LARGE-N GAUGE THEORIES: LATTICE PERSPECTIVES AND CONJECTURES

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I summarise what recent lattice calculations tell us about the large-N limit of SU(N) gauge theories in 3+1 dimensions. The focus is on confinement, how close SU(∞) is to SU(3), new stable strings at larger N, deconfinement, topology and θ-vacua. I discuss the effective string theory description, as well as master fields, space-time reduction and non-analyticity.

1. Introduction, Overview, Conjectures

It is 30 years since it was proposed that it might be useful to think of QCD as a perturbation in $1/N$ around the $N = \infty$ theory [1]. As is apparent from the talks at this meeting, this has been a very fruitful idea [2]. However we still do not have a quantitative control of the SU($N = \infty$) theory and the phenomenology needs to assume, for example, that it is confining and, of course, 'close to' SU(3). Lattice simulations can attempt to answer such questions directly (albeit never exactly) and there has been substantial progress in doing so this last decade, first in D=2+1 dimensions [3], which I do not discuss here, and then in the physically interesting case of D=3+1. (For a review of work in the earlier 80's, when lattice calculations were not yet precise enough to be so useful, see e.g. [4].) Here I focus on what these 'modern' lattice calculations teach us about the properties of SU(N) gauge theories at large N. I will begin with some motivation for these calculations.

A gluon loop on a gluon propagator comes with a factor of g^2N. One easily sees that g^2 is in fact the smallest power of the coupling that comes with a factor of N. So if one wants an $N \to \infty$ limit that is not given by either a free field theory or by infinite order diagrams on all length scales (in neither case would we get something like QCD), then one needs to take the limit keeping g^2N fixed [1]. In Section 6 we shall see that this standard all-order perturbative statement also appears to hold non-perturbatively. Using 't Hooft's double-line notation for gluons, diagrams can be categorised as lying on surfaces of different topology, with more handles corresponding to higher powers of $1/N$, so that in the $N = \infty$ limit only planar diagrams survive [1]. As the coupling g^2N becomes strong, the vertices of the diagram fill the surface more densely, defining a world-sheet of the kind one might expect in a string theory. This suggests that perhaps as $N \to \infty$ the gauge theory can be described as a weakly interacting string theory [1]. If the theory is, as one expects, linearly confining – and in Section 3 we present evidence that it is – then the confining flux tube will behave like a string at long distances, described by some effective string theory. At large N this will presumably coincide with the string theory that describes the SU(N) gauge theory. In Section 3 we shall discuss what a study of the string spectrum teaches us about this string theory. These 'old' string theory ideas [5] have been recently complemented by the realisation that at $g^2N \to \infty$ and $N \to \infty$ gauge theories have a dual string description that is analytically tractable [6]. Determining the effective string theory numerically should provide useful hints about what the dual theory might be in the physical weak-coupling limit, $g^2N \to 0$.

In a confining theory there are no decays at $N = \infty$. This is in contrast to what would happen in a non-confining theory where a coloured state could have a finite decay width into other coloured states. But once we constrain states to be colour singlet we reduce the density of final states by factors of N so that all decay widths vanish. In addition there is no scattering between the colour singlet states. Think of two propagating mesons. A meson propagator is like a closed quark loop. Exchange two gluons between these two closed loops and you clearly gain (up to) a factor of N, but at the cost of $g^4 \propto 1/N^2$. So, no scattering at $N = \infty$. However it is also easy to see that within a single closed loop, planar interactions give factors of g^2N, are not suppressed, and so there are non-trivial bound states. So we have what looks like a free theory, but it has a complex bound state spectrum and so is non-trivial. If one is going to find room in D=3+1 for notions of e.g. integrability, it is here in the $N = \infty$ limit that they might find a suitable home.

Is this theory with no decays and no scattering similar to the observed world of the strong interactions? First we need the SU(∞) theory to be linearly confining – and we provide evidence for this in Section 3. Now we can ask: is this confining theory close to SU(3)? In Section 4 we calculate the lightest masses in the spectrum for several values of N, and find that the SU(3) mass spectrum is indeed very close to the (extrapolated) spectrum at $N = \infty$. This provides support for the phenomenological relevance of the SU(∞) theory. And this provides motivation for trying to understand that theory much better. In Section 3 we also see that the effective string theory describing long confining flux tubes appears to be in the bosonic string universlity class. More surprisingly, the energy of shorter strings is close to the Nambu-Goto prediction and at smaller N, where this question can be addressed, the string condensation temperature is very close to that of the Nambu-Goto string action. A new phenomenon for $N > 3$ is the existence of new stable strings. In Section 7 we summarise the latest lattice calculations of the corresponding string tensions and find they lie between the 'Casimir scaling' [7] and 'MQCD' [8] conjectures. We remark how these new strings can contribute to an N-dependence of the mass spectrum even in the confining phase, contrary to naive expectations. In Section 8 we learn how the large-N gauge theory deconfines. Contrary to some speculations that the $N = \infty$ transition might be second order, partly motivated by the weakness of the first order transition in SU(3), we will show it to be robustly first order.

