FLUIDS, FLOWS AND COMPLEXITY

PROBLEM SET 2

Julia Yeomans

Comments and corrections to julia.yeomans@physics.ox.ac.uk please.

1. **Rankine vortex**

   (a) A rotating impeller of radius a spins up a steady, two-dimensional vortex in a large container of water, in which the flow consists of a core of uniform angular velocity Ω for $r < a$ and irrotational flow with velocity given by $u_\theta = \Omega a^2/r$ for $r > a$. Determine the profile of vorticity $\omega(r)$ across the vortex and sketch the variations of $u_\theta(r)$ and $\omega(r)$. You may neglect viscous damping.

   (b) Use the Navier-Stokes equations in (cylindrical polars) to show that the difference in pressure between the centre of the vortex core and $r \to \infty$ is $-\rho\Omega^2 a^2$. Can you use Bernoulli's theorem for this calculation?

   (c) The water layer is bounded above by a free surface. Obtain an expression for the height $h(r)$ of the free surface across the vortex, and sketch the variation of h with r. Determine the change in the level of the free surface between the vortex core and large r for an impeller of radius 5 cm located at the bottom of the tank and rotating at 120 revolutions per minute. The container is 1.5 m deep. What is the maximum rotation rate of the impeller if the upper surface of the water is not to expose the impeller to the air?

2. **The Bernoulli function**

   (a) Starting from the Navier-Stokes equation, show that for incompressible flow

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla H = \mathbf{u} \times \omega + \nu \nabla^2 \mathbf{u}, \tag{1}$$

   where $H = |\mathbf{u}|^2/2 + p/\rho$ is the Bernoulli function and $\omega = \nabla \times \mathbf{u}$ is the vorticity. Hence determine the conditions under which H is invariant along streamlines. Under what conditions is H constant everywhere?

   (b) Form a vorticity equation from (1), and hence show that vorticity is conserved along streamlines if the flow is steady, two-dimensional, inviscid and incompressible.

   (c) How high can water rise up one's arm hanging in the river from a punt?

   [Assume the following vector identities:

$$(\mathbf{F}.\nabla)\mathbf{F} = \frac{1}{2}\nabla|\mathbf{F}|^2 - \mathbf{F} \times (\nabla \times \mathbf{F})$$

$$\nabla \times (\nabla^2 \mathbf{F}) = \nabla^2 (\nabla \times \mathbf{F}) \quad]$$

3. **Circulation and Lift**

   Consider an irrotational, inviscid, incompressible two-dimensional flow with clockwise swirl around a cylinder of radius a. The flow is defined by the velocity potential,

$$\varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta - A\theta, \qquad r > a.$$

(a) Find expressions for the velocity components perpendicular and parallel to the walls of the cylinder.

(b) Using Bernoulli's equation, show that the pressure on the cylinder is

$$p = \text{constant} - 2\rho U^2 \sin^2\theta - \frac{2\rho U A}{a}\sin\theta.$$

(c) Show that the lift force on the cylinder is

$$L = -\rho U \Gamma_C$$

where Γ_C is the circulation around a circle lying just outside the cylinder.

4. **Surface waves**

(a) Small-amplitude waves are generated at the surface of a shallow channel filled with water to depth h. For two-dimensional, inviscid, irrotational flow in the water layer, the boundary conditions satisfied are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at the surface } y = h, \text{ and}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at the bottom of the channel at } y = 0,$$

where η is the vertical displacement of the surface and ϕ the velocity potential in the water layer. Give a physical justification for these boundary conditions.

(b) Given that ϕ satisfies Laplace's equation in x and y, show that the waves satisfy the dispersion relation

$$\omega^2 = gk \tanh kh. \tag{1}$$

(c) Show that, for wavelengths for which kh is small but finite, this dispersion relation may be approximated by

$$\omega \simeq c_0 \left(k - \frac{k^3h^2}{6}\right),$$

where $c_0 = (gh)^{1/2}$.

(d) The Korteweg–de Vries equation

$$\frac{\partial \eta}{\partial t} + c_0\frac{\partial \eta}{\partial x} + \left(\frac{3c_0}{2h}\right)\eta\frac{\partial \eta}{\partial x} + \frac{c_0h^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0$$

is a model equation for weakly dispersive, nonlinear waves on the surface of a channel of water of depth h and surface elevation η. Without detailed mathematical derivation, give a brief physical justification of the origin of each term and its effect on the resultant dispersion relation and shape of the solution.

