $\psi(1,2) \to \psi(2,1)$ (1)
- If particles are *indistinguishable*, this cannot change any observables, so the probability density cannot change under the swapping operation:

$$|\psi(2,1)|^2 = |\psi(1,2)|^2 \qquad (2)$$

Therefore $\psi(2,1) = e^{i\phi}\,\psi(1,2)$ (3)

- Apply the swapping operation twice:

$$\psi(2,1) = e^{i\phi}\psi(1,2) = e^{2i\phi}\psi(2,1)$$

$$\Downarrow$$

$$e^{2i\phi} = 1 \Rightarrow e^{i\phi} = \pm 1$$ [technically speaking, this is only true in 3D world: see Blundell §29.1]

So this tells us that there are 2 types of particles:

1) $$\boxed{\psi(2,1) = \psi(1,2)} \tag{4}$$ bosons - can prove these are particles with integer spin, e.g. photons (1), ^{4}He atoms (0)

2) $$\boxed{\psi(2,1) = -\psi(1,2)} \tag{5}$$ fermions - half-integer spin, e.g. e, n, p, ^{3}He, ... ($\frac{1}{2}$)

For these, there is the Pauli exclusion principle:

suppose states 1 and 2 are the same, then

$$\psi(1,1) = -\psi(1,1) = 0$$

so two fermions cannot be in the same state

[see Landau & Lifshitz Quantum Mechanics §61-62 for rigorous generalisation to N-particle wave functions and the derivation of the connection between spin and exchange symmetry]

What does this mean for statistical mechanics of systems composed of these particles?

Recall that the microstates of a box of ideal gas were specified in terms of occupation #'s n_i of the single-particle states i.

What we have just learned about exchange symmetry imposes constraints on what the occupation #'s can be:

$$
\begin{aligned}
&\underline{\text{for bosons}}, \quad n_i = 0,1,2,3,\ldots \ \text{any integer \#} \\
&\underline{\text{for fermions}}, \quad n_i = 0 \text{ or } 1 \ \ \text{no more than 1 particle in each state!}
\end{aligned}
\qquad (6)
$$

The latter is precisely an example of quantum correlations: even though fermions in an ideal gas are non-interacting, the system as a whole "knows" which single-particle states are occupied and so unavailable to further particles.

16.2 Partition Function

Recall that

$$\mathcal{Z} = \sum_{\alpha} e^{-\beta(E_\alpha - \mu N_\alpha)} \qquad (7)$$

microstates:
$\alpha = \{n_1, n_2, \ldots, n_i, \ldots\}$
occupation #'s of single-particle states indexed by i
$[i = (\underset{\text{momentum}}{\vec{p}}, \underset{\text{spin}}{s}, \ldots)]$

total energy levels
$E_\alpha = \sum_i n_i \varepsilon_i$
single-particle energy levels

total # of particles:
$N_\alpha = \sum_i n_i$

Thus,

$$\mathcal{Z} = \sum_{\{n_i\}} e^{-\beta \sum_i n_i(\varepsilon_i - \mu)} = \qquad (8)$$

all possible sets of occupation #'s, so $\sum_{n_1}\sum_{n_2}\sum_{n_3}\cdots$

$$= \sum_{\{n_i\}} \prod_i e^{-\beta n_i(\varepsilon_i-\mu)} = \prod_i \sum_{n_i=0,1,\dots} \left[e^{-\beta(\varepsilon_i-\mu)}\right]^{n_i}$$

all single-particle states

product of sums = sum of products

all possible values for the occupation #'s

Fermions: $n_i = 0, 1$, so

$$\mathcal{Z} = \prod_i \left[1 + e^{-\beta(\varepsilon_i-\mu)}\right]$$

Bosons: $n_i = 0, 1, 2, 3, \dots$, so

$$\mathcal{Z} = \prod_i \sum_{n_i=0}^{\infty} \left[e^{-\beta(\varepsilon_i-\mu)}\right]^{n_i} = \prod_i \frac{1}{1-e^{-\beta(\varepsilon_i-\mu)}},$$

sum of geometric progression!