At $N = \infty$ the expectation value of a product of gauge invariant operators factorises into the product of the respective expectation values – by the same argument that there is no scattering. This suggests that a single gauge orbit – Witten's Master Field [9] – dominates the Path Integral calculation of all the physics in the confined phase. Since the physics is translation invariant, so must the Master Field be (for gauge invariant quantities). This suggests that all we need to know is the field in an arbitrarily small region to know it everywhere – even on one point if that can be made precise by a suitable regularisation. On the lattice this is achieved through (twisted) Eguchi-Kawai reduction [10].

In the deconfinement transition one sees explicitly how the large-N behaviour of various quantities – latent heat, interface tension, fluctuations – means that a 'phase transition' occurs on ever smaller volumes as $N \to \infty$. We shall also see, by a heuristic but physical argument, why the imposition of twisted boundary conditions is required to remain in the right phase. Precisely at the deconfining temperature, T_c, there is a different Master Field of the Euclidean Path Integral for the confined and deconfined phases. Through hysteresis this extends either side of this temperature; conceivably to all T. Indeed the Euclidean system possesses N different phases, and hence Master Fields in the deconfined phase. This multiplication of master fields is not peculiar to deconfinement. For example, intertwining θ-vacua [11,12] would lead to N non-degenerate vacua at $\theta = 0$ which become absolutely stable at $N = \infty$, as discussed in Section 9. Each of these vacua will have its corresponding master field.

For $N \geq 5$ one finds, in the lattice gauge theory with the standard Wilson plaquette action, a first order 'bulk' phase transition at a particular value of the inverse 't Hooft coupling $\lambda_c(a) = g^2(a)N \simeq 1/3$. (We write the bare coupling as a running coupling on the scale a.) This is essentially the same as the $D = 1 + 1$, $N = \infty$ Gross-Witten phase transition [13]. For $\lambda(a) > \lambda_c(a)$ the vacuum is non-perturbative on all lengths scales, so that the confining string tension is $O(1)$ in lattice units. For $\lambda(a) < \lambda_c(a)$ the vacuum is perturbative on the shortest distance scales, and hence asymptotically free as $a \to 0$, and the string tension is $O(1)$ in physical units. This transition appears as a lattice peculiarity but, as has recently been discovered [14,15], there appears to be an analogous non-analyticity at $N = \infty$ that occurs as we increase the size of a Wilson loop: at a certain critical size the eigenvalue spectrum of the loop changes non-analytically [14,15]. This is possible because at $N = \infty$ the number of physically relevant degrees of freedom per unit volume is infinite. It appears that this critical size is fixed in physical units and will survive in the continuum limit.

There are, of course, other non-analyticities as $N \to \infty$. For example, we shall see in Section 9 that the instanton size distribution exactly vanishes for sizes up to some critical size. However this can be understood as due to the factor $\exp(-8\pi^2/g^2) \propto \exp(-cN)$ that dominates the weighting of small instantons. We do not know of any such simple argument in the case of Wilson loops. Indeed we might conjecture that this non-analyticity provides an explanation for the puzzlingly rapid transition between short and long distance physics that is observed experimentally [16]; i.e. as soon as one is at values of Q^2 where one can apply perturbation theory, one finds that there is little room for the higher twist operators that one might expect to parametrise the transition from perturbative to non-perturbative physics – 'precocious scaling'. The non-analyticity discovered in [14,15] suggests that for $l < l_c$ we can calculate the wilson loop perturbatively, while for $l > l_c$ it is confining and non-perturbative. At $N = \infty$ the transition is infinitely sharp; in SU(3) and QCD it might become a very rapid cross-over, explaining the phenomenon of precocious scaling.

The presence of such a 'phase transition' as we increase the distance, might effectively disconnect the confining theory from its short distance perturbative framework. In this disembodied confining theory the coupling is never small, it is confining on all available length scales and one never needs to discuss gluons. This raises, for example, the possibility of a dual string theory in which the coupling need not be large. Such a dual theory might be analytically tractable. It also raises the possibility that at $N = \infty$ the same confining theory can have different ultraviolet completions. That is to say, to solve the theory we do not necessarily need to solve the full non-Abelian gauge theory. These conjectures are speculative, of course, but they certainly provide motivation for clarifying [17] the nature of this remarkable non-analyticity.

