Statistical Physics

Second year physics course

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Problem Set 4: Kinetic Theory

PROBLEM SET 4: Collisions and Transport

This Problem Set is vacation work and so is longer than usual and contains some revision problems (at the end).

Mean Free Path

4.1 Consider a gas that is a mixture of two species of molecules: type-1 with diameter d_1, mass m_1 and mean number density n_1 and type-2 with diameter d_2, mass m_2 and mean number density n_2. If we let them collide with each other for a while (for how long? answer this after you have solved the rest of the problem), they will eventually settle into a Maxwellian equilibrium and the temperatures of the two species will be the same.

a) What will be the rms speeds of the two species?

b) Show that the combined pressure of the mixture will be $p = p_1 + p_2$ (Dalton's law).

c) What is the cross-section for the collisions between type-1 and type-2 molecules?

d) What is the mean collision rate of type-1 molecules with type-2 molecules? (here you will need to find the mean relative speed of the two types of particles, a calculation analogous to one in the lecture notes)

Hint. You will find the answers in Pauli's book, but do try to figure them out on your own.

4.2 Consider particles in a gas of mean number density n and collisional cross-section σ, moving with speed v (let us pretend they all have exactly the same speed).

a) What is the probability $P(t)$ for a particle to experience *no* collisions up to time t? Therefore, what is the mean time between collisions?

Hint. Work out the probability for a particle not to have a collision between t and $t+dt$. Hence work out $P(t+dt)$ in terms of $P(t)$ and the relevant parameters of the gas. You should end up with a differential equation for $P(t)$, which you can then solve. [You will find this derivation in Blundell & Blundell, but do try to figure it out yourself!]

b) What is the the probability $P(x)$ for a particle to travel a distance x between two subsequent collisions? Show that the root mean square free path is given by $\sqrt{2}\lambda_{\rm mfp}$ where $\lambda_{\rm mfp}$ is the mean free path.

c) What is the most probable free path length?

d) What percentage of molecules travel a distance greater than (i) $\lambda_{\rm mfp}$, (ii) $2\lambda_{\rm mfp}$, (iii) $5\lambda_{\rm mfp}$?

4.3 Given that the mean free path in a gas at standard temperature and pressure (S.T.P.) is about 10^3 atomic radii, *estimate* the highest allowable pressure in the chamber of an atomic beam apparatus 10^{-1} m long (if one does not want to lose an appreciable fraction of atoms through collisions).

4.4 A beam of silver atoms passing through air at a temperature of 0°C and a pressure of 1 $\mathrm{Nm^{-2}}$ is attenuated by a factor 2.72 in a distance of 10^{-2} m. Find the mean free path of the silver atoms and estimate the effective collision radius.

4.5 (*) Recall the example (discussed in the lecture notes) of billiard balls sensitive to the gravitational pull of a passerby. Consider now a room filled with air. Work out how long it will take for the trajectories of the molecules to be completely altered by the gravitational interaction with a stray electron appearing out of nowhere at the edge of the Universe (ignore all non-A-level physics involved).

Conductivity, Viscosity, Diffusion

4.6 a) Obtain an expression for the thermal conductivity of a classical ideal gas. Show that it depends only on temperature and the properties of individual gas molecules.

b) The thermal conductivity of argon (atomic weight 40) at S.T.P. is 1.6×10^{-2} $\mathrm{Wm^{-1}K^{-1}}$. Use this to calculate the mean free path in argon at S.T.P. Express the mean free path in terms of an effective atomic radius for collisions and find the value of this radius. Solid argon has a close packed cubic structure, in which, if the atoms are regarded as hard spheres, 0.74 of the volume of the structure is filled. The density of solid argon is 1.6 g $\mathrm{cm^{-3}}$. Compare the effective atomic radius obtained from this information with the effective collision radius. Comment on the result.

4.7 a) Define the coefficient of viscosity. Use kinetic theory to show that the coefficient of viscosity of a gas is given, with suitable approximations, by

$$\eta = K\rho\langle v\rangle\lambda_{\mathrm{mfp}}$$

where ρ is the density of the gas, λ_{mfp} is the mean free path of the gas molecules, $\langle v\rangle$ is their mean speed, and K is a number which depends on the approximations you make.

b) In 1660 Boyle set up a pendulum inside a vessel which was attached to a pump which could remove air from the vessel. He was surprised to find that there was no observable change in the rate of damping of the swings of the pendulum when the pump was set going. Explain the observation in terms of the above formula.