or, to write this compactly,

$$\boxed{\ln \mathcal{Z} = \pm \sum_i \ln\left[1 \pm e^{-\beta(\varepsilon_i-\mu)}\right]} \qquad \begin{matrix} \oplus \text{ fermions} \\ \ominus \text{ bosons} \end{matrix} \qquad (9)$$

16.3 Occupation # Statistics

The probability of a given set $\{n_i\}$ of occupation #'s is given by the grand canonical distribution:

$$p_\alpha \equiv p(n_1, n_2, \dots) = \frac{1}{\mathcal{Z}} e^{-\beta \sum_i n_i(\varepsilon_i-\mu)} \qquad (10)$$

Therefore, the mean occupation # of single-particle state j is

$$\bar{n}_j = \sum_{\{n_i\}} n_j\, p(n_1, n_2, \dots) = \frac{1}{\mathcal{Z}} \sum_{\{n_i\}} n_j e^{-\beta \sum_i n_i(\varepsilon_i-\mu)} =$$

$$= -\frac{1}{\beta}\frac{\partial \ln \mathcal{Z}}{\partial \varepsilon_j} \underset{\substack{\uparrow \\ \text{use}(9)}}{=} \mp \frac{1}{\cancel{\beta}} \frac{\pm e^{-\beta(\varepsilon_j-\mu)}\cdot(\cancel{-\beta})}{1\pm e^{-\beta(\varepsilon_j-\mu)}} = \frac{1}{e^{\beta(\varepsilon_j-\mu)}\pm 1}$$

Thus,

$$\boxed{\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i-\mu)}\pm 1}} \qquad (11)$$

(+) Fermi-Dirac statistics

(−) Bose-Einstein statistics

So we can predict how many particles will be in any given state (i), on average (= same as calculating the distribution function),

provided we know

- the corresponding single particle energy levels ε_i
- the chemical potential $\mu(T,V,\bar{N})$, the equ. for which is

$$\bar{N} = \sum_i \bar{n}_i \left[\underset{\uparrow}{=} \frac{1}{\beta}\frac{\partial \ln \mathcal{Z}}{\partial \mu} \right] \qquad (12)$$

eq.(7) p.149

Ex. Show this is the same for (11) & (9)

We can also calculate

- grand potential $\Phi = -k_B T \ln \mathcal{Z}$ (13) — eq.(9)
- equation of state $P = -\frac{\Phi}{V}$ (14)
- energy $U = \sum_i \varepsilon_i \bar{n}_i \left[= -\left(\frac{\partial \ln \mathcal{Z}}{\partial \beta}\right) + \mu \bar{N} \right]$ (15) — eq.(6) p.149
- entropy $S = \frac{U - \Phi - \mu\bar{N}}{T} \left[= -\left(\frac{\partial \Phi}{\partial T}\right)_{V,\mu} \right]$ (16) — eq.(15) p.151

↳ heat capacities ...

16.4 Preview of various interesting results to come

1) Classical limit: we must be able to recover this at high enough temperature if our earlier assumptions were correct, viz. [see p.107]

$$\underbrace{n\lambda_{th}^3}_{\approx e^{\beta\mu}} \ll 1 \qquad (17)$$

[see eq.(20) p.154]

Then $$\bar{n}_i = \frac{1}{\underset{\text{Large}}{\boxed{e^{-\beta\mu}}}\, e^{\beta\varepsilon_i} \pm 1} \approx e^{\beta\mu} e^{-\beta\varepsilon_i} \qquad (18)$$

Maxwell's distribution
$\varepsilon_i = mv^2/2$

[precisely as anticipated on p.110, eq.(29)]

The interesting (new) physics happens when (17) fails, i.e. $e^{\beta\mu}$ is not small:

