Statistical Physics

Oxford Physics

Second year physics course

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Problem Set 7: Statistical Mechanics

Some useful constants

Boltzmann's constant	k_B	$1.3807 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Stefan–Boltzmann constant	σ	$5.67 \times 10^{-8}\ \mathrm{Wm^{-2}K^{-4}}$
Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Standard molar volume		$22.414 \times 10^{-3}\,\mathrm{m^3\,mol^{-1}}$
Molar gas constant	R	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
1 pascal (Pa)		$1\,\mathrm{N\,m^{-2}}$
1 standard atmosphere		$1.0132 \times 10^5\,\mathrm{Pa\ (N\,m^{-2})}$
1 bar (= 1000 mbar)		$10^5\,\mathrm{N\,m^{-2}}$

PROBLEM SET 7: Quantum Gases

This Problem Set can be attempted during Weeks 6 and 7 of Hilary Term, with the tutorial or class on this material held in Week 7 or later. Some of the questions (1, 2, 5-7, 8, 9) require quite long calculations, so start working on this Problem Set early! Note that gaining good command of these techniques is crucial both for your understanding of this course and for several 3rd-year papers.

Basic Calculations for Quantum Gases

7.1 *Ultrarelativistic Quantum Gas.* Consider an ideal quantum gas (Bose or Fermi) in the ultrarelativistic limit.

(a) Find the equation that determines its chemical potential (implicitly) as a function of density n and temperature T.

(b) Calculate the energy U and grand potential Φ and hence prove that the equation of state can be written as

$$PV = \frac{1}{3}U,$$

regardless of whether the gas is in the classical limit, degenerate limit or in between.

(c) Consider an adiabatic process with the number of particles held fixed and show that

$$PV^{4/3} = \text{const}$$

for any temperature and density (not just in the classical limit).

(d) Show that in the hot, dilute limit (large T, small n), $e^{\mu/k_BT} \ll 1$. Find the specific condition on n and T that must hold in order for the classical limit to be applicable. Hence derive the condition for the gas to cease to be classical and become degenerate. Estimate the minimum density for which an electron gas can be simultaneously degenerate and ultrarelativistic.

(e) Find the Fermi energy ε_F of an ultrarelativistic electron gas and show that when $k_BT \ll \varepsilon_F$, its energy density is $3n\varepsilon_F/4$ and its heat capacity is

$$C_V = Nk_B\pi^2\frac{k_BT}{\varepsilon_F}.$$

Sketch the heat capacity of an ultrarelativistic electron gas as a function of temperature, from $T \ll \varepsilon_F/k_B$ to $T \gg \varepsilon_F/k_B$.

7.2 *Degenerate Bose Gas in 2D.*

(a) Show that Bose condensation does not occur in 2D.

(b) Calculate the chemical potential as a function of n and T in the limit of small T. Sketch $\mu(T)$ from small to large T.

(c) Show that the heat capacity (at constant area) is $C \propto T$ at low temperatures and sketch $C(T)$ from small to large T.

Non-relativistic Fermi Gases at Zero Temperature

7.3 Calculate the Fermi energies ε_F (in units of eV) and the corresponding Fermi temperatures T_F for:

(a) Liquid ^{3}He (density $0.0823\,\mathrm{g\,cm^{-3}}$).

(b) Electrons in aluminium (valence 3, density $2.7\,\mathrm{g\,cm^{-3}}$).

(c) Neutrons in the nucleus of ^{16}O. The radius r of a nucleus scales roughly as $r \approx 1.2A^{1/3} \times 10^{-15}\,\mathrm{m}$, where A is the atomic mass number.

7.4 (i) Consider a system where N electrons are constrained to move in two dimensions in a region of area A. What is the density of states for such a system? Obtain an expression for the Fermi energy of such a system as a function of the surface density $n = N/A$.

(ii) The electrons in a GaAs/AlGaAs heterostructure (this just means they act as though they are in 2D!) have a density of $4 \times 10^{11}\,\mathrm{cm^{-2}}$. The electrons act as free particles, but their interaction with the lattice means that they "appear" to have a mass of only 15% of their normal mass (don't worry about this — this property is something you will come across in the Solid State course). What is the Fermi energy of the electrons?

(iii) Derive an expression for the density of states and Fermi energy for electrons confined in 1D.

(iv) There are certain long-chain molecules which contain mobile electrons. The electrons can move freely along the chain, and the system is a 1D organic conductor, with n electrons per unit length. A typical molecule of this type has a spacing of 2.5Å between the atoms that donate electrons, and "on average" each atom donates 0.5 electrons. What is the Fermi energy of this system?