Make a rough order of magnitude estimate of the lower limit to the pressure which Boyle obtained; use reasonable assumptions concerning the apparatus which Boyle might have used. [The viscosity of air at atmospheric pressure and at 293 K is 18.2 μN s $\mathrm{m^{-2}}$.]

Explain why the damping is nearly independent of pressure despite the fact that fewer molecules collide with the pendulum as the pressure is reduced.

4.8 Two plane disks, each of radius 5 cm, are mounted coaxially with their adjacent surfaces 1 mm apart. They are in a chamber containing Ar gas at S.T.P. (viscosity 2.1×10^{-5} $\mathrm{N\,s\,m^{-2}}$) and are free to rotate about their common axis. One of them rotates with an angular velocity of 10 rad $\mathrm{s^{-1}}$. Find the couple which must be applied to the other to keep it stationary.

4.9 Measurements of the viscosity η of argon gas (^{40}Ar) over a range of pressures yield the following results at two temperatures:

$$\begin{array}{ll} \text{at 500 K} & \eta \approx 3.5 \times 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1} \\ \text{at 2000 K} & \eta \approx 8.0 \times 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1} \end{array}$$

The viscosity is found to be approximately independent of pressure. Discuss the extent to which these data are consistent with (i) simple kinetic theory, and (ii) the diameter of the argon atom (0.34 nm) deduced from the density of solid argon at low temperatures.

4.10 a) Argue qualitatively or show from elementary kinetic theory that the coefficient of self-diffusion D, the thermal conductivity κ and the viscosity η of a gas are related via

$$D \sim \frac{\kappa}{c_V} \sim \frac{\eta}{\rho},$$

where c_V is the heat capacity per unit volume ($3nk_B/2$ for ideal monatomic gas) and ρ is the mass density of the gas.

b*) Starting from the kinetic equation for the distribution function $F^*(t, \mathbf{r}, \mathbf{v})$ of some labelled particle admixture in a gas, derive the self-diffusion equation

$$\frac{\partial n^*}{\partial t} = D \frac{\partial^2 n^*}{\partial z^2}$$

for the number density $n^*(t, z) = \int d^3\mathbf{v} F^*(t, z, \mathbf{v})$ of the labelled particles (which we assume to change only in the z direction). Derive also the expression for the self-diffusion coefficient D, given that

—the molecular mass of the labelled particles is m^*,

—the temperature of the unlabelled ambient gas is T (assume it is uniform),

—collision frequency of the labelled particles with the unlablelled ones is ν_c^*.

Assume that the ambient gas is static (no mean flows), that the density of the labelled particles is so low that they only collide with the unlabelled particles (and not each other) and that the frequency of these collisions is much larger than the rate of change of any mean quantities. Use the Krook collision operator, assuming that collisions relax the distribution of the labelled particles to a Maxwellian F_M^* with density n^* and the same velocity (zero) and temperature (T) as the ambient unlabelled gas.

Hint. Is the momentum of the labelled particles conserved? You should discover that self-diffusion is related to the mean velocity u_z^* of the labelled particles. You can calculate this velocity either directly from $\delta F^* = F^* - F_M^*$ or from the momentum equation for the labelled particles.

c*) Derive the momentum equation for the mean flow of the labelled particles and obtain the result you have known since school: friction force (collisional drag exerted on labelled particles by the ambient population) is proportional to the mean velocity (of the lablelled particles). What is the proportionality coefficient? This, by the way, is the "Aristotelian equation of motion" — Aristotle thought force was generally proportional to velocity. It took a while for another man to figure out the more general formula.

Show from the momentum equation you have derived that the flux of labelled particles is proportional to their pressure gradient: $n^* u_z^* = -(1/m^* \nu_c^*) \partial p^* / \partial z$.

Heat Diffusion Equation

4.11 a) A cylindrical wire of thermal conductivity κ, radius a and resistivity ρ uniformly carries a current I. The temperature of its surface is fixed at T_0 using water cooling. Show that the temperature $T(r)$ inside the wire at radius r is given by

$$T(r) = T_0 + \frac{\rho I^2}{4\pi^2 a^4 \kappa}(a^2 - r^2).$$

b) The wire is now placed in air at temperature $T_{\rm air}$ and the wire loses heat from its surface according to Newton's law of cooling (so that the heat flux from the surface of the wire is given by $\alpha(T(a) - T_{\rm air})$ where α is a constant. Find the temperature $T(r)$.