2) Photon gas: photons are bosons, $\varepsilon_i = pc$ ($m=0$), and their # is not fixed at all, so

$$\mu = 0 \quad (\text{no constraint on } \bar{N})$$

So we'll get

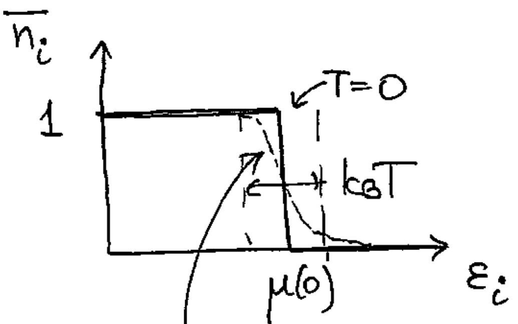
$$\bar{n}(p) = \frac{1}{e^{\beta pc} - 1} \qquad (19)$$

Planck distribution

[see A. Boothroyd's lecture to follow mine; all the results for bosons will apply to photons provided we set $\mu = 0$ and work in the ultrarelativistic limit]

3) Fermi gas at low temperatures ("degenerate")

$T\to 0$, or $\beta\to\infty$: $\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} \longrightarrow \begin{cases} 1 \text{ if } \varepsilon_i<\mu(\overset{T=0}{0}) \\ 0 \text{ if } \varepsilon_i>\mu(0) \end{cases}$



... so the fermions "stack up" to energy $\varepsilon_F=\mu(0)$ (20)

"Fermi energy"

at low but finite T (i.e. $k_BT\ll\varepsilon_F$) the "step" gets smeared by $\sim k_BT$

The distribution is very simple, so we'll be able to calculate things. This applies to electrons in metals (even at room temperature!) and also to the structure of white dwarves and neutron stars ...

4) Bosons at low temperature

$$\bar{n}_i = \frac{1}{\underbrace{e^{\beta(\varepsilon_i-\mu)}}_{\vee}-1} \qquad (21)$$

1 in order for $\bar{n}_i>0 \Rightarrow \mu<\overset{=0\,(\text{say})}{\varepsilon_0}$ - lowest level

As $\beta\to\infty$ $(T\to 0)$, eq. (21) suggests that particles will condense in the lowest-energy state. [Bose condensation]: $\bar{n}_0 = \frac{1}{e^{-\beta\mu}-1}\to N$ as $\beta\to\infty$

But we can't say more at this stage as we first have to learn how to calculate $\mu(T, n)$.

OK, enough talk, let's learn how to do some real calculations!

(Lecture 14 ended here)

16.5 Real Calculations (Continuous Limit)

We will follow the programme outlined on p.171.
The first order of business is to compute [eq.(12)]

$$\bar{N} = \sum_i \bar{n}_i = \sum_i \frac{1}{e^{\beta(\varepsilon_i-\mu)}\pm 1} \qquad (22)$$

$\hookrightarrow \mu = \mu(T,n)$, $n = \frac{\bar{N}}{V}$

from now on, we will drop overbars, $\bar{N} \to N$; our results are perfectly valid for a fixed N [see p.149]

The single-particle states are $i = (\vec{p}, S_z)$

momentum

$\vec{p} = \hbar\vec{k}$ in a box of volume $V = L_x \times L_y \times L_z$

$\vec{k}$ is discrete with "mesh size" $\frac{2\pi}{L_x} \times \frac{2\pi}{L_y} \times \frac{2\pi}{L_z}$ (see p.100)

spin projection
$S_z = -S, \dots, S$
$2S+1$ values,
S = integer or half integer
($2S+1 = 2$ for $S = \frac{1}{2}$)

for non-relativistic gas ($k_BT \ll mc^2$),

$$\boxed{\varepsilon_i = \varepsilon(k) = \frac{\hbar^2 k^2}{2m}} \qquad (23) \qquad \Longleftarrow$$

independent of spin or direction of $\vec{k}$

[more generally, $\varepsilon(k) = \sqrt{m^2c^4 + \hbar^2k^2c^2}$

$\varepsilon(k) \approx \hbar kc$ for the ultrarelativistic case ($\hbar kc \gg mc^2$) (e.g. photons)]

