Introduction to Cosmology

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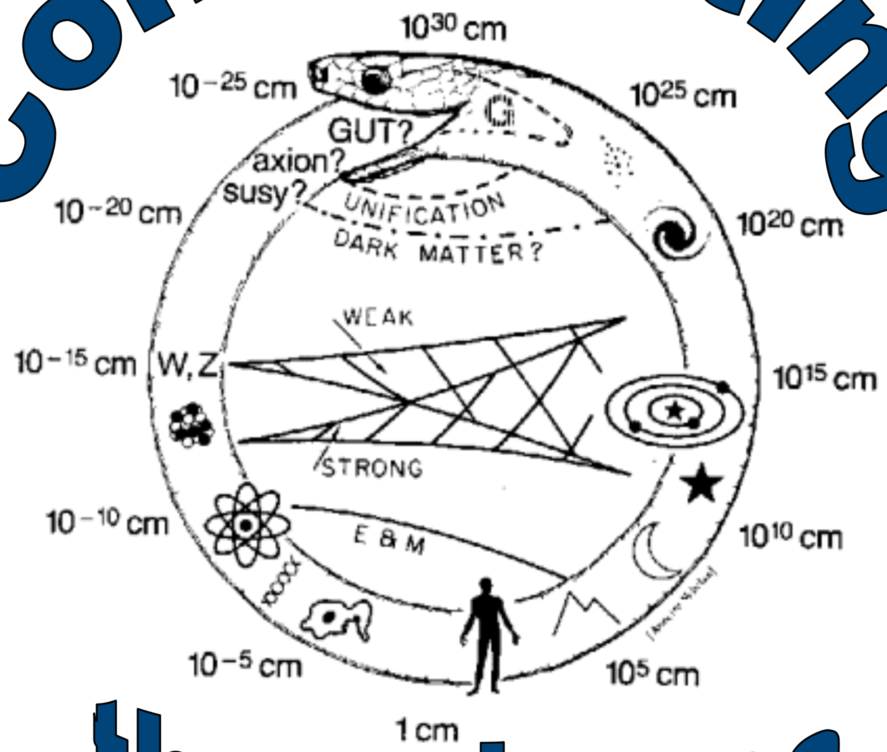
- **Seeing the edge of the Universe:** From speculation to science
- **Constructing the Universe:** Relativistic world models
- **The history of the Universe:** Decoupling of the relic radiation and nucleosynthesis of the light elements
- **The content of the Universe:** Dark matter & dark energy
- **Making sense of the Universe:** Fundamental physics & cosmology

http://www-thphys.physics.ox.ac.uk/user/SubirSarkar/cernlectures.html

Lecture 2

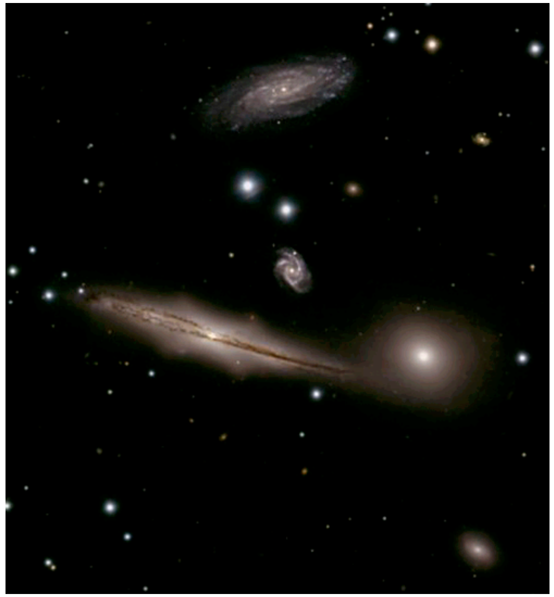
Constructing

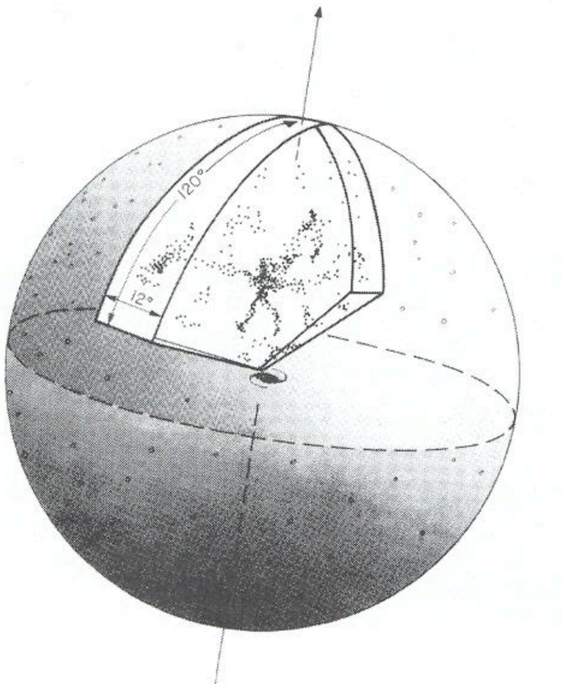


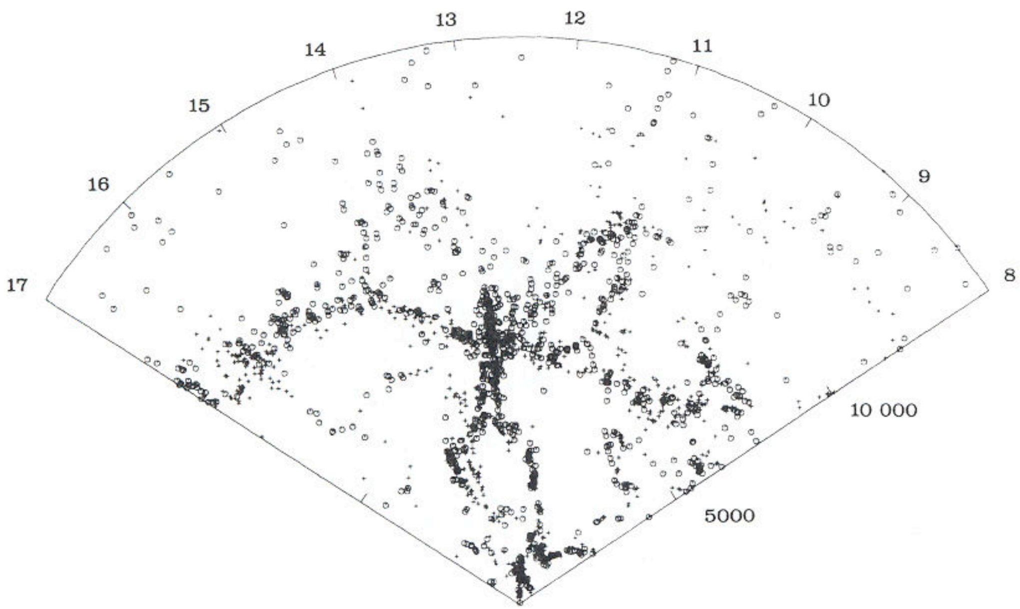
the universe

relativistic world-models

The universe appears complex and structured on many scales ....