Stability of Stars

7.5 A solar-mass star ($M_\odot = 2 \times 10^{30}\,\mathrm{kg}$) will eventually run out of nuclear fuel (all the fusion processes stop). At this point it will collapse into a white dwarf, and comprise a degenerate electron gas (i.e. one that obeys quantum statistics) with the nuclei neutralizing the charge and providing the gravitational attraction. The radius of the star can be found by balancing the gravitational energy with the energy of the electrons. The electrons can, to a good approximation, be treated as though $T = 0$ (because, as we shall find, the Fermi energy is very large). The real calculation of this problem is quite sophisticated, and we are only going to use a very crude method to illustrate the basic physics.

(i) Assume a star of mass M and radius R is of uniform density (clearly this will not be the case in reality — and the next best method uses a pressure balance equation — but we are going to do the easiest calculation that gets answers in the right ball park). Show that the gravitational potential energy of the star is

$$U_{\mathrm{grav}} = -\frac{3GM^2}{5R}, \tag{1}$$

where G is the gravitational constant. [Hint: "build" the star up out of spherical shells].

(ii) Assume that the star contains equal numbers of electrons, protons, and neutrons. If the electrons can be treated as being non-relativistic, show that the total energy of the degenerate electrons is given by

$$U_{\text{electrons}} = 0.0088 \frac{h^2 M^{5/3}}{m_{\text{e}} m_{\text{p}}^{5/3} R^2}, \tag{2}$$

where the mass of a neutron or proton is m_{p}.

(iii) The white dwarf will have a radius, R, that minimizes the total energy. Sketch U_{total} as a function of R, and derive an expression for the equilibrium radius $R(M)$.

(iv) Show that the equilibrium radius for a solar-mass white dwarf is of order the radius of the earth.

(v) Evaluate the Fermi energy. Do you think we were correct in treating the electrons as being non-relativistic?

7.6 (i) If the electrons in the white dwarf considered in the previous question were relativistic, show that the total energy of the electrons in the star would scale as R^{-1} rather than R^{-2} as in the non-relativistic case.

(ii) If the electrons are relativistic, the star is no longer stable and will collapse further. This will happen when the average energy of an electron is of order its rest mass. Above what mass would you expect a white dwarf to be unstable? (This limit is called the Chandrasekhar limit, and was first derived by him when he was aged 19, during his voyage from India to England — it was published in 1931).

7.7 A white dwarf with a mass above the Chandrasekhar limit collapses to such a high density that the electrons and protons react to form neutrons (+neutrinos): the star comprises neutrons only, and is called a *neutron star*.

(i) Using the same methods as in the two previous questions, find the mass-radius relationship of a neutron star, assuming the neutrons are non-relativistic.

(ii) What is the radius of a neutron star of 1 solar mass?

(iii) Again, if the average energy of the neutrons becomes relativistic, the star will be unstable. When a neutron star collapse occurs, a black hole is formed. Make an *estimate* of the critical mass of a neutron star.

Further Applications of Quantum Statistics

7.8 *Pair Production.* At relativistic temperatures, the number of particles can stop being a fixed number, with production and annihilation of electron-positron pairs providing the number of particles required for thermal equilibrium. The reaction is

$$e^+ + e^- \Leftrightarrow \text{photon(s)}.$$

(a) What is the condition of "chemical" equilibrium for this system?

(b) Assume that the numbers of electrons and positrons are the same (i.e., ignore the fact that there is ordinary matter and, therefore, a surplus of electrons). This allows you to assume that the situation is fully symmetric and the chemical potentials of electrons and positrons are the same. What are they equal to? Hence calculate the densitiy of electrons and positrons $n^{\pm}$ as a function of temperature, assuming $k_B T \gg m_e c^2$. You will need to know that

$$\int_0^\infty \frac{dx\, x^2}{e^x + 1} = \frac{3}{2}\,\zeta(3), \quad \zeta(3) \approx 1.202$$

(see, e.g., Landau & Lifshitz §58 for the derivation of this formula).

(c) To confirm an *a priori* assumption you made in (b), show that at ultrarelativistic temperatures, the density of electrons and positrons you have obtained will always be larger than the density of electrons in ordinary matter. This will require you to come up with a simple way of estimating the upper bound for the latter.

(d*) Now consider the non-relativistic case, $k_B T \ll m_e c^2$, and assume that temperature is also low enough for the classical (non-degenerate) limit to apply. Let the density of electrons in matter, without the pair production, be n_0. Show that the density of positrons due to spontaneous pair production, in equilibrium, is exponentially small:

$$n^+ \approx \frac{4}{n_0}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^3 e^{-2m_e c^2/k_B T}.$$

Hint. Use the law of mass action. Note that you can no longer assume that pairs are more numerous than ordinary electrons. Don't forget to reflect in your calculation the fact that the energy cost of producing an electron or a positron is $m_e c^2$.