Since $\bar{n}_i$ only depends on k, we can convert the sum over single-particle states i in (22) as follows:

$$\sum_i = (2S+1)\sum_{\vec{k}} \overset{}{\underset{\uparrow}{=}} (2S+1)\frac{V}{(2\pi)^3}\int d^3\vec{k} \underset{\uparrow}{=} (2S+1)\frac{V}{(2\pi)^3}\, 4\pi\int_0^\infty dk\, k^2$$

same as on p.101 convert sum to integral

polar coordinates in $\vec{k}$ space

$$= \frac{(2S+1)V}{2\pi^2}\int_0^\infty dk\,k^2 \equiv \int_0^\infty dk\,g(k), \qquad (24)$$

where the density of states is

$$g(k) = \frac{(2S+1)V}{2\pi^2}\,k^2$$

– same as defined on p.102 [eq.(10)] except for the spin factor [before, we tacitly assumed spinless particles – if they have spin, this is analogous to having $Z_1^{(\text{internal})} = 2S+1$].

In fact: $\bar{n}_i = \bar{n}(\varepsilon)$, so it is convenient to convert from k to ε integration: using eq.(23),

$$k = \frac{\sqrt{2m\varepsilon}}{\hbar} \quad \text{and} \quad dk = \frac{1}{\hbar}\sqrt{\frac{m}{2\varepsilon}}\,d\varepsilon \qquad (25)$$

So $$g(k)\,dk = \frac{(2S+1)V}{\cancel{2}\pi^2}\,\frac{\cancel{2}m\varepsilon}{\hbar^2}\,\frac{1}{\hbar}\sqrt{\frac{m}{2\varepsilon}}\,d\varepsilon = g(\varepsilon)\,d\varepsilon$$

where $$\boxed{g(\varepsilon) = \frac{(2S+1)V m^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}\sqrt{\varepsilon}} = \frac{2(2S+1)}{\sqrt{\pi}}\,\frac{V}{\lambda_{th}^3}\,\frac{\sqrt{\varepsilon}}{(k_BT)^{3/2}} \qquad (26)$$

$$\lambda_{th} = \hbar\sqrt{\frac{2\pi}{mk_BT}}$$

Thus, from (24),

$$\sum_i = \int_0^\infty d\varepsilon\,g(\varepsilon) \qquad (27)$$

and eq. (22) becomes

$$N = \int_0^\infty \frac{d\varepsilon\,g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}\pm 1} = \frac{2(2S+1)}{\sqrt{\pi}}\,\frac{V}{\lambda_{th}^3}\int_0^\infty \frac{d\varepsilon\,\sqrt{\varepsilon}\,\beta^{3/2}}{e^{\beta(\varepsilon-\mu)}\pm 1}$$

$$= \frac{2(2S+1)}{\sqrt{\pi}}\,\frac{V}{\lambda_{th}^3}\int_0^\infty \frac{dx\,\sqrt{x}}{e^{x-\beta\mu}\pm 1} \qquad (28)$$

$(x = \beta\varepsilon)$

This is an implicit equation for $\mu(n,T)$:

$$\frac{n}{n_Q} = \boxed{\frac{n\lambda_{th}^3}{2S+1} = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx\sqrt{x}}{e^{x-\beta\mu}\pm 1}} \qquad (29)$$

↑ quantum concentration

Energy: similar calculation: from eq. (15),

$$U = \sum_i \varepsilon_i \bar{n}_i \overset{\text{eq.(27)}}{=} \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)\varepsilon}{e^{\beta(\varepsilon-\mu)}\pm 1} \overset{\text{eq.(26)}}{=}$$

$$= \frac{2(2S+1)}{\sqrt{\pi}}\frac{V}{\lambda_{th}^3}k_BT\int_0^\infty \frac{d\varepsilon\,\varepsilon^{3/2}\beta^{5/2}}{e^{\beta(\varepsilon-\mu)}\pm 1}$$

$$= Nk_BT\frac{n_Q}{n}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx\,x^{3/2}}{e^{x-\beta\mu}\pm 1} \qquad (30)$$