2. Lattice

We will calculate Euclidean Feynman Path Integrals numerically. This requires a finite number of degrees of freedom, so we discretise continuous space-time and make the volume finite by going to a hypercubic lattice on a 4-torus. Since the theory is renormalisible and has a mass gap, the errors induced by this should rapidly disappear as the lattice spacing is reduced and the volume enlarged. The lattice spacing is a and the size of the μ-torus is $l_\mu = aL_\mu$. The degrees of freedom are SU(N) matrices, U_l, defined on the links l of the lattice. The partition function is

$$\mathcal{Z}(\beta) = \int \prod_l dU_l e^{-\beta \sum_p \{1 - \frac{1}{N}\mathrm{ReTr}u_p\}} \quad ; \quad \beta = \frac{2N}{g^2} \tag{1}$$

where u_p is the ordered product of matrices around the boundary of the elementary square (plaquette) labelled by p and g^2 is the bare coupling. This is the standard Wilson plaquette action and one can easily see that for smooth fields it reduce to the usual continuum gauge theory. Since the theory is asymptotically free and since the bare coupling is a running coupling on length scale a, the continuum limit is approached by tuning $\beta = 2N/g^2(a) \to \infty$. As we remarked earlier, one expects from the diagrammatic analysis that for large N the value of a is fixed in physical units (e.g. in units of the mass gap) if one keeps the 't Hooft coupling $\lambda(a) \equiv g^2(a)N$ fixed i.e. $\beta \propto N^2$. This will be confirmed below.

The lattice path integral in eqn(1) is no easier to calculate analytically than the original continuum version. However because the number of integrations is now finite, we can attempt a numerical evaluation. The number of integrations is large and so the natural method to use is the (Markovian) Monte Carlo. The Monte Carlo generates 'points' in the integration space.

Each such 'point' is an explicit lattice gauge field i.e. an SU(N) matrix on every link of the lattice. These fields are generated with the measure

$$\mathcal{D}U = \prod_l dU_l e^{-\beta \sum_p \{1 - \frac{1}{N}\mathrm{ReTr}u_p\}} \tag{2}$$

so if we generate n_c such 'points', i.e. $\{U_\mu(n); \mu = 0,..,3; n = 1,...,L^4\}^I$; $I = 1,...,n_c$, then the expectation value of Ψ will be just the average over these fields:

$$\langle \Psi_L(U) \rangle = \frac{1}{n_c} \sum_{I=1}^{n_c} \Psi_L(U^I) \pm O(\frac{1}{\sqrt{n_c}}). \tag{3}$$

I have made explicit here the statistical error which decreases as the square root of the number of field configurations – as one would expect for such a probabilistic estimate.

We calculate masses from Euclidean correlation functions

$$C(t = an_t) \equiv \langle \phi^\dagger(t)\phi(0) \rangle = \sum_n |\langle n|\phi(0)|vac\rangle|^2 e^{-aE_n n_t} \quad ; \quad E_i \leq E_{i+1} \tag{4}$$

which we evaluate numerically as just described. Note that all energies will be obtained in lattice units, aE_n. At large n_t the lightest state will dominate $C(t = an_t)$ and can be easily extracted. Unfortunately the statistical error in the calculation of eqn(3), is more-or-less independent of n_t, since the average fluctuation squared around the correlator is itself a higher-order correlator which, one can easily verify, has a disconnected piece. Thus the error to signal ratio grows exponentially with n_t and one needs $C(t = an_t)$ to be dominated by the lightest state at small n_t i.e. one needs ϕ to be a good wave-functional for the desired state. Standard techniques now exist to achieve this, and can be used within a variational calculation, based on the $\exp(-aH)$ implicit in $C(t = an_t)$, to obtain excited as well as ground state energies [18]. However it should be apparent that the larger the energy, the less accurate the calculation.

To calculate glueball masses we use operators that are based on contractible Wilson loops. We calculate the string tension from the energy of the lightest flux loop that winds around a spatial torus, and use operators based on a Wilson line that encircles the torus. In Fig. 1 I show an example of the latter [19]. The calculation is in SU(6) on a 12×14^3 lattice with the flux loop winding around the x-torus. Shown also is the best single exponential fit (actually a cosh because of the periodicity in t). It is clear that it dominates the correlator from very small t – indeed the overlap of the

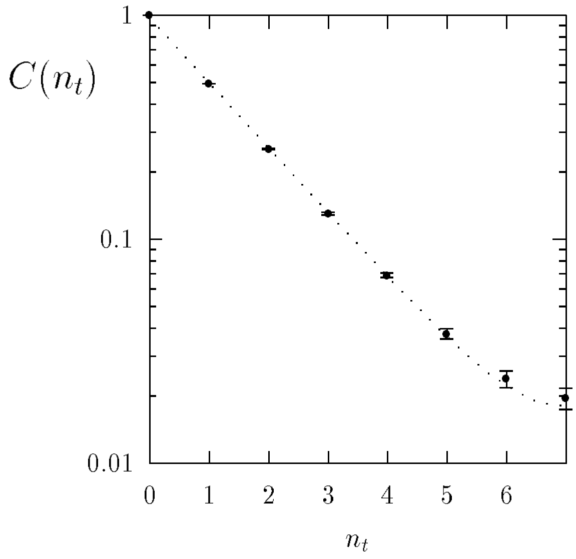

Figure 1. Correlation function where the lightest state is a flux loop that winds around the spatial torus. Best single energy cosh fit shown.

operator on the flux loop is $\simeq 0.97$. This is achieved by iterative smearing of the fields and by a variational calculation (see e.g. [18] for details).