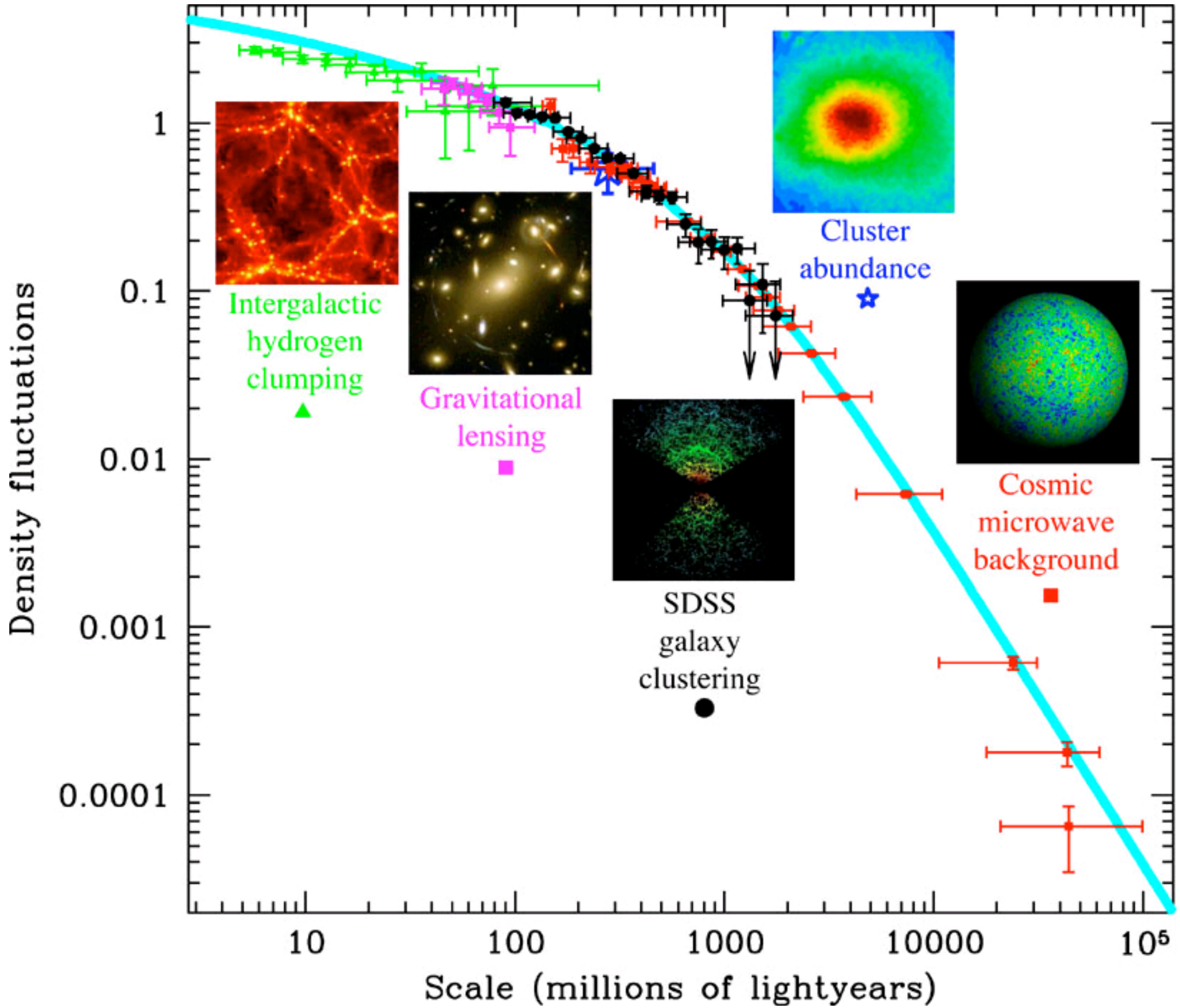






How can we possibly describe it by a simple mathematical model?

Although the universe is lumpy, it seems to become smoother and smoother when averaged over larger and larger scales …

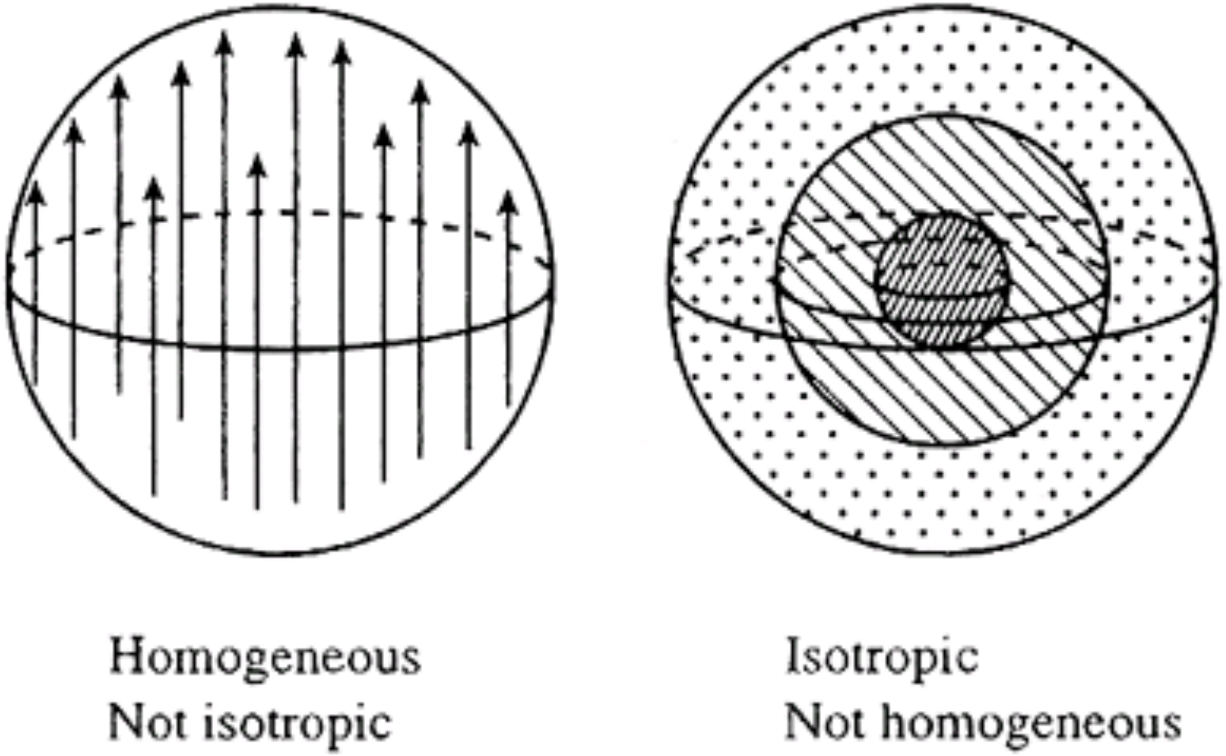


The universe certainly looks *isotropic* around us …

e.g. this is the distribution of the 31000 brightest radio sources at $\lambda \sim 6$ cm

But is the universe *homogeneous*?

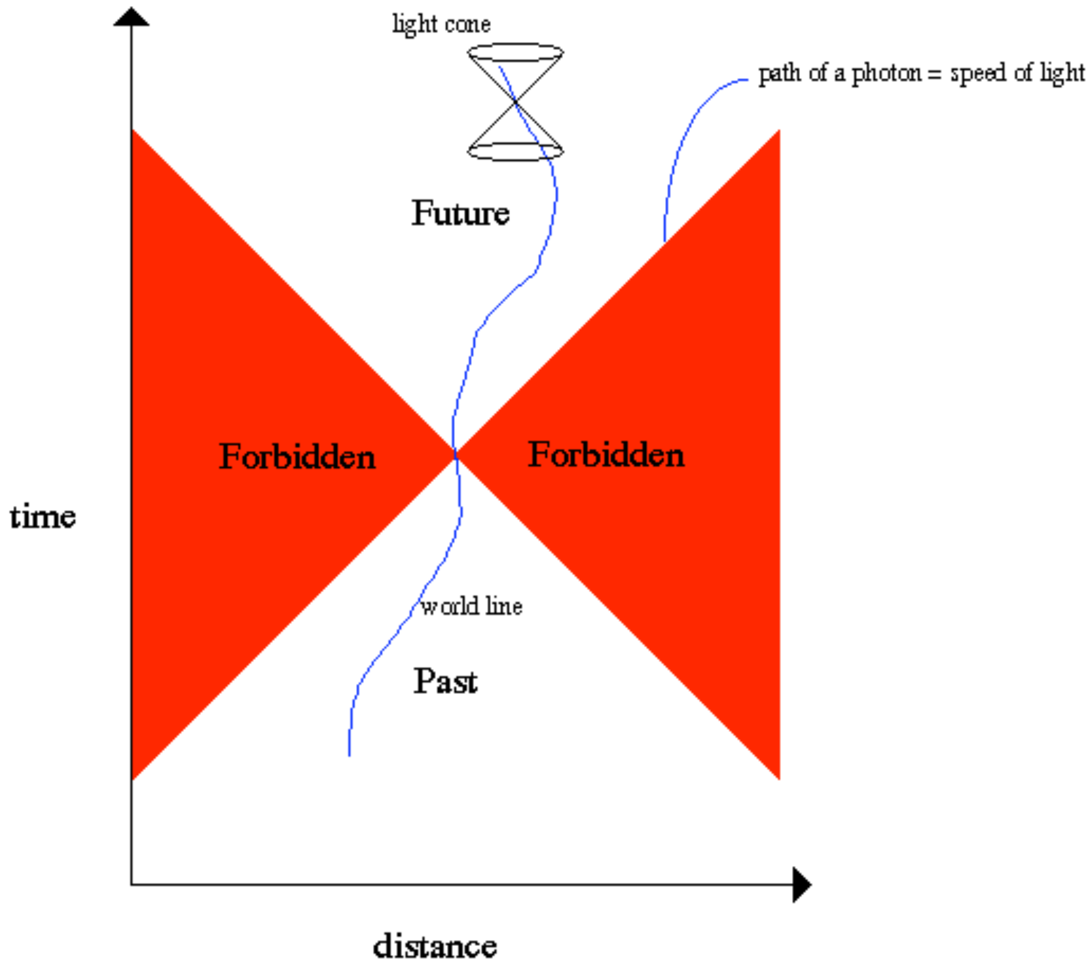
Isotropy does not necessarily imply homogeneity …



… unless it is so about *every* point in space

But we cannot move (very far) in space so must *assume* that our position is typical - "The Cosmological Principle" (**Milne 1935**)

All we can ever learn about the universe is contained within our past light cone



We *cannot* move over cosmological distances and check that the universe looks the same from 'over there' as it does from here … so there are ***fundamental limits* to what we can know about the universe**