4.12 A microprocessor has an array of metal fins attached to it, whose purpose is to remove heat generated within the processor. Each fin may be represented by a long thin cylindrical copper rod with one end attached to the processor; heat received by the rod through this end is lost to the surroundings through its sides.

The internal energy density ε of the rod is related to its temperature T via $\varepsilon = \rho c_m T$, where ρ is mass density, c_m the specific (i.e., per unit mass) heat capacity of the metal (not $3k_B/2m$; you will learn what it is later in the course). Show that the temperature $T(x,t)$ at location x along the rod at time t obeys the equation

$$\rho c_m \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{2}{a} R(T),$$

where a is the radius of the rod, and $R(T)$ is the rate of heat loss per unit area of surface at temperature T.

The surroundings of the rod are at temperature T_0. Assume that $R(T)$ has the form (Newton's law of cooling)

$$R(T) = A(T - T_0).$$

In the steady state:

(a) obtain an expression for T as a function of x for the case of an infinitely long rod whose hot end has temperature $T_{\rm m}$;

(b) show that the heat that can be transported away by a long rod of radius a is proportional to $a^{3/2}$, provided that A is independent of a.

In practice the rod is not infinitely long. What length does it need to have for the results above to be approximately valid? The radius of the rod is 1.5 mm.

[The thermal conductivity of copper is $380\,\mathrm{W\,m^{-1}\,K^{-1}}$. The cooling constant $A = 250\,\mathrm{W\,m^{-2}\,K^{-1}}$.]

4.13 One face of a thick uniform layer is subject to a sinusoidal temperature variation of angular frequency ω. Show that damped sinusoidal temperature oscillations propagate into the layer and give an expression for the decay length of the oscillation amplitude.

A cellar is built underground covered by a ceiling which is 3 m thick made of limestone. The outside temperature is subject to daily fluctuations of amplitude 10°C and annual fluctuations of 20°C. Estimate the magnitude of the daily and annual temperature variations within the cellar. Assuming that January is the coldest month of the year, when will the cellar's temperature be at its lowest?

[The thermal conductivity of limestone is 1.6 $\mathrm{Wm^{-1}K^{-1}}$, and the heat capacity of limestone is $2.5 \times 10^6 \mathrm{JK^{-1}m^{-3}}$.]

Pressure, Energy, Effusion *(revision)*

4.14 Consider an insulated cylindrical vessel filled with monatomic ideal gas, closed on one side and plugged by a piston on the other side. The piston is very slowly pulled out (its velocity u is much smaller than the thermal velocity of the gas molecules). Show *using kinetic theory, not thermodynamics,* that during this process the pressure p of the gas inside the vessel and its volume V are related by $pV^{5/3}$ =const.

Hint. Consider how the energy of a gas particle changes after each collision with the piston and hence calculate the rate of change of the internal energy of the gas inside the vessel.

4.15 Consider two chambers of equal volume separated by an insulating wall and containing an ideal gas maintained at two distinct temperatures $T_1 = 225$ K and $T_2 = 400$ K. Initially the two chambers are connected by a long tube whose diameter is much larger than the mean free path in either chamber and equilibrium is established (while maintaining T_1 and T_2). Then the tube is removed, the chambers are sealed, but a small hole is opened in the insulating wall, with diameter $d \ll \lambda_{\mathrm{mfp}}$ (for either gas).

a) In what direction will the gas flow through the hole: $1 \to 2$ or $2 \to 1$?

b) If the total mass of the gas in both chambers is M, what is the mass of the gas that will be transferred through the hole from one chamber to the other before a new equilibrium is established?

Some Useful Constants

Boltzmann's constant	k_{B}	$1.3807 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Proton rest mass	m_{p}	$1.6726 \times 10^{-27}\,\mathrm{kg}$
Avogadro's number	N_{A}	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Standard molar volume		$22.414 \times 10^{-3}\,\mathrm{m^3\,mol^{-1}}$
Molar gas constant	R	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
1 pascal (Pa)		$1\,\mathrm{N\,m^{-2}}$
1 standard atmosphere		$1.0132 \times 10^5\,\mathrm{Pa\ (N\,m^{-2})}$
1 bar (= 1000 mbar)		$10^5\,\mathrm{N\,m^{-2}}$