3. Confinement and Strings

Consider one spatial torus of size l and all the other tori large. We calculate the mass of the lightest flux loop that winds once around this torus. We expect [20] its energy to be

$$m(l) \stackrel{l\to\infty}{=} \sigma l - \frac{\pi(d-2)}{6}\frac{c}{l} \tag{5}$$

where c (times the dimensional factor $d-2$) is the central charge of the effective string theory that describes the long-distance properties of the confining flux tube. In Fig. 2 I show the results of a calculation of $am(l)$ in SU(6) at a fixed value of a [19]. We see linear confinement, and a fit with eqn(5) to $l \geq 10, 12$ gives $c = 1.16(8), 1.09(10)$ respectively. This tells us that the effective string theory at long distances is a simple bosonic theory in the universality class of the Nambu-Goto action. Indeed, if we fix the coefficient of the $1/l$ term to the bosonic string value $\pi/3$, then we find that

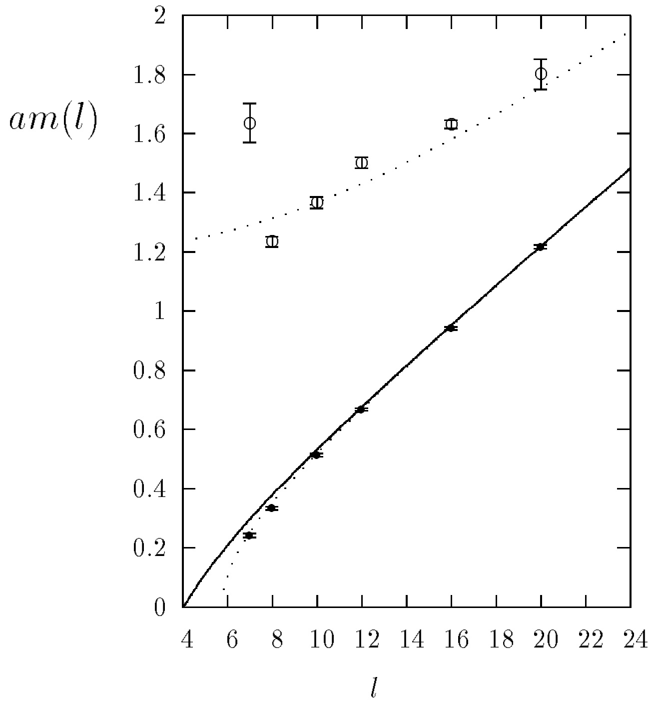

Figure 2. The masses of the lightest, •, and first excited, ∘, $k = 1$ flux loops that wind around a spatial torus of length l in the SU(6) calculation at $\beta = 25.05$. The dotted lines are the predictions of the Nambu-Goto string action, as in eqn(7). The dynamical lower bound on the string length is $l_{min} = 1/aT_c \simeq 6.63$.

we can obtain an acceptable fit to our whole range of l:

$$aE_0(l) = 0.06358(20)l - \frac{\pi}{3l} - \frac{19.4(3.2)}{l^3} \quad : \quad l \geq 7 \quad , \chi^2/n_{df} = 0.9 \qquad (6)$$

This is remarkable: since there is a minimum length for a periodic flux loop, which is $l_{min} = 1/aT_c \simeq 6.63$ in the present calculation, the fit in eqn(6) essentially works all the way down to the shortest possible strings. (If we fit the $O(1/l)$ term as well, then its coefficient comes to 0.94(16).) Since the corrections in the pure gauge theory to $N = \infty$ are $O(1/N^2)$, we can assume that all this is also true of the SU($N = \infty$) theory. In physical units the lattice spacing is small, $a\sqrt{\sigma} \simeq 0.25$, so we can assume that this is also true of the continuum limit. Finally, the longest string is $20a \simeq 5/\sqrt{\sigma}$ so we can assume we are seeing the asymptotic behaviour of a long string.

In the Nambu-Goto string theory the spectrum is given by [21,22]

$$E_n(l) = \sigma l \Big\{ 1 + \frac{8\pi}{\sigma l^2}\big(n - \frac{d-2}{24}\big)\Big\}^{\frac{1}{2}} \tag{7}$$

In Fig. 2 we show that the one parameter fit with $E_0(l)$ to the lightest string mass works not too badly all the way down to the minimum possible string length, $l_c = 1/T_c$. The χ^2 is too large to be acceptable, but is much smaller than for SU(4). This leaves open the intriguing (and unexpected) possibility that the Nambu-Goto string action describes confining strings on all length scales at $N = \infty$.