(e) A possible (soliton) solution to this equation is of the form

$$\eta = \frac{2hV}{c_0}\operatorname{sech}^2\left[\frac{1}{2}\left\{\frac{V}{\sigma}\right\}^{1/2}\{x - (c_0 + V)t\}\right],$$

where $\sigma = c_0h^2/6$ and V is a parameter. Sketch the form of this solution, clearly labelling its amplitude and extent in x, and indicating its propagation speed. How do the properties of this solution differ from those of small-amplitude linear waves which satisfy the dispersion relation (1)?

(f) Scott Russell first observed the formation of such a soliton wave on the surface of a canal in 1834 when a barge stopped abruptly, and determined its amplitude a and propagation speed C to be $a \approx 0.3\,\text{m}$ and $C \approx 3.5\,\text{m}\,\text{s}^{-1}$. Estimate the depth h of the canal.

5. **Stokes drag**

The flow and pressure field for Stokes flow (Re=0) past a sphere of radius a, in spherical polar co-ordinates with z along the direction of the incoming flow, are

$$u_r = u_0 \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right),$$

$$u_\theta = -u_0 \sin\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right),$$

$$p = p_0 - \frac{3\eta u_0 a}{2r^2}\cos\theta.$$

(a) Sketch the flow field.
(b) Confirm that the solution obeys (i) the Stokes equations (ii) the correct boundary conditions.
(c) The components of the viscous stress tensor in spherical polar co-ordinates are:

$$\sigma_{rr} = -p + 2\eta\frac{\partial u_r}{\partial r}$$

,

$$\sigma_{\theta\theta} = -p + 2\eta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right),$$

$$\sigma_{\varphi\varphi} = -p + 2\eta\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right),$$

$$\sigma_{r\theta} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right),$$

$$\sigma_{r\varphi} = \eta\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right),$$

$$\sigma_{\theta\varphi} = \eta\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\right).$$

Calculate the viscous stress tensor at $r = a$ and hence identify the stress (force per unit area) acting on the sphere.
(d) By integrating over the sphere derive Stokes law for the drag D on the sphere:

$$D = 6\pi\eta a u_0.$$

(e) Assuming Stokes law, calculate the terminal velocity of a raindrop of radius 1mm falling in air. Discuss whether the use of Stokes law is in fact valid in this case.

6. **Thin film approximation**

A circular disk of radius a initially sticks to a flat ceiling at $z = 0$ by means of a thin film of viscous, incompressible liquid of dynamic viscosity η and thickness $h(t) \ll a$. Effects of surface tension may be neglected.

(a) Given that h and $W(t) = dh/dt$ are very small, explain briefly why the radial velocity u_r in the thin film obeys the approximate equation in cylindrical polar coordinates

$$\frac{\partial p}{\partial r} = \eta\frac{\partial^2 u_r}{\partial z^2}.$$

(b) Show that $p \approx p(r,t)$, and that u_r is given in this approximation by

$$u_r = \frac{1}{2\eta}\frac{\partial p}{\partial r} z(z+h).$$

(c) By integrating the incompressibility condition downwards in z from $z=0$ to $z=-h$, show that

$$W = -\frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right).$$

(d) Hence show that the pressure distribution in the fluid film above the disk is given by

$$p - p_0 = \frac{3\eta W}{h^3}(a^2 - r^2),$$

where p_0 is atmospheric pressure.

(e) Show that the total downwards force on the disk needed to pull it away from the ceiling at speed W is

$$F = \frac{3\pi}{2}\frac{\eta a^4 W}{h^3}.$$

(f) Spiderman, of mass 50 kg, wishes to hang suspended from the ceiling by being attached to a circular disk of radius 10 cm on the underside of a thin film of a fluid of dynamic viscosity 300 kg m^{-1} s^{-1}. The film has an initial thickness 1 mm, and initially fills the entire space between the disk and the ceiling. The volume of the film V remains constant at $V = \pi a_0^2 h_0$, where a_0 and h_0 are the initial radius and thickness of the film. Estimate the length of time spiderman can remain attached to the ceiling. Comment on the likely accuracy of your estimate for a real fluid film (a real man?).