Grand potential: from eq. (13) and (9),

$$\Phi = -k_BT\ln\mathcal{Z} = \mp k_BT\sum_i \ln\left[1\pm e^{-\beta(\varepsilon_i-\mu)}\right]$$

$$\overset{\text{eq.(27)}}{=} \mp k_BT\int_0^\infty d\varepsilon\, g(\varepsilon)\ln\left[1\pm e^{-\beta(\varepsilon-\mu)}\right] =$$

$$\overset{\text{eq.(26)}}{=} \mp\frac{2(2S+1)}{\sqrt{\pi}}\frac{V}{\lambda_{th}^3}k_BT\int_0^\infty dx\underbrace{\sqrt{x}}_{\boxed{\frac{2}{3}\frac{d}{dx}x^{3/2}\text{ and integrate by parts}}}\ln\left[1\pm e^{-x+\beta\mu}\right]$$

$$= \mp Nk_BT\frac{n_Q}{n}\frac{2}{\sqrt{\pi}}\frac{2}{3}(-1)\int_0^\infty dx\,x^{3/2}\frac{\mp e^{-x+\beta\mu}}{1\pm e^{-x+\beta\mu}} =$$

$$= -Nk_BT\frac{n_Q}{n}\frac{2}{\sqrt{\pi}}\frac{2}{3}\int_0^\infty \frac{dx\,x^{3/2}}{e^{x-\beta\mu}\pm 1} = -\frac{2}{3}U \quad \text{-cf. (30)!}$$

↑ NB: This # depends on the power of ε in $g(\varepsilon)$

Thus, $\boxed{\Phi = -\frac{2}{3} U}$ (31)

Since $\Phi = -PV$ [eq. (32) p.158],

(32) $\boxed{P = \cancel{3}\,\frac{2}{3}\frac{U}{V}}$ completely generally for non-relativistic quantum gas (not just in the classical limit!)

pressure

energy density

and using (30), we get the equation of state:

$$\boxed{P = \frac{2}{3}(nk_BT)\frac{n_Q}{n}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx\,x^{3/2}}{e^{x-\beta\mu}\pm 1}}, \quad (33)$$

where $\mu^{(n,T)}$ is given by eq. (29).

Finally, entropy is [eq. (16)]

$$S = \frac{U-\Phi-\mu N}{T} \underset{\uparrow \atop \text{eq.}(31)}{=} \frac{\frac{5}{3}U-\mu N}{T}, \quad (34)$$

whence it follows that for adiabatic process (S = const, N = const), $\boxed{PV^{5/3} = \text{const}}$ (35)

again completely generally for non-relativistic gas.

Proof. From eq. (34),

$$\frac{S}{N} = \underbrace{\frac{5}{3}\frac{U}{N}}\frac{1}{T} - \frac{\mu}{T} = \begin{array}{c}\text{function of}\\ \mu/T \text{ only}\end{array} = \text{const} \Rightarrow \frac{\mu}{T} = \text{const}$$

from (30), this is a function of $\frac{\mu}{T}$ only because n/n_Q is a function of μ/T only [from (29)]

But $\left(\frac{\mu}{T} = \text{const} \Rightarrow \frac{n}{n_Q} = \text{const} \right.$ by eq. (29) $\left. \right)$

$\Rightarrow n\lambda_{th}^3 = \text{const}$

$\Rightarrow VT^{3/2} = \text{const} \quad \left(n = \frac{N}{V},\ \lambda_{th} = \hbar\sqrt{\frac{2\pi}{mk_BT}} \right)$

$\Downarrow$

$\frac{P}{nk_BT} = \text{const}$

by eq. (33)

$\Rightarrow PVT^{-1} = \text{const}$

$\boxed{PV^{5/3} = \text{const}}$ q.e.d.

NB: While the exponent $\frac{5}{3}$ turns out to be general (not just classical), it is not in general equal to $\frac{C_p}{C_V}$ (C_p and C_V have to be calculated, as usual, from eq. (30) and the eq. of state (33))

Now we are ready to look at what happens in various limits (high T, low T, fermions, bosons...)