4. Spectrum

The lightest $J^{PC} = 0^{++}$ and 2^{++} glueballs turn out to be the lightest states in the $a = 0, V = \infty$ SU(N) gauge theory. In Fig. 3 I plot the ratios of these masses to the (simultaneously calculated) string tension, obtained after a continuum extrapolation of the lattice results for each value of N [18]. We see that a modest $O(1/N^2)$ correction suffices to fit the ratios for $N \geq 3$: for these quantities SU(3) is indeed close to SU(∞).

This is for only two states of course. For a much more detailed comparison of SU(3) and SU(8) see [23].

5. Pomeron

In [23] you will also find estimates for higher spin glueballs. This requires novel lattice techniques because of the reduced cubic rotational symmetry of our lattice. This calculation enables us to ask if the Pomeron trajectory is the leading glueball trajectory (ignoring mixing). Fig. 4 contains a Chew-Frautschi plot for the $PC = ++$ sector of the D=3+1 SU(3) gauge theory. Recall that $1/2\pi\sigma \simeq 1GeV$ is roughly the slope of the usual mesonic Regge trajectories. Assuming linear trajectories, the best fit to the intercept α_0 and slope α' of the leading trajectory is [23]

$$2\pi\sigma\alpha' = 0.281(22) \quad , \quad \alpha_0 = 0.93(24) \tag{8}$$

This provides quite convincing evidence for the old conjecture that the Pomeron is in fact the leading glueball trajectory. It is interesting to note that in 2+1 dimensions the leading glueball trajectory has a very low interecept and so is unimportant at high energies [23,24]. This is the case not only for SU(3) but for larger N as well [23].

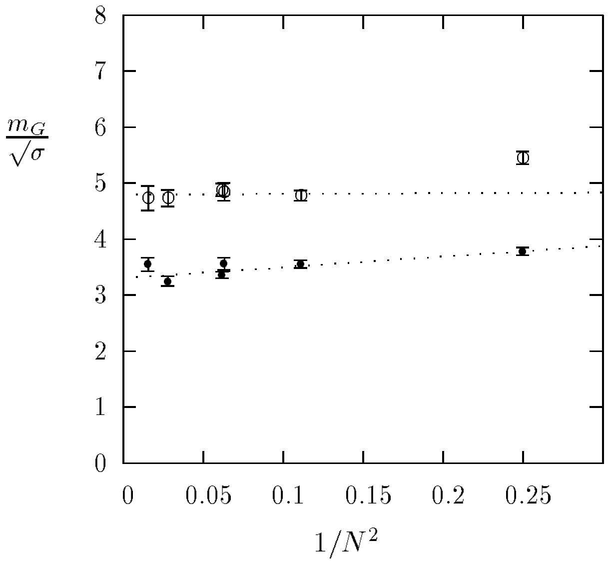

Figure 3. The lightest 0^{++}, $\bullet$, and 2^{++}, $\circ$ glueball masses expressed in units of the fundamental string tension, in the continuum limit, plotted against $1/N^2$. Dotted lines are extrapolations to $N = \infty$.

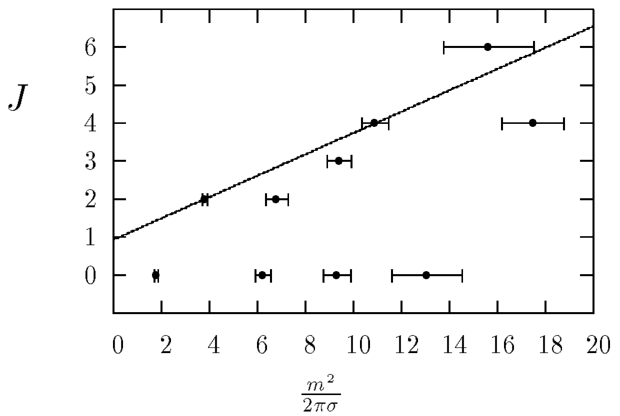

Figure 4. Chew-Frautschi plot of $PC = ++$ states in the continuum SU(3) gauge theory. The leading Regge trajectory is shown.

6. Coupling

The diagrammatic expectation is that one keeps fixed the ‘t Hooft coupling $\lambda = g^2 N$ for a smooth large-N limit. For theories with running couplings (as here) this becomes

$$\lambda(l\mu) = g^2(l\mu)N \stackrel{N\to\infty}{=} \text{ind of } N \tag{9}$$

where the length scale l is expressed in units of μ, which is some physical mass in the SU(N) gauge theory that one expects to have a smooth, non-zero large-N limit. We expect that if we plot $\lambda(l\mu)$ against $l\mu$ then as $N \to \infty$ all the curves for different N will lie on top of each other.