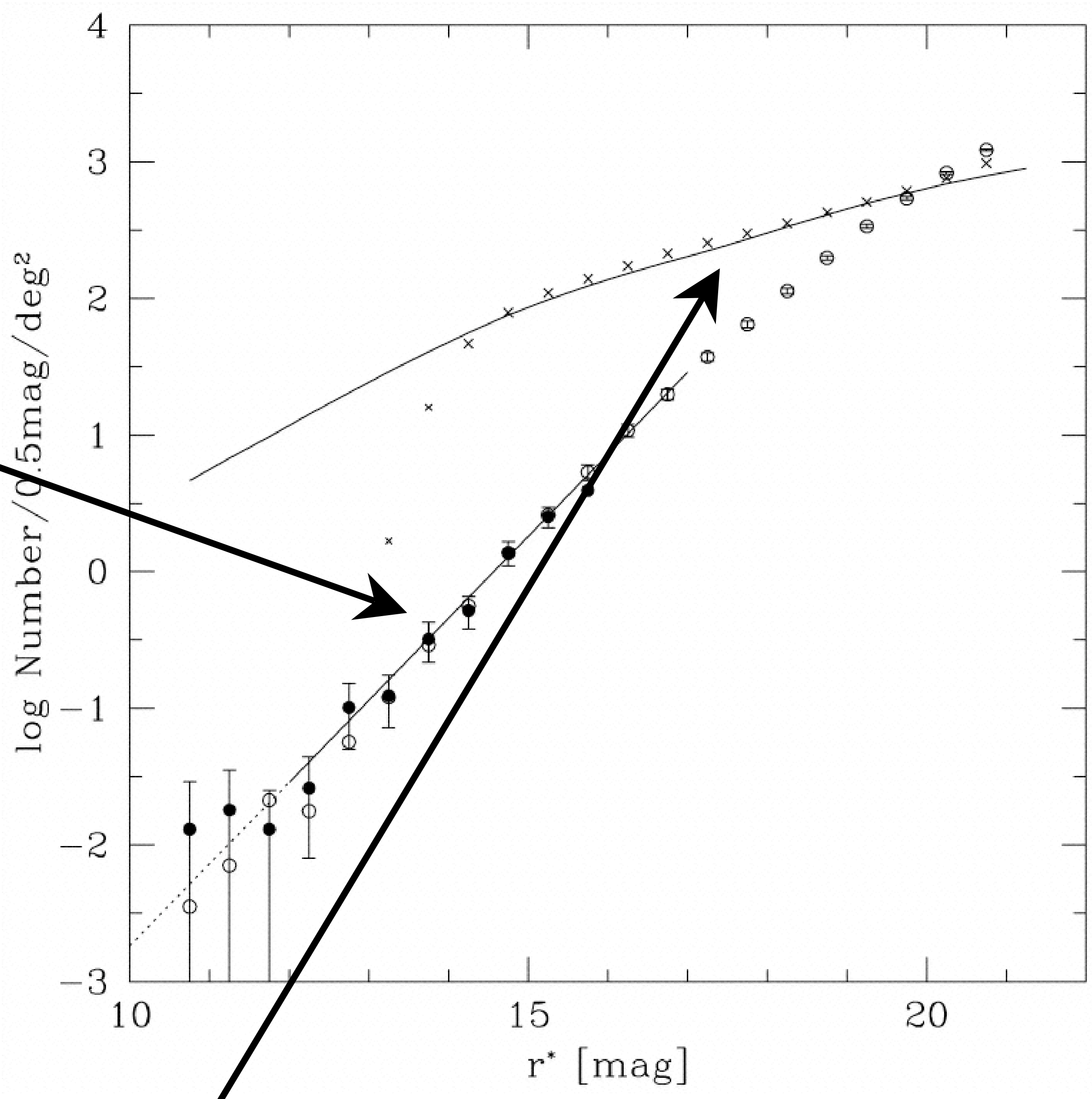
Hubble showed that the distribution of galaxies is **homogeneous**,

i.e. $N(>S) \propto S^{-3/2}$

$\Rightarrow N(<m) \propto 10^{0.6m}$

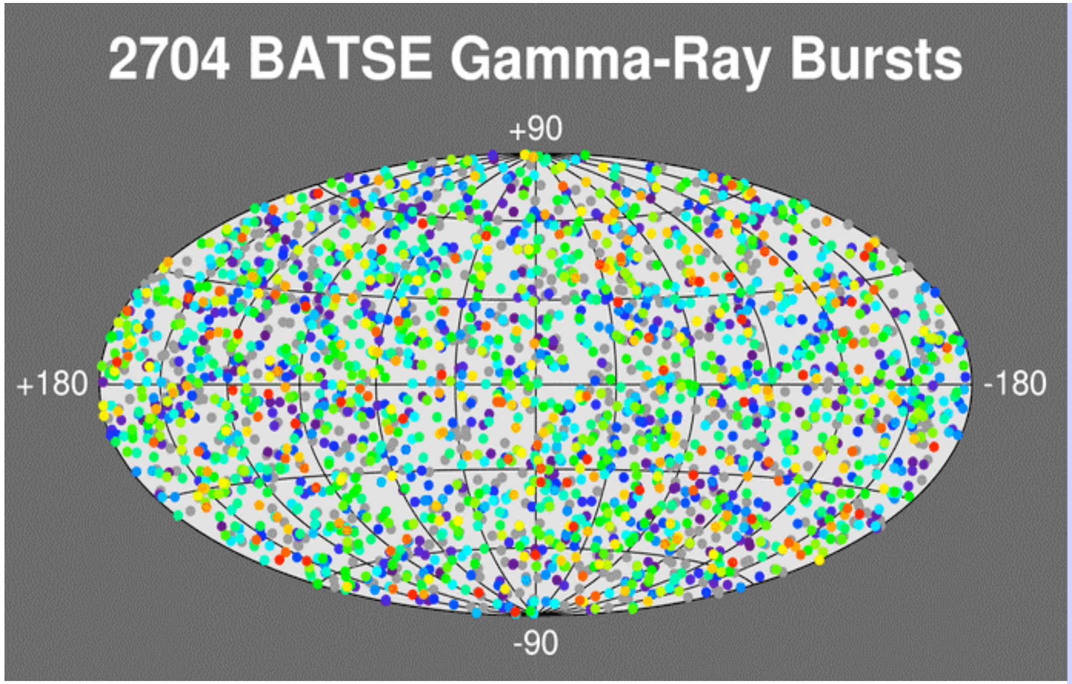
where $m \equiv -2.5 \log (S/S_0)$

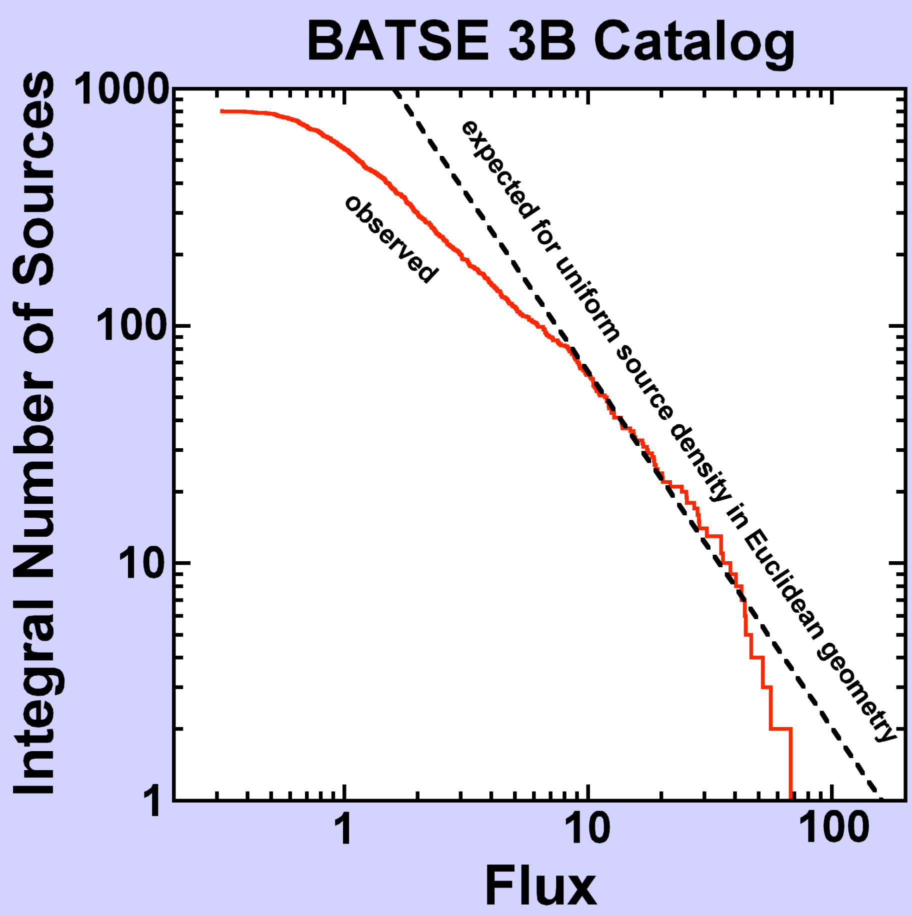
Here is the modern version of this test for galaxies in the **Sloan Digital Sky Survey**



Note that for stars, $N(<m) \propto 10^{0.4m}$, reflecting their 2D distribution

This is a test routinely carried out for all new classes of sources
e.g. it shows that γ-ray bursts are homogeneously distributed therefore presumably at cosmological distances

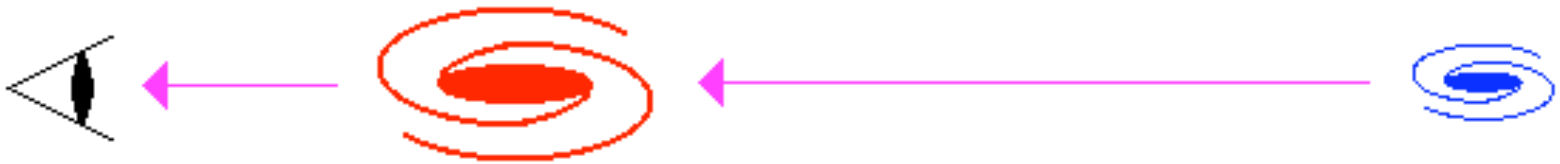




Note deviation from the $S^{-3/2}$ expectation at the faint end - are we actually seeing the 'edge' of the distribution?