16.6 Classical Limit

First of all, we make sure we can recover classical ideal gas results at high temperatures. Eq. (29):

$$\underbrace{\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{dx\sqrt{x}}{e^{x-\beta\mu}\pm 1}}_{\substack{|||\\ f(\beta\mu)}} = \frac{n\lambda_{th}^3}{2S+1} = \frac{n\hbar^3}{2S+1}\left(\frac{2\pi}{mk_BT}\right)^{3/2} \to 0 \quad (36)$$

if $n \to 0$ and/or $T \to \infty$ (hot dilute gas)

$f(\beta\mu) \to 0$ if $e^{-\beta\mu} \to \infty$: indeed, in this case

$$f(\beta\mu) \approx \frac{2}{\sqrt{\pi}}\, e^{\beta\mu} \underbrace{\int_0^\infty dx\,\sqrt{x}\,e^{-x}}_{\overset{\shortparallel}{\frac{\sqrt{\pi}}{2}}\ (\text{ex.})} = e^{\beta\mu} \ll 1 \text{ indeed.}$$

So, from (36),

$$e^{\beta\mu} \approx \frac{n\lambda_{th}^3}{2S+1} \Rightarrow \boxed{\mu \approx k_BT \ln\left[\frac{n\lambda_{th}^3}{2S+1}\right]} \qquad (37)$$

precisely eq. (20) p.154 with $Z_1^{(\text{internal})} = 2S+1$ (spins)

Thus, we have recovered the classical formula for the chemical potential of the ideal gas.

Ex. By a similar approximation, show from eq. (33) that $P \approx nk_BT$ when $n\lambda_{th}^3 \ll 1$ (classical eq. of state)

Partition function: from eq. (9), as $e^{\beta\mu} \ll 1$,

$$\ln \mathcal{Z} \approx \sum_i e^{\beta\mu} e^{-\beta\varepsilon_i} = e^{\beta\mu} \underset{\substack{\uparrow \\ (2S+1)\frac{V}{\lambda_{th}^3}}}{Z_1} \Rightarrow \mathcal{Z} \approx e^{Z_1 e^{\beta\mu}}$$

eq. (17) p.154

Furthermore, if N is fixed, from eq. (8) [p.149], we get

$$Z = \frac{\mathcal{Z}}{(e^{\beta\mu})^N} \approx \Bigg(\underbrace{\frac{2S+1}{n\lambda_{th}^3}}_{\overset{\shortparallel}{Z_1/N}}\Bigg)^N e^{\underbrace{Z_1\left(\frac{n\lambda_{th}^3}{2S+1}\right)}_{\overset{\shortparallel}{N}}} = \frac{Z_1^N}{N^N e^{-N}} \qquad (38)$$

$$\approx \frac{Z_1^N}{N!}$$

as we surmised for classical ideal gas without taking quantum correlations into account.

Maxwell's Distribution : as anticipated on p.172,

$$\overline{n_i} = \frac{1}{e^{\beta(\varepsilon_i-\mu)}\pm 1} \underset{\substack{\uparrow\\ e^{-\beta\mu}\gg 1}}{\approx} e^{\beta\mu}\, e^{-\beta\varepsilon_i} \underset{\substack{\uparrow\\ \text{eq.}(37)}}{=} \frac{n\lambda_{th}^3}{2s+1}\, e^{-\beta\varepsilon_i} \qquad (39)$$

Now $\overline{n_i}$ is the # of particles per microstate i, so the # of particles per unit volume per wave # is

$$\frac{1}{V}\underbrace{g(k)\,dk}_{(2s+1)\frac{V}{(2\pi)^3}\underbrace{d^3\vec{k}}_{d^3\left(\frac{m\vec{v}}{\hbar}\right)}} \cdot \frac{n\lambda_{th}^3}{2s+1}\, e^{-\beta\underbrace{\varepsilon(k)}_{\frac{\hbar^2k^2}{2m}=\frac{mv^2}{2}}} =$$

$$= \frac{1}{\cancel{V}}\cancel{(2s+1)}\frac{\cancel{V}}{(2\pi)^3}\,\frac{m^3}{\hbar^3}\,\frac{n\hbar^3}{\cancel{2s+1}}\left(\frac{2\pi}{mk_BT}\right)^{3/2} e^{-\frac{mv^2}{2k_BT}}\, d^3\vec{v}$$

$$= \frac{n}{(2\pi k_BT/m)^{3/2}}\, e^{-\frac{mv^2}{2k_BT}}\, d^3\vec{v} = \underbrace{\frac{n e^{-v^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}}_{F(\vec{v})\ \text{Maxwellian, q.e.d.}}\, d^3\vec{v} \qquad (40)$$

[precisely as anticipated on p.110!]