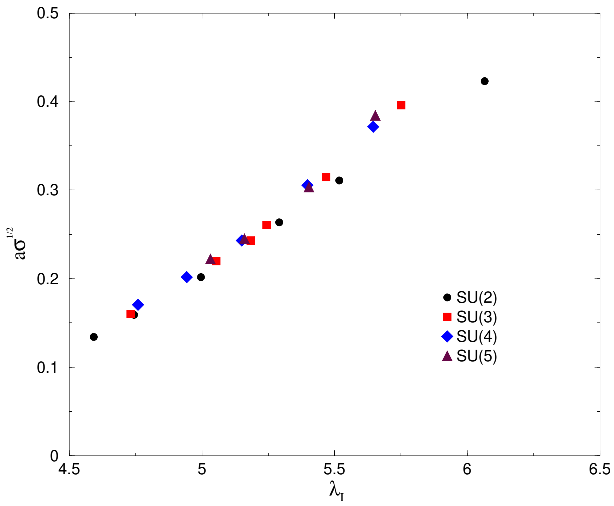

Figure 5. The square root of the string tension in lattice units, $a\sqrt{\sigma}$, plotted against the ’t Hooft coupling, $\lambda_I \equiv g_I^2 N$.

Let us choose $\mu = \sqrt{\sigma}$ and $l = a$. Then we can use $\beta = 2N/g^2$ to define a coupling $g^2(a)$. This is not ideal because of large lattice spacing corrections. It is well known that the ‘mean field improved’ coupling, $\beta\langle \mathrm{ReTr}\,u_p\rangle/N = 2N/g_I^2(a)$, is much closer to standard definitions. Defining $\lambda_I(a) = g_I^2(a)N$ we can plot its value against the value of $a\sqrt{\sigma}$ obtained in the calculation with the corrsponding β. We plot the results [25] in Fig.5 for $2 \le N \le 5$. This provides nice non-perturbative confirmation of the usual perturbative result.

7. k-strings

Sources that transform as $\psi \to z^k \psi$ under a gauge transformation of the centre, $z \in Z_N \subset SU(N)$, cannot be screened by gluons to a source that transforms otherwise. Thus each sector k has its own stable string tension, σ_k. Lattice calculations [26] have shown these are bound, $\sigma_k < k\sigma$, and have focussed on a range of values that includes the Casimir scaling [7] and 'MQCD' conjectures [8]. Results from the most recent work on this topic [18] are listed in Table 1. The results lie between the Casimir Scaling and MQCD predictions.

σ_k/σ			
(N,k)	Casimir scaling	this paper	'MQCD'
(4,2)	1.333	1.370(20)	1.414
(4,2)	1.333	1.358(33)	1.414
(6,2)	1.600	1.675(31)	1.732
(6,3)	1.800	1.886(61)	2.000
(8,2)	1.714	1.779(51)	1.848
(8,3)	2.143	2.38(10)	2.414
(8,4)	2.286	2.69(17)	2.613

The values in Table 1 are obtained after an extrapolation to the continuum limit:

$$\frac{\sigma_k}{\sigma}(a) = \frac{\sigma_k}{\sigma}(0) + ca^2\sigma \tag{10}$$

from a range of a where the leading correction proves sufficient. (It is known that the leading correction for the plaquette action is $O(a^2)$.) I show in Fig. 6 an example [18] from SU(4), which superimposes two separate calculations. It is clear that the continuum extrapolation is very well determined. This is also true of the glueball calculations discussed earlier.

All this has interesting implications for the mass spectrum. One expects that (most) highly excited glueballs can be described as closed loops of flux i.e. closed strings. Normally one thinks of flux tubes in the fundamental representation. However closed loops of k-strings will do just as well. These will scale with σ_k. Thus as N increases we expect the mass spectrum to acquire new towers of states, based the stable strings with $k \leq N/2$. These towers will be copies of each other scaled up from the fundamental by $\sqrt{\sigma_k/\sigma}$. There is a hint of such a state in the comparison between SU(3)

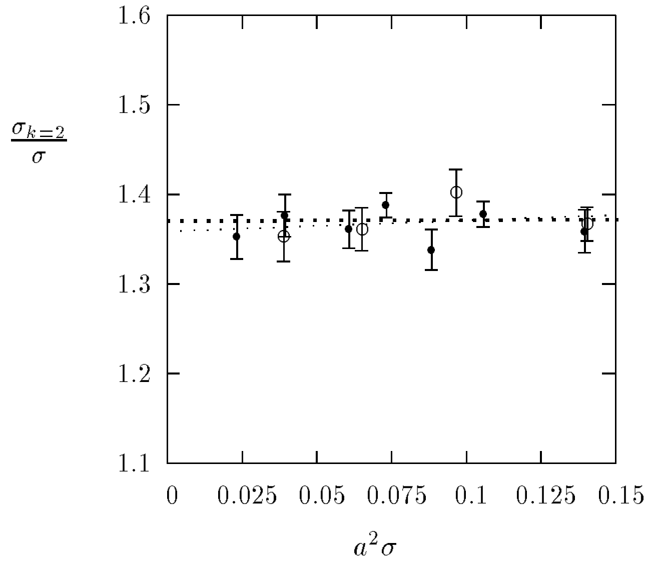

Figure 6. Ratio of the tension of the $k = 2$ string to that of the fundamental ($k = 1$) string in SU(4), for the anisotropic ($\circ$) and isotropic ($\bullet$) calculations. Plotted against the square of the (spatial) lattice spacing. Continuum extrapolations are shown.