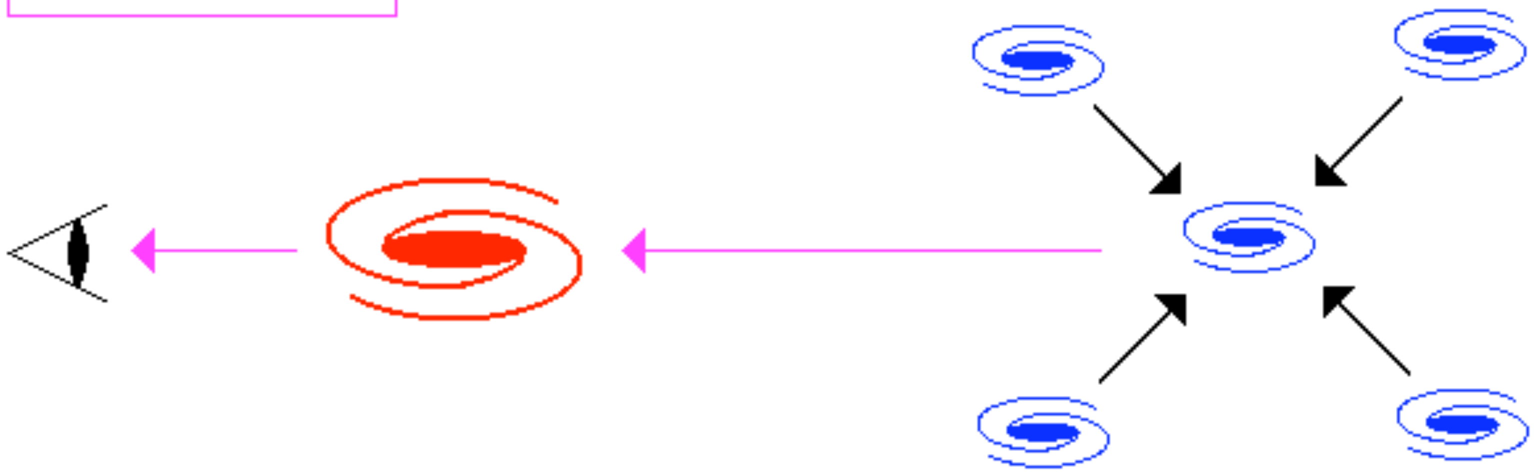
Such tests are complicated however by *evolution effects*

Color Evolution



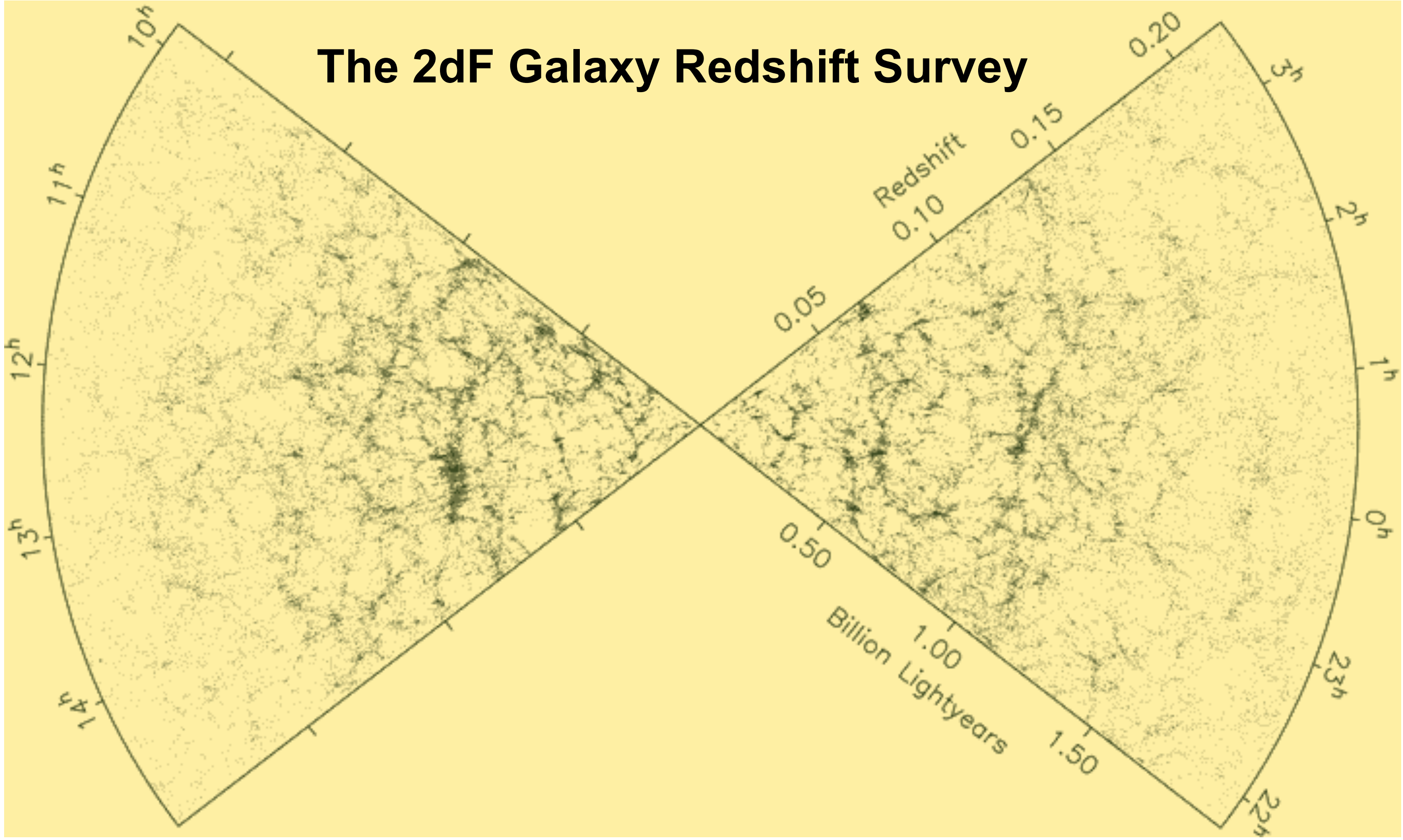
distant galaxies are bluer since we are looking back in time, and are seeing them at a younger age, younger stars = hotter stars = bluer stars

Number Evolution



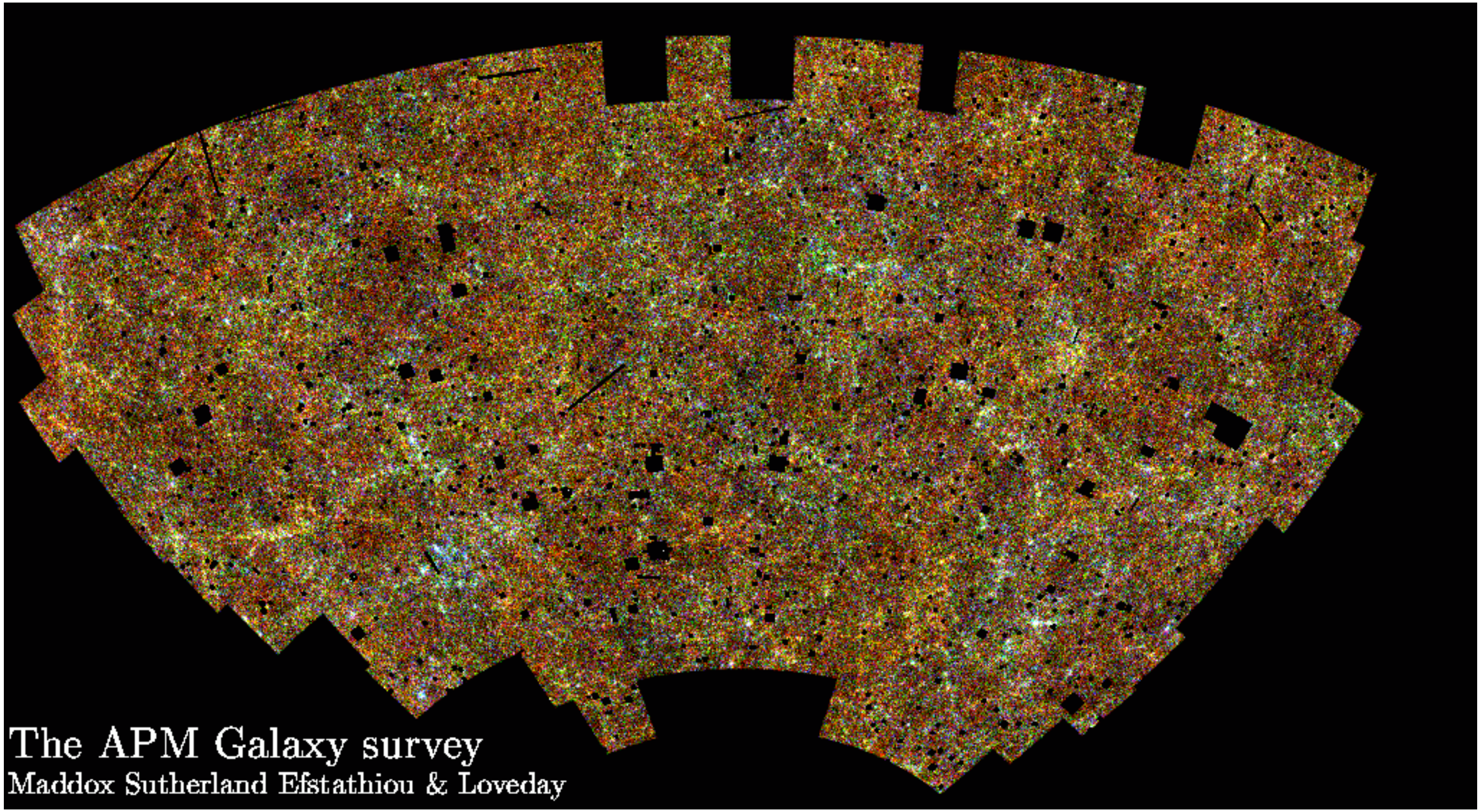
small galaxies merger at early epochs to form present–day galaxies. More galaxies are seen as we look back into the past.