7.9 *Paramagnetism of a Degenerate Electron Gas (Pauli Magnetism).* Consider a fully degenerate non-relativistic electron gas in a weak magnetic field. Since the electrons have two spin states (up and down), take the energy levels to be

$$\varepsilon(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} \pm \mu_B B,$$

where $\mu_B = e\hbar/2m_e c$ is the Bohr magneton (in cgs-Gauss units). Assume the field to be sufficiently weak so that $\mu_B B \ll \varepsilon_F$.

(a) Show that the magnetic susceptibility of this system is

$$\chi = \left(\frac{\partial M}{\partial B}\right)_{B=0} = \frac{3^{1/3}}{4\pi^{4/3}} \frac{e^2}{m_e c^2}\, n^{1/3},$$

where M is the magnetisation (total magnetic moment per unit volume) and n density (the first equality above is the definition of χ, the second is the answer you should get).

Hint. Express M in terms of the grand potential Φ. Then use the fact that energy enters the Fermi statistics in combination $\varepsilon - \mu$ with the chemical potential μ. Therefore, in order to calculate the individual contributions from the spin-up and spin-down states to the integrals over single-particle states, we can use the unmagnetised formulae with μ replaced by $\mu \pm \mu_B B$, viz., the grand potential, for example, is

$$\Phi(\mu, B) = \frac{1}{2}\,\Phi_0(\mu + \mu_B B) + \frac{1}{2}\,\Phi_0(\mu - \mu_B B),$$

where $\Phi_0(\mu) = \Phi(\mu, B = 0)$ is the grand potential in the unmagnetised case. Make sure to take full advantage of the fact that $\mu_B B \ll \varepsilon_F$.

(b) Show that in the classical (non-degenerate) limit, the above method recovers Curie's law. Sketch χ as a function of T, from very low to very high temperature.

(c*) Show that at $T \ll \varepsilon/k_B$, the finite-temperature correction to χ is quadratic in T and negative (i.e., χ goes down as T increases).

Thermal radiation

7.10 Thermal radiation can be treated thermodynamically as a gas with internal energy $U = u(T)V$, pressure $p = u/3$ and chemical potential $\mu = 0$. Starting from the fundamental equation $\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N$, show that

(i) the energy density $u \propto T^4$,

(ii) the temperature $T \propto V^{-1/3}$ in an adiabatic expansion.

The Universe is filled uniformly with radiation called the cosmic microwave background (CMB) which is left over from an early stage of development when the Universe contained a hot dense plasma of electrons and baryons. The CMB became thermally *decoupled* from the plasma when the Universe was very young, and since then has expanded adiabatically. The temperature of the CMB is currently 2.73 K, and the *cosmic scale factor* which describes the relative expansion of the Universe is approximately 1,100 times larger today than it was at the time of decoupling. Determine the temperature of the Universe at the time of decoupling.

7.11 Outline the steps leading to the formula for the number of photons with angular frequencies between ω and $\omega + \mathrm{d}\omega$ in blackbody radiation at a temperature T:

$$n(\omega)d\omega = 2 \times \frac{V}{2\pi^2 c^3} \frac{\omega^2\,\mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1}.$$

Show that $n(\omega)$ has a peak at a frequency given by $\omega = 1.59 k_\mathrm{B}T/\hbar$. Show further that the spectral energy densities u_λ and u_ω peak at $\lambda_\mathrm{max} = hc/(4.97 k_\mathrm{B}T)$ and $\omega_\mathrm{max} = 2.82 k_\mathrm{B}T/\hbar$, respectively.

7.12 (i) The gas pressure at the centre of the Sun is 4×10^{11} atmospheres, and the temperature is 2×10^7 K. Estimate the radiation pressure and show that it is very small compared with the gas pressure.

(ii) The surface temperature of the Sun is 5,700 K, and the spectrum of radiation it emits has a maximum at a wavelength of 510 nm. Estimate the surface temperature of the North Star, for which the corresponding maximum is 350 nm.

(iii) Assuming the Sun (radius 6.955×10^8 m) emits radiation as a black body, calculate the solar power incident on a thin black plate of area $1\,\mathrm{m}^2$ facing the Sun and at a distance of 1 Astronomical Unit ($= 1.496 \times 10^{11}$ m) from it. Find the temperature of the plate given that it emits radiation from both surfaces. Neglect any radiation incident on the surface of the plate facing away from the Sun.