Not surprising, but it's nice how neatly it all works out, isn't it?

16.7 Degeneration

So, for $n\lambda_{th}^3 \ll 1$ (hot, dilute gas), we are in classical limit. Obviously, we did not go to all this trouble of calculating quantum statistics just to recover this! The new and exciting things will happen when this breaks down, viz., $n\lambda_{th}^3 \sim 1$.

When does this happen and what does it mean?

↳ let's start from the classical limit, use $p = nk_BT$, and see when it breaks down:

$$n\lambda_{th}^3 = \frac{p}{k_BT}\hbar^3\left(\frac{2\pi}{mk_BT}\right)^{3/2} \approx$$

$$\approx \left(\frac{p}{1\text{atm}}\right)\left(\frac{T}{300K}\right)^{-5/2}\left(\frac{m}{m_p}\right)^{-3/2}\cdot 2.5\cdot 10^{-5} \qquad (41)$$

Air at S.T.P: $n\lambda_{th}^3 \sim 10^{-6} \ll 1$ safely classical

^{4}He at 4K and 1atm: $n\lambda_{th}^3 \sim 0.15$ getting dangerous...

Electrons in metals (using $n\sim 10^{28}\,m^{-3}$, not $p = nk_BT$)

$n\lambda_{th}^3 \sim 10^4 \gg 1$ even at $T = 300K$

Thus, they are completely degenerate even in everyday conditions! It does indeed turn out that you can't correctly calculate heat capacity of metals based on classical models.

So this is one clear application of Fermi statistics

Lecture 15 ended here.

in the quantum ("degenerate") limit.

Note that this teaches that "low" and "high" temperature limits do not necessarily happen at temperatures naïvely appearing low or high from our everyday perspective. Thus, for electrons in metals, temperature stops being "low" (i.e. classical limit is approached) when

$$n\lambda_{th}^3 \sim 1 \quad \text{or} \quad T \sim \frac{2\pi n^{2/3}\hbar^2}{m_e k_B} \sim 10^4 K \qquad (42)$$

↑ high because density is high (n) and particles are very light (m_e)

[Chandrasekhar's theory of stability of stars - see PS-7] →

Another famous application of degenerate Fermi gas is to white dwarves and neutron stars, where densities are so high that even relativistic temperatures ($T > mc^2/k_B$) can be "low" from the point of view of quantum effects being dominant!

What is the physical meaning of degeneration?

1) Well, as I explained before, $n\lambda_{th}^3 \sim 1$ means that

$$\text{\# of quantum states per particle} \sim \frac{V}{\lambda_{th}^3} \sim N \ \text{\# of particles}$$

So it is not the case that particles are unlikely to compete for the same microstate ($\bar{n}_i$'s are no longer small!)

2) Another intuitive argument is as follows (cf. p. 22)

In a classical Maxwellian gas, mean energy per particle is

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{3}{2} k_B T$$

and so $\langle p^2 \rangle = 3 k_B T m$. (43)

In order for this to be true, quantum uncertainty in the determination of momentum must be

$$\delta p \ll \langle p^2 \rangle^{1/2} \sim \sqrt{m k_B T} \qquad (44)$$

(otherwise the spread in the distribution of momenta would be larger and so $\langle p^2 \rangle$ would be larger!)

Uncertainty principle : $\delta r \, \delta p \sim \hbar$, so

$$\delta r \gg \frac{\hbar}{\sqrt{m k_B T}} \sim \lambda_{th} \qquad (45)$$

But if the gas is to be modelled classically,

$$\delta r \ll \left(\frac{V}{N}\right)^{1/3} = n^{-1/3} \qquad (46)$$

mean distance between particles

$$\boxed{n \lambda_{th}^3 \ll 1}$$ classical limit

When this breaks down, the particles become completely blurred — they are everywhere, the gas is degenerate...