Einstein "anticipated" (without any data!) that the universe is homogeneous and isotropic when averaged over large scales



The distribution of galaxies is in fact *fractal* on small scales ...
but when averaged over very large scales (>$10^{8\text{-}9}$ light years)
the galaxy distribution does *seem* to become homogeneous
although there is *still* structure on such scales ('walls', 'voids')

A *consistency test* of homogeneity is the *scaling* of the galaxy angular correlation function with the survey depth



If the distribution is homogeneous on large scales (with overdensities on small scales), then the characteristic angular scale of clustering should be *smaller* for fainter galaxies (which are on average further away) than for the (nearby) brighter ones …

This is indeed found to be the case for the APM survey which measured the positions of 2 million galaxies reaching upto ~600 Mpc …

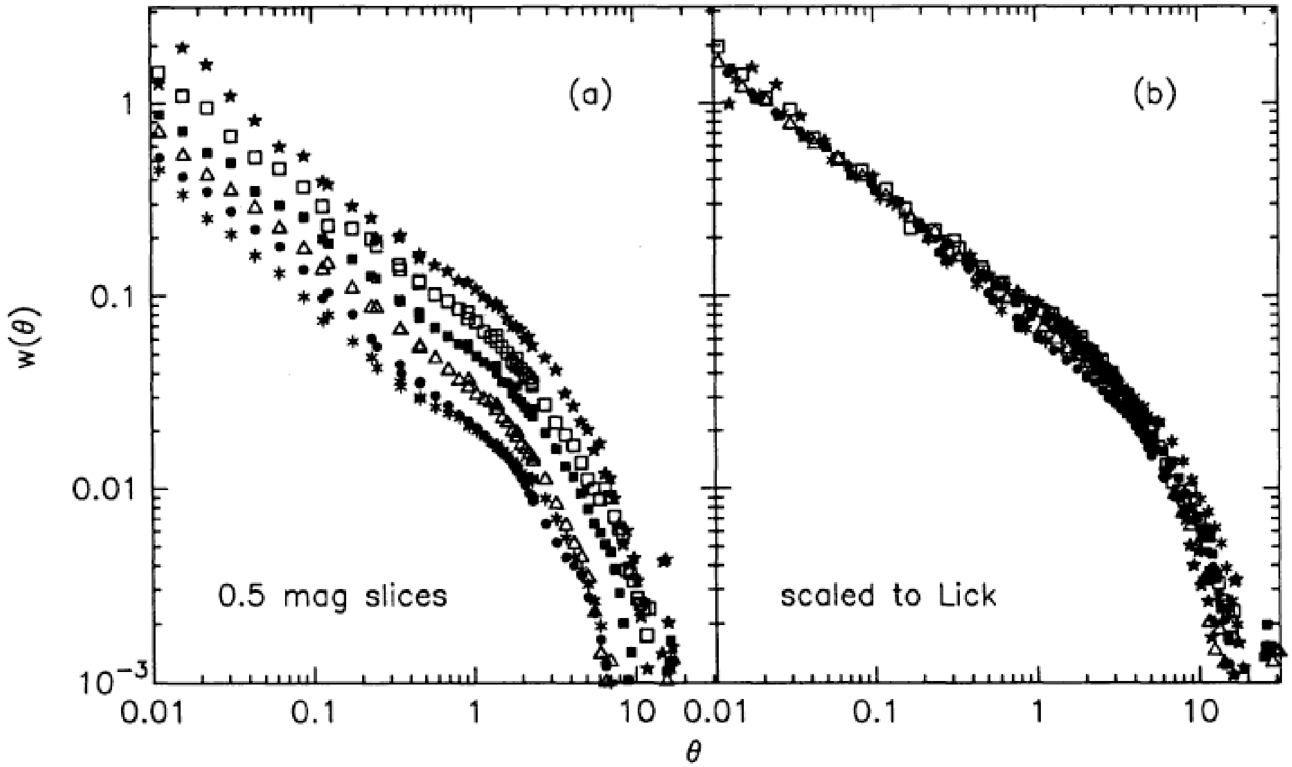


Figure 2. (a) Shows angular correlation functions for six 0.5 mag slices in the range $17.5 \le b_J \le 20.5$. (b) Shows the results from (a) scaled to the depth of the Lick survey as described in the text.

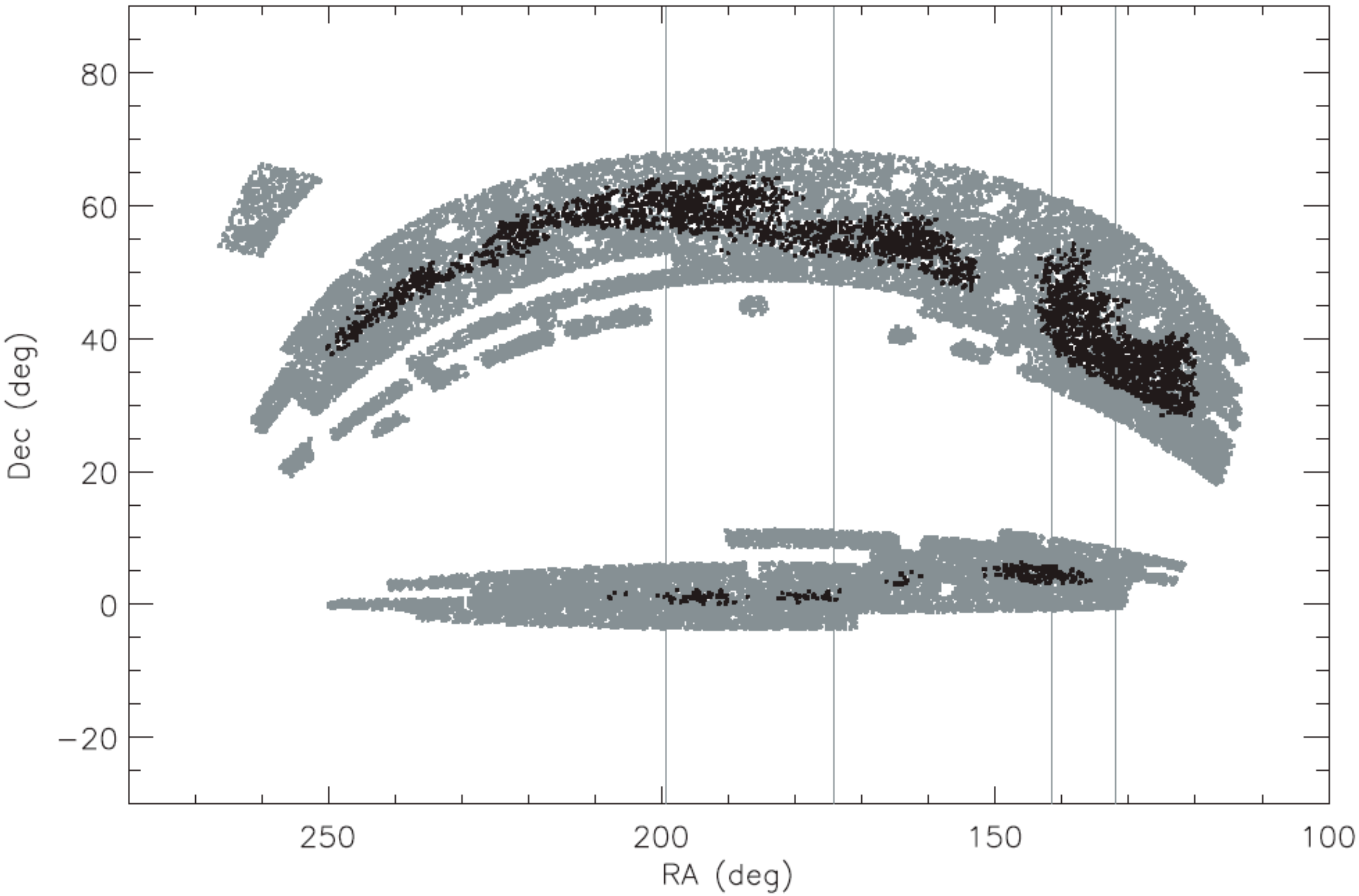
(Maddox *et al* 1990)

