YEAR 2: ELECTRICITY AND MAGNETISM

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PROBLEM SET 5: GUIDED WAVES

* Standard bookwork, included because you need to understand and learn this material, but your tutor may not want it handed in.

† Trickier problems, for people who have finished all the others.

A. <u>Transmission lines</u>

1.* (a) Derive the 'Telegraph Equations' for a transmission line

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI, \quad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV$$

where the symbols have their usual meanings, defined per unit length of line.

(b) For a loss-free line ($G = R = 0$) show that V and I satisfy the wave equation and determine the wave velocity v.

(c) If $V(x,t) = f(x - vt) + g(x + vt)$ find I, and the characteristic impedance of the line.

2. A transmission line consists of

(a) an air-filled coaxial cable of inner radius a and outer radius b.

(b) a pair of cylindrical conductors of radius a and separation d, in air ($a \ll d$).

(c) a cylindrical conductor of radius a at a distance d in air from a conducting plane ($a \ll d$).

In each case derive the capacitance per unit length C and the inductance per unit length L and check that $1/\sqrt{LC} = c$. Explain why you obtain this value.

3. If Z_1 and Z_2 are the input impedances of a given length l of transmission line when terminated by an open or closed circuit respectively, show that $Z_1 Z_2 = Z_0^2$, independent of l, where Z_0 is the characteristic impedance of the line.

4. A transmission line of characteristic impedance Z_1 is to be matched to another of impedance Z_3 (ie the connection should not generate a reflected wave). Show that this is possible if the connection is a quarter wavelength section of line of impedance $Z_2 = \sqrt{Z_1 Z_3}$. (cf Set 4, Q2)

5. A wave travels along a loss-free transmission line of impedance Z_1 which is terminated by a load of impedance Z_2. Show that a fraction

$$\frac{4Z_1 \ Re(Z_2)}{\mid Z_1 + Z_2 \mid^2}$$

of the incident power is transmitted into the load.

6. A leak develops at one point in an infinite transmission line. The resistance of the leak is equal to the characteristic impedance of the line. Show that $1/9$ of the power in the incident wave is reflected and $4/9$ is dissipated in the leak.

B. Rectangular Waveguides†

7. (a) Derive the equations for the longitudinal components of the electric and magnetic fields in a hollow, loss-free waveguide of constant cross section.
(b) Work out the theory of TE modes for a loss-free rectangular waveguide of dimensions $a \times b, \quad a \geq b$. In particular, find the longitudinal magnetic field, the cutoff frequencies, and the phase and group velocities.
(c) For TM modes find the longitudinal electric field and the cutoff frequencies. Show that the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given waveguide, is

$$\frac{\omega_{min}^{TM}}{\omega_{min}^{TE}} = \left(1 + \frac{a^2}{b^2}\right)^{1/2} .$$

(d) Why can TEM modes not propagate in this (or any other hollow) waveguide?

8. Consider the resonant cavity produced by closing off the two ends of a rectangular wave guide at $z = 0$ and $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by

$$\omega_{lmn} = v\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}$$

for integers l, m and n. Find the associated electric and magnetic fields.