and SU(8) in [23]. If σ_k follows Casimir scaling then this will lead to an 'unexpected' $O(1/N)$ variation in some mass ratios. It also implies that the 'entropy' in the confined phase is not N-independent, in contradiction to usual assumptions (although this will not upset usual conclusions).

8. Deconfinement and String Condensation

It is well-known that the SU(2) deconfining transition is second order and that SU(3) is 'weakly' first order. Recent numerical studies of deconfinement in SU($N > 3$) gauge theories [27] have shown that the transition becomes more strongly first order as $N \uparrow$ and it is clear that it is robustly first order in the $N = \infty$ limit.

In Fig. 7 I plot the value of the deconfining temperature in units of the string tension, for $2 \leq N \leq 8$. We see that a modest $O(1/N^2)$ correction suffices to fit these values. This is perhaps puzzling given that SU(2) is second order.

If nothing else happens as one increases T, one expects a string condensation deconfining transition to occur [28] simply because for strings of

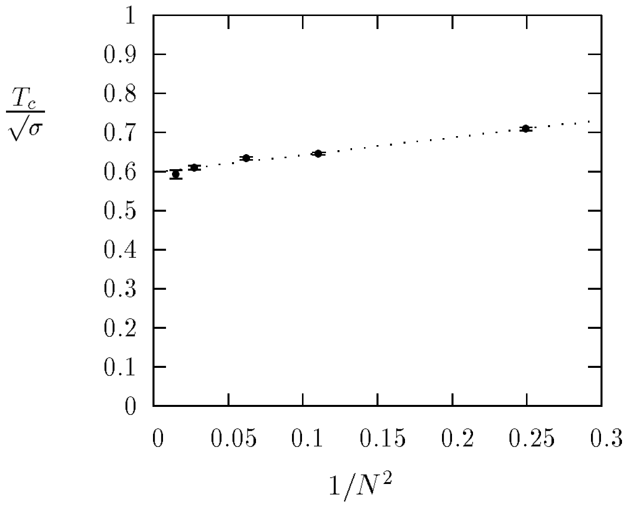

Figure 7. The deconfining temperature in units of the string tension for various SU(N) gauge theories. Large N extrapolation shown.

length l the thermodynamic suppression $\exp\{-E/T\} \sim \exp\{-\sigma l/T\}$ is outweighed by the number of strings $\propto \exp\{+c_s l\}$ once $T > \sigma/c = T_c^s$, where we call T_c^s the string condensation temperature. Thus any second order deconfining transition may be driven by string condensation and the value of T_c^s/σ will then tell us something about the effective string theory. For SU(N) gauge theories we have

$$\frac{T_c}{\sqrt{\sigma}} = \begin{cases} 0.709(4) & : \quad SU(2), d=4 \\ 1.12(1) & : \quad SU(2), d=3 \\ 0.98(2) & : \quad SU(3), d=3 \end{cases} \tag{11}$$

and it is interesting to compare it to the Nambu-Goto value

$$\frac{T_c^s}{\sqrt{\sigma}} = \sqrt{\frac{3}{(d-2)\pi}} \simeq \begin{cases} 0.691 & : \quad d=4 \\ 0.977 & : \quad d=3 \end{cases} \tag{12}$$

obtained from the vanishing of $E_0(l)$ in eqn(7).

As expected, one finds [27] that the latent heat $L_h \propto N^2$ and that the interface tension, σ_{cd}, grows with N. It is easy to see [27] that if $\sigma_{cd} \propto N$ then at $N = \infty$ the transition becomes infinitely sharp, i.e. a real phase transition, even on a finite volume and there is no hysteresis. If $\sigma_{cd} \propto N^2$

then the hysteresis may be large – perhaps even infinite! In that case, as we increase T past T_c in the confined phase, we may continue to remain confined to a high enough T that we encounter the string condensation transition T_c^s. If so, then we can study string condensation in the interesting $N = \infty$ limit.

Although one finds that one can calculate T_c on ever smaller volumes as $N \uparrow$, one also knows that once one of the spatial sizes drops below $\sim 1/T_c$, flux loops that wind around that torus will condense, and this spontaneous symmetry breaking will alter the thermal physics. This looks like an obstacle to reducing the volume further as $N \uparrow$. However one knows how to prevent this – twisted boundary conditions can force a domain wall that restores the symmetry [29]. For small sizes the wall unravels and the system sits in the symmetric core. Then one can continue to reduce V with N. This provides a physical interpretation of twisted Eguchi-Kawai.