The angular correlation function w(θ) - defined as the *excess* probability over average of finding two galaxies within an angle θ of each other - is found to scale with the depth of the survey D* as: $w(\theta) = (r_0/D_*)\, W(\theta\, D_*/r_0)$

… as is expected for a homogeneous distribution (with clustering scale $\mathbf{r_0}$)

For a **fractal distribution** (with *no* intrinsic scale), w(θ) should *not* change with D_*

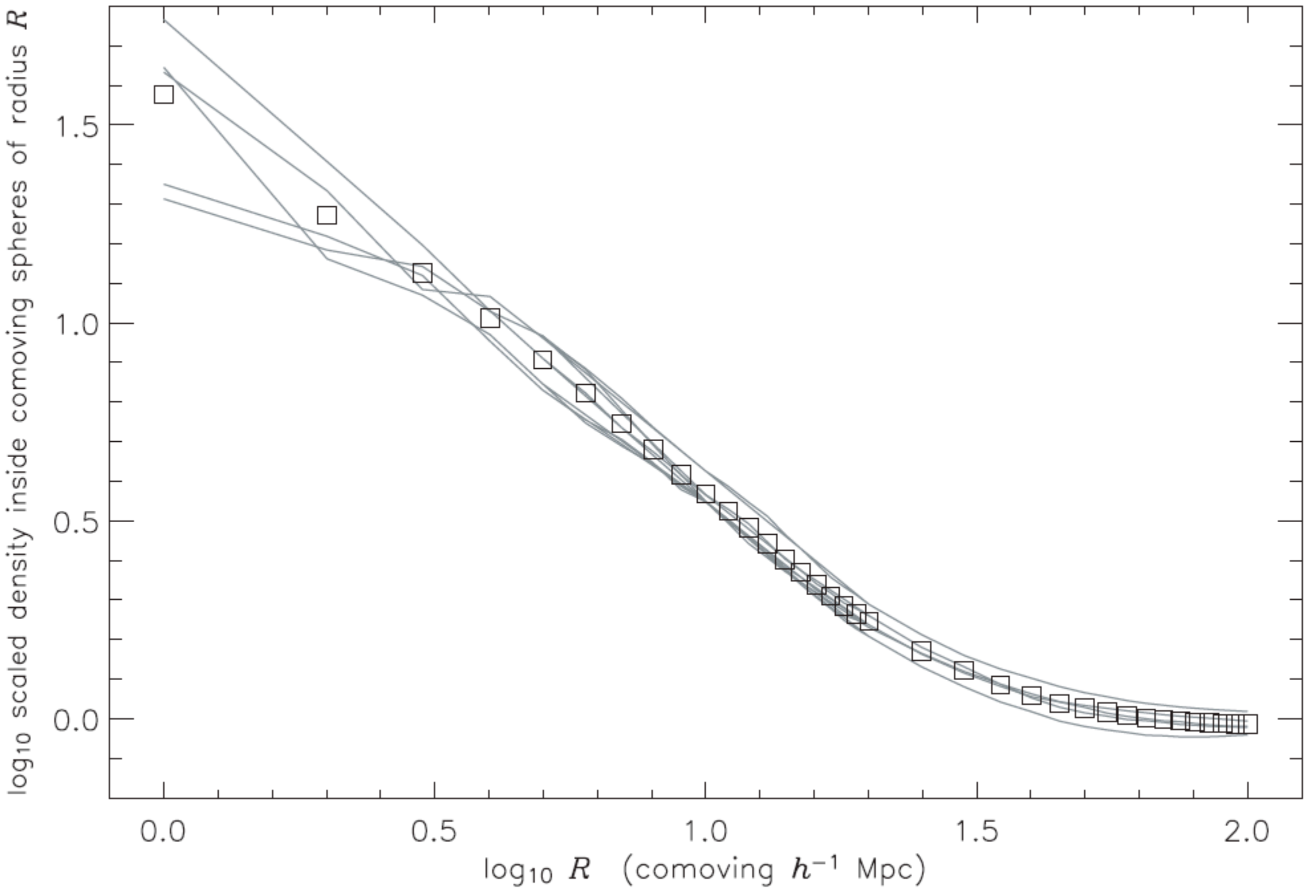
Alternatively simply count the number of galaxies in spheres of increasing radius centred on each galaxy in the survey - this ought to grow as r^3 *beyond* the **homogeneity scale**



(Hogg *et al* 2005)

This test has been performed on a sample of 3658 Luminous Red Galaxies with $0.2<z<0.4$ (occupying a volume 2 Gpc^3) in the **Sloane Digital Sky Survey**

Actual counts in the SDSS grow as ~r^2 on small scales where the distribution is fractal, but tend to homogeneity beyond ~100 Mpc …



(Hogg *et al* 2005)

FIG. 2.—Average comoving number density (i.e., number counted divided by expected number from a homogeneous random catalog) of LRGs inside comoving spheres centered on the 3658 LRGs shown in Fig. 1, as a function of comoving sphere radius R. The average over all 3658 spheres is shown with squares, and the averages of each of the five R.A. quantiles are shown as separate lines. At small scales, the number density drops with radius, because the LRGs are clustered; at large scales, the number density approaches a constant, because the sample is homogeneous. See however Sylos-Labini *et al* (2008) for a contrary view

Special Relativity

- $$ds^2 = \sum_{i,j} g_{ij}\, dx^i dx^j$$ ... interval between 2 space-time events x^i and x^j $(i, j = 0, 1, 2, 3)$

- $$g_{ij}(x) = g_{ji}(x)$$ $\Rightarrow$ 10 independent functions

- **Minkowski form:**

$$g_{ij} = \eta_{ij} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \frac{\partial g_{ij}}{\partial x^k} = 0$$

$\Rightarrow$ $$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$ ... invariant under Lorentz velocity transformations

→ equivalent to the local inertial co-ordinates of Newtonian Mechanics

- General Relativity:

$g_{ij}(x)$ is related to the distribution of matter

... however, $g_{ij} = \eta_{ij}$ is a solution in the absence of matter

contrary to Mach's principle (viz. inertial frames are determined relative to the motion of the rest of the matter in the universe)

→ add boundary conditions to eliminate anti-Machian solution

e.g. far away from all matter,

$$g_{ij} = \begin{bmatrix} 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

(De Sitter 1916)

... phenomenologically unacceptable

or

→ postulate that the matter distribution is homogeneous (in the average) and that matter causes space to curve so as to close in on itself (3-D analogue of 2-D surface of balloon)

⇒ Spatial volume finite but no boundaries and a non-singular metric everywhere (Einstein 1919)

- **Einstein's world model:**

homogeneity $\Rightarrow \frac{dN}{dm} \propto 10^{0.6m}$ ... as observed later (Hubble '26)

↓

'Cosmological Principle' (Milne '35)

$$ds^2 = dt^2 + g_{\alpha\beta}\,dx^\alpha dx^\beta$$

... synchronous gauge (dense set of comoving observers)

... picture the spatial part as S^3 (3-d analogue of balloon surface) embedded in flat 4-d space (cartesian co-ordinates x, y, z, ω)

$$d\ell^2 = dx^2 + dy^2 + dz^2 + d\omega^2$$

... distance between neighbouring points

$$R^2 = x^2 + y^2 + z^2 + \omega^2$$

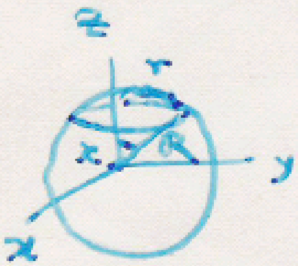
... set of points defining S^3

Then,

$$d\ell^2 = dx^2 + dy^2 + dz^2 + \frac{r^2 dr^2}{R^2 - r^2} \quad , \quad r^2 \equiv x^2 + y^2 + z^2$$

≠

Then,



$$dl^2 = dx^2 + dy^2 + dz^2 + \frac{r^2 dr^2}{R^2 - r^2} \quad , \quad r^2 \equiv x^2 + y^2 + z^2$$

$$= dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{r^2 dr^2}{R^2 - r^2} \quad ,$$

$z = r\cos\theta$
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$

$$= \frac{dr^2}{1 - r^2/R^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$\Rightarrow \quad ds^2 = dt^2 - R^2\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta \, d\phi^2)\right] \quad , \quad r = R\sin\chi$$

... generalization of 2-d line element in polar co-ordinates to 3-dim. (Einstein 1916)

Note that this approaches the Minkowski form at $r \ll R$ and displays the behaviour of the surface of a sphere for $r \sim R$

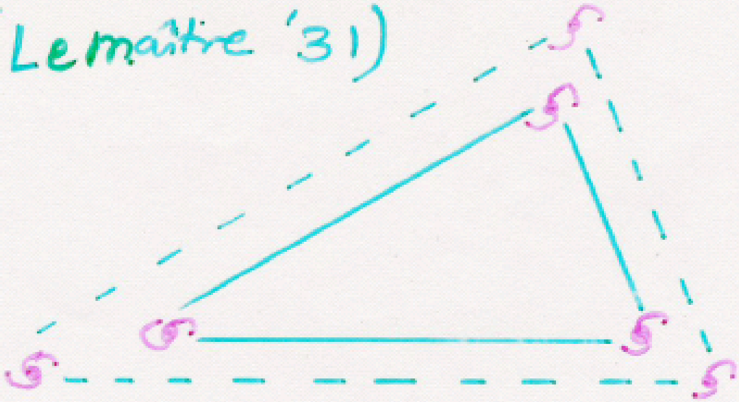
... e.g. $\delta = \frac{D}{R\sin\chi}$ ... reaches minimum $\left(= \frac{D}{R}\right)$ at $\chi = \pi/2$ and diverges to fill the sky at $\chi = \pi$

(δ: angular size; D: proper diameter)

$\epsilon = \frac{A}{R}\cot\chi$ ... vanishes at $\chi = \pi/2$

(ϵ: parallax; A: shift of observer's position)

Expanding universe (Lemaître '31)



$$\ell(t) = \ell_0 \, a(t)$$

scale-factor

... generalize line-element

$$ds^2 = dt^2 - a^2(t)\, g^0_{\alpha\beta}\, dx^\alpha dx^\beta$$ ... no off-diagonal terms

$$= dt^2 - a^2(t)\left[\frac{dr^2}{1 - r^2/R^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$= dt^2 - a^2(t)\, R^2 [d\chi^2 + \sin^2\chi\, d\Omega] \quad , \quad r \equiv R \sin\chi$$

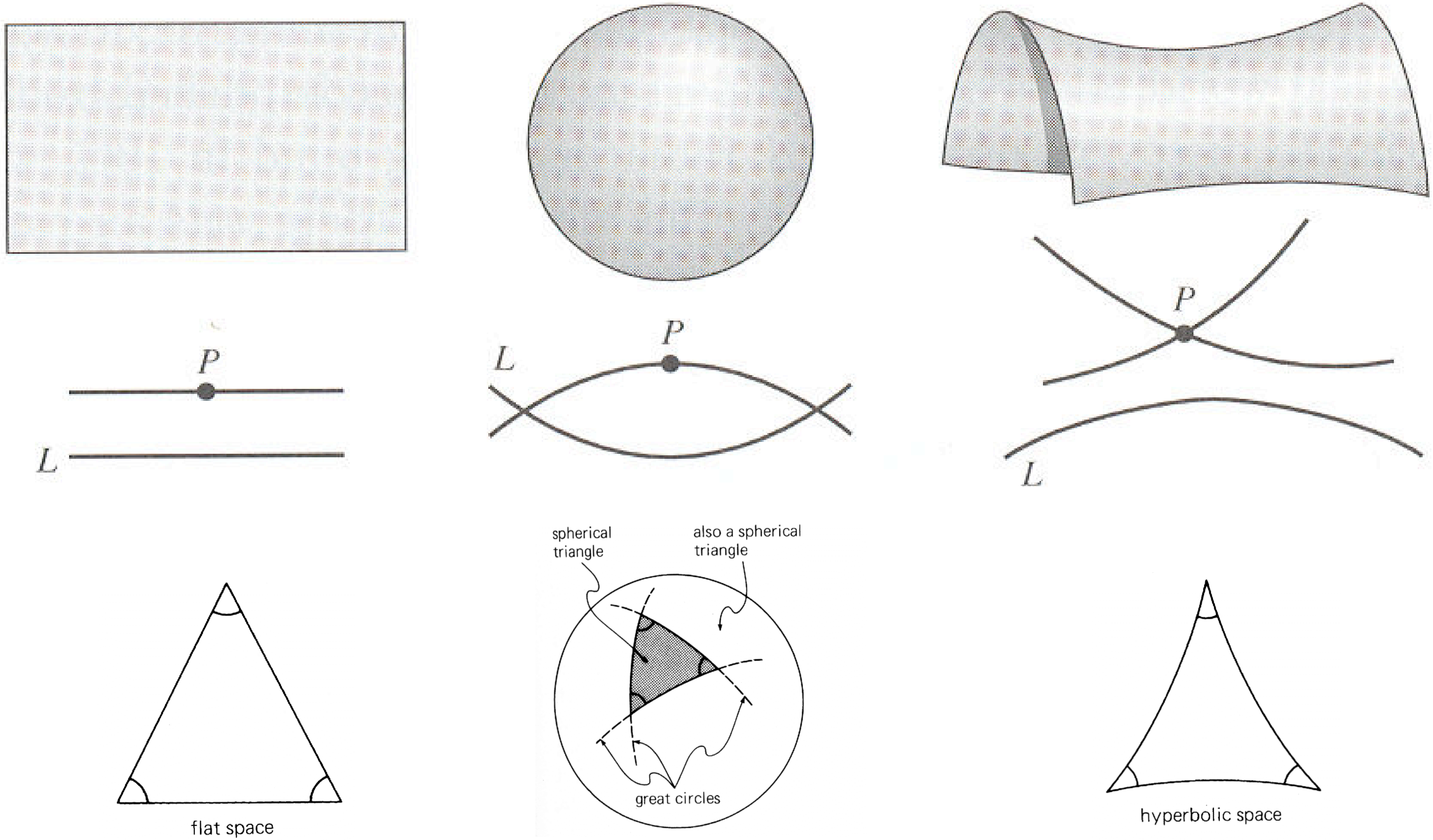
This describes a spatially closed universe

... to obtain an open universe, $\chi \to i\chi$, $R \to iR$

$$\Rightarrow \quad ds^2 = dt^2 - a^2(t)\, R^2 [d\chi^2 + \sinh^2\chi\, d\Omega]$$

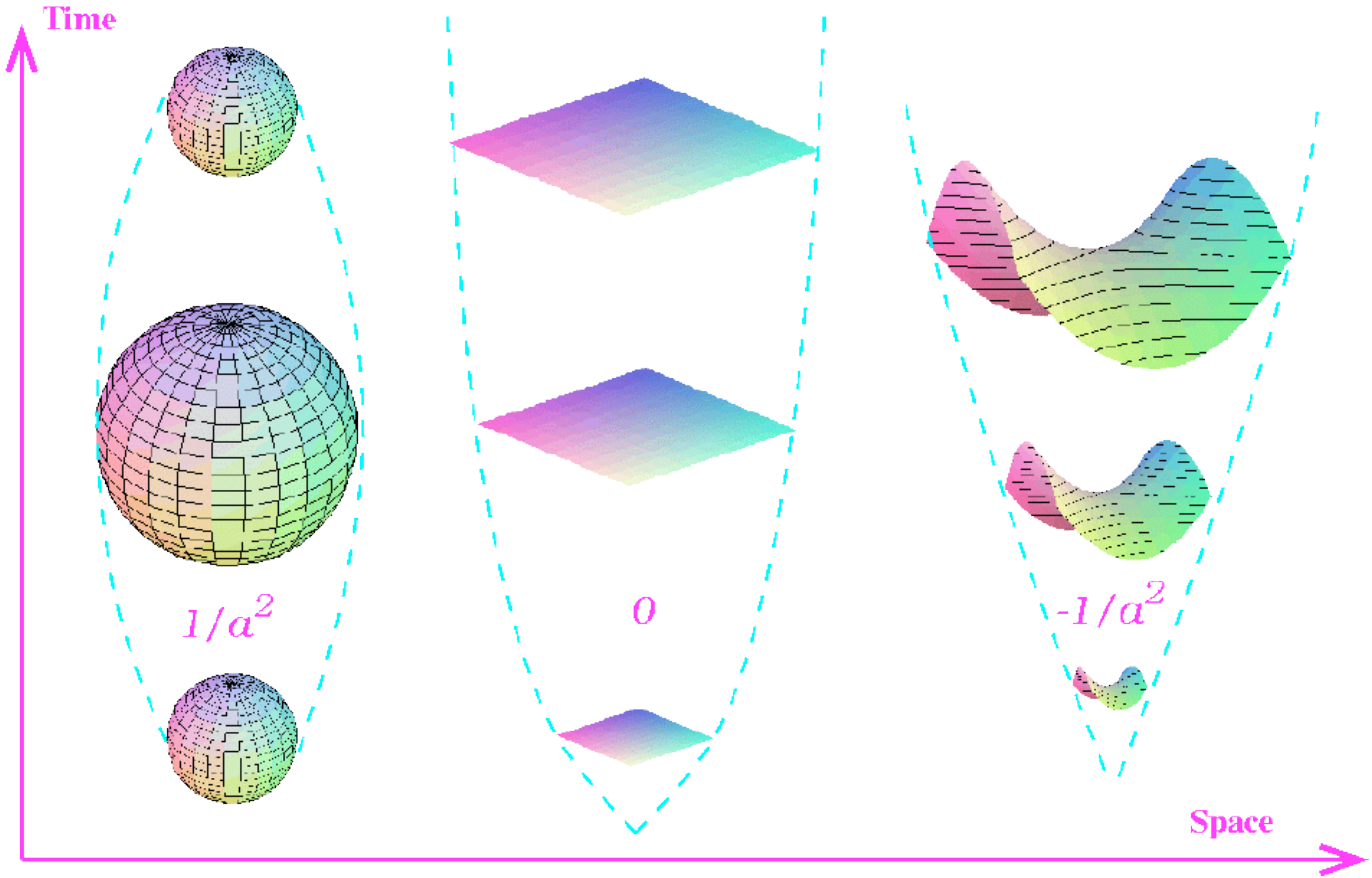
This is the Robertson-Walker line element