9. Topology, Instantons, θ-vacua

The topological susceptibility χ_t has a non-zero limit at $N = \infty$ with a value that fits in with the Witten-Veneziano analysis of the η' mass [25,30]. This is of course a non-leading effect [31] since $\langle Q^2 \rangle = \langle Q \rangle^2 = 0$ to leading order.

In lattice calculations a very small instanton is a very big spike in the $F\tilde{F}$ density and is easy to see. So in the instanton density plot [31] in Fig. 8 the rapid suppression of small instantons with N is reliable – and of course expected from the

$$D(\rho) \overset{\rho \to 0}{\propto} \frac{1}{\rho^5} e^{-\frac{8\pi^2}{g^2(\rho)}} \propto \left\{ \frac{1}{\rho} \right\}^{-(\frac{11N}{3}-5)} \tag{13}$$

factor that dominates the weight as $\rho \to 0$. This has consequences for the numerical calculation. Since the Monte Carlo deforms the fields 'continuously' (one link matrix at a time) the only way to change Q is to shrink an (anti)instanton out of the lattice (or the reverse). Thus the probability to change Q is linked to the probability to find an instanton of size $\rho \sim$ few$a$ and so is suppressed as $\propto a^{11N/3-5}$. So for large N a lattice calculation gets stuck in a fixed topological sector [25]. You could then try to calculate $\langle Q^2 \rangle$ from a large sub-volume of a very large lattice volume.

Larger instantons are more ambiguous but Fig. 8 certainly suggests the interesting possibility

$$D(\rho) \overset{N \to \infty}{\longrightarrow} \delta(\rho - \rho_c) \quad ; \quad \rho_c \sim \frac{1}{T_c} \tag{14}$$

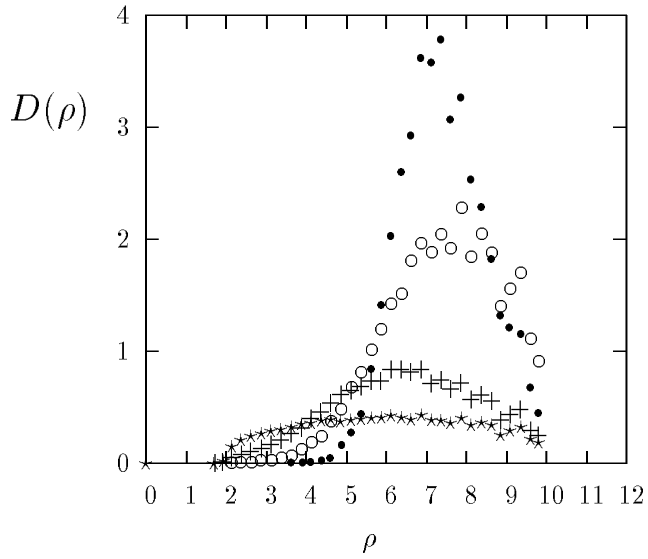

Figure 8. The instanton size density, $D(\rho)$, for $N = 2(\star), 3(+), 4(\circ), 8(\bullet)$ on 16^4 lattices with $a \simeq 1/8T_c$.

Usual counting tells us that the physics should be a function of θ/N and so periodic in $2\pi N$. However Q integer tells us it is periodic in 2π for all N. This implies [11] that there are in fact N vacua that intertwine as θ varies. At $\theta = 0$ there are N 'vacua' which become stable (exponentially fast) as $N \to \infty$. One can see these as a continuation from $\mathcal{N} = 1$ SUSY of the N degenerate vacua associated with gluino condensation [12]. As a first step, on the lattice side, it has been shown [30] that the moments of Q at $\theta = 0$ do suggest a vacuum periodicity of $2\pi N$ rather than 2π. What one would like to see are the near-stable 'vacua' directly. It is easy to calculate their energies

$$E_n(\theta = 0) = \frac{1}{2}\chi_t(2\pi n)^2 V \quad ; \quad N \to \infty \tag{15}$$

where $\chi_t \simeq 0.39\sqrt{\sigma} \simeq 2T_c/3$. Thus sometimes one might tumble into an $n \neq 0$ vacuum from the deconfined phase at T_c, but it should be easier from the bulk phase. This could be done on the lattice.

10. Some directions ...

It would be very nice to have the spectrum of $QCD_{N=\infty}$ with light quarks: glueballs and mesons, ground state and excited, all stable and non-mixing. Since this can be approached through extrapolating quenched theories in N, this is feasible.

As I pointed out, the intertwining θ-vacua might well be directly accessible in lattice calculations. It would be nice to follow them back to $\mathcal{N}=1$ SUSY by varying, say, a gluino mass parameter.

Space-time reduction and the non-analyticities at $N=\infty$ might have important phenomenological and theoretical implications as conjectured earlier in this talk.

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