The three possible geometries of the Universe



180° **> 180°** **< 180°**

Homogeneous and isotropic world models



- **Einstein's Field Equation:**

$$R_{ij} - \frac{1}{2} g_{ij} R_c = 8\pi G_N T_{ij}$$

$R_{ij} \equiv g^{\lambda k} R_{\mu\nu\lambda k}$ — Ricci tensor

$R_c \equiv g^{\mu\nu} R_{\mu\nu}$ — curvature scalar

- **Ideal fluid:**

$$T_{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

- **Poisson's equation:**

$$\nabla \cdot g = -4\pi G (\rho + 3p)$$

... in small enough region Newtonian gravity is recovered ..

- **Birkhoff's theorem:** If $T_{ij} = 0$ in some region within a spherically symmetric distribution of matter, then the solution in the hole $\Rightarrow$ flat space-time

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- **Dynamics** : ... consider sphere of radius ℓ

$$\ddot{\ell} = -\frac{G_N M}{\ell^2} = -\frac{4}{3}\pi G_N (\rho + 3p)\ell \qquad \text{(McCrea \& Milne '34)}$$

also: $dU = -p\,dV = \rho\,dV + V\,d\rho$ (by differentiating $U = \rho V$)

$$\Rightarrow \quad \dot{\rho} = -(\rho + p)\frac{\dot{V}}{V} = -3(\rho + p)\frac{\dot{\ell}}{\ell} \quad \text{... energy eqn. for ideal fluid}$$

$$\therefore \quad \ddot{\ell} = \frac{8\pi}{3} G_N \rho \ell + \frac{4}{3}\pi G_N \dot{\rho}\frac{\ell^2}{\dot{\ell}} \quad \Rightarrow \quad \dot{\ell}^2 = \frac{8}{3}\pi G_N \rho \ell^2 + K$$

... for a static solution (ℓ = constant), $K = -\ell^2/R^2$ and

$$\frac{4}{3}\pi G_N(\rho + 3p) = 0 \;, \quad \frac{8}{3}\pi G_N \rho - \frac{1}{R^2} = 0$$

↳ ... i.e. $p = -\rho/3$!

Einstein's "biggest blunder"

so modify field equation: $R_{ij} - \frac{1}{2} g_{ij} R_c - \Lambda g_{ij} = 8\pi G_N T_{ij}$, $\rho_\Lambda = \frac{\Lambda}{8\pi G_N}$, $p_\Lambda = -\rho_\Lambda$

The static solution is in fact *unstable* (metric perturbations grow exponentially) but we *cannot*, as Einstein said, just "do away with the cosmological constant"!

- **Dynamics:**

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G_N(\rho + 3p) = -\frac{4}{3}\pi G_N(\rho_b + 3p_b) + \frac{\Lambda}{3}$$

'background' (i.e. ordinary matter or radiation) — 'vacuum'

$$\dot{\rho}_b = -3(\rho_b + p_b)\frac{\dot{a}}{a}$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8}{3}\pi G_N \rho_b \pm \frac{1}{a^2 R^2} + \frac{\Lambda}{3}$$

(open / closed) .. **Friedmann equations**

Two interesting solutions describing an expanding universe:

- **Einstein – De Sitter universe:** $p_b \ll \rho_b$, $\Lambda \ll \rho_b$, $\frac{1}{a^2 R^2} = 0$

$$\Rightarrow a(t) \propto t^{2/3}, \quad t = \frac{2}{3H} = \frac{1}{\sqrt{6\pi G_N \rho}}$$

- **De Sitter universe:** $\rho_b = p_b = 0 \Rightarrow a(t) = e^{H_\Lambda t}$, $H_\Lambda = \sqrt{\frac{\Lambda}{3}}$

The De Sitter universe was "motion without matter" as opposed to Einstein's static universe which was "matter without motion"!

The Robertson-Walker metric can be written as

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

NB: $a(t)$ now has dimensions of length

the redshift follows in a straightforward manner as:

$$\int_t^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dr}{(1-kr^2)^{1/2}} = \text{constant (for a given galaxy) in comoving co-ordinates}$$

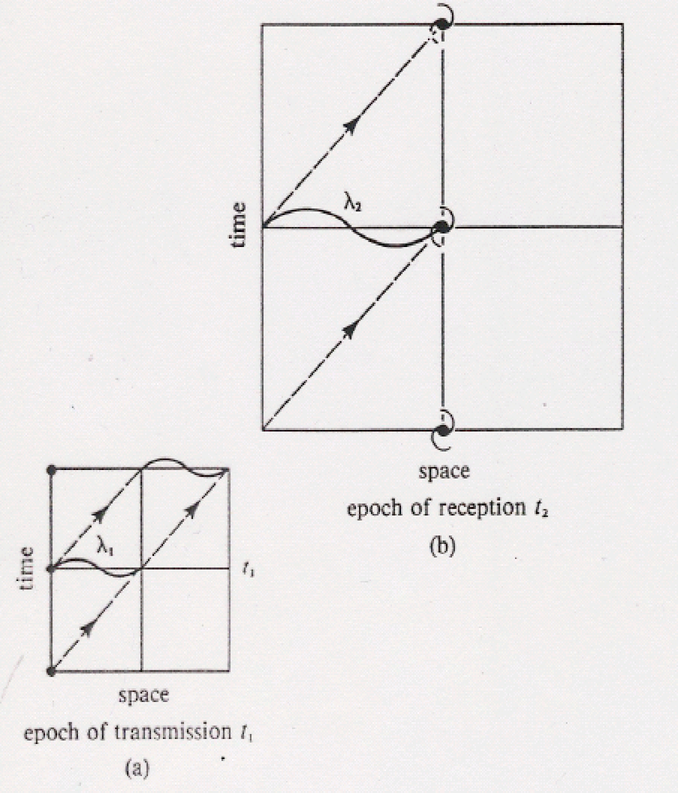
... so crests of adjacent waves, separated by Δt at emission will be received with separation Δt_0, with

$$\frac{\Delta t_0}{\Delta t} = 1 + \frac{\Delta\lambda}{\lambda_0} = 1+z = \frac{a(t_0)}{a(t)}$$

Everything is not expanding (how would we know?) certainly not atoms or planets or galaxies …

It is the large-scale *smoothed* space-time metric which is stretching with cosmic time …

The expansion is in a sense *illusory* … we can always transform to a "comoving" coordinate system where *galaxies are at rest* wrt each other



The Robertson-Walker metric describes *maximally symmetric* space-time

$$ds^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

... Robertson-Walker metric (does not reduce to Minkowskii metric as $r \to \infty$, cf. Schwarzchild metric)

$$\to \quad ds^2 = a^2(\eta)\left\{d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

... conformal time : $d\eta \equiv dt/a(t)$

($k=0$ R-W metric is conformal to Minkowskii metric globally)

$$\to \quad ds^2 = a^2(\eta)\left\{d\eta^2 - d\chi^2 - \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\}$$

... spherical co-ordinate system

… a less symmetric possibility is e.g. the Lemaitre-Tolman-Bondi metric which describes an universe that is *inhomogeneous but isotropic* around our position

Using the RW metric we can define *observational quantities* to be measured

... expand in Taylor series

$$\frac{a(t)}{a(t_0)} = 1 + H_0 (t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \cdots$$

$H_0 \equiv \dot{a}(t_0)/a(t_0)$, $\quad q_0 \equiv -\ddot{a}(t_0)\, a(t_0)/\dot{a}^2(t_0)$

... invert to obtain

$$z = H_0 (t_0 - t) + (1 + q_0/2) H_0^2 (t_0 - t)^2 + \cdots$$

i.e. $$(t_0 - t) = H_0^{-1} \left[z - (1 + q_0/2) z^2 + \cdots \right]$$

$\Rightarrow$ **co-ordinate distance**: $$r_e = a^{-1}(t_0) H_0^{-1} \left[z - \frac{1}{2}(1 + q_0) z^2 + \cdots \right]$$

using $$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^{r_e} \frac{dr}{(1 - kr^2)^{1/2}} = \begin{cases} \sin^{-1} r_e, & k = +1 \\ r_e, & k = 0 \\ \sinh^{-1} r_e, & k = -1 \end{cases}$$

$\Rightarrow$ **'Hubble law'**: $$H_0 d_L = z + \frac{1}{2}(1 - q_0) z^2 + \cdots$$

$d_L^2 \equiv a^2(t_0)\, r_e^2 (1 + z)^2$