COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

PROBLEM SET 1

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1. Change to polar form

(i) $-i$, (ii) $\frac{1}{2} - \frac{\sqrt{3}i}{2}$, (iii) $-3 - 4i$, (iv) $1 + i$, (v) $1 - i$, (vi) $(1 + i)/(1 - i)$.

2. For (a) $z_1 = 1 + i,\ z_2 = -3 + 2i$ and (b) $z_1 = 2e^{\frac{i\pi}{4}},\ z_2 = e^{\frac{-3i\pi}{4}}$ find

(i) $z_1 + z_2$, (ii) $z_1 - z_2$, (iii) $z_1 z_2$, (iv) z_1/z_2, (v) $|z_1|$, (vi) z_1^*.

3. For $z = x + iy$ find the real and imaginary parts of

(i) z^2, (ii) $1/z$, (iii) i^{-5}, (iv) $(2 + 3i)/(1 + 6i)$, (v) $e^{\frac{i\pi}{6}} - e^{-\frac{i\pi}{6}}$.

4. Draw in the complex plane.

(i) $3 - 2i$,
(ii) $4e^{-\frac{i\pi}{6}}$,
(iii) $|z - 1| = 1$,
(iv) $\text{Re}\,(z^2) = 4$,
(v) $\arg\,(z + 3i) = \pi/4$,
(vi) $|z + 1| + |z - 1| = 8$,
(vii) $z = te^{it}$ for real values of the parameter t,
(viii) $\arg\left(\frac{z-4}{z-1}\right) = \frac{3\pi}{2}$.

5. Use de Moivre's theorem to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Deduce that

$$\cos(\pi/8) = \left(\frac{2 + \sqrt{2}}{4}\right)^{1/2}$$

and write down an expression for $\cos(3\pi/8)$.

6. Express $\sin^6\theta$ as a sum of terms in $\cos n\theta$ for integer n.

7. Find (i) $(1 + i)^9$, (ii) $(1 - i)^9/(1 + i)^9$.

8. Find all the values of the following roots

(i) $\sqrt[4]{\frac{-1-\sqrt{3}i}{2}}$,

(ii) $(-8i)^{\frac{2}{3}}$,

(iii) $\sqrt[8]{16}$.

9. (i) Show that the sum of the n n^{th} roots of any complex number is zero.

(ii) By considering the roots of $z^{2n+1} + 1 = 0$, with n a positive integer, show that

$$\sum_{k=-n}^{n} \cos\left(\frac{2k+1}{2n+1}\right)\pi = 0.$$

10. Find the roots of the equation $(z-1)^n + (z+1)^n = 0$. Hence solve the equation $x^3 + 15x^2 + 15x + 1 = 0$.

11. Show that

$$\sum_{n=0}^{\infty} 2^{-n} \cos n\theta = \frac{1 - \frac{1}{2}\cos\theta}{\frac{5}{4} - \cos\theta}.$$

12. Prove that

$$\sum_{r=1}^{n} {}^nC_r \sin 2r\theta = 2^n \sin n\theta \cos^n \theta.$$

Hint: express the left-hand side as $\text{Im}\left\{e^{in\theta} \sum_{r=1}^{n} {}^nC_r e^{i(2r-n)\theta}\right\}$.

13. Find the real and imaginary parts of:

(i) $e^{3ln2-i\pi}$, (ii) $\ln i$, (iii) $\ln(-e)$, (iv) $(1+i)^{iy}$, (v) $\sin(i)$,
(vi) $\cos(\pi - 2i\ln 3)$, (vii) $\tanh(x+iy)$, (viii) $\tan^{-1}(\sqrt{3}i)$, (ix) $\sinh^{-1}(-1)$.

Answers

1. (i) $e^{\frac{-i\pi}{2}}$, (ii) $e^{\frac{-i\pi}{3}}$, (iii) $5e^{i\theta}, \theta = \tan^{-1}\frac{4}{3} + \pi$,
(iv) $\sqrt{2}e^{\frac{i\pi}{4}}$, (v) $\sqrt{2}e^{\frac{-i\pi}{4}}$, (vi) $e^{\frac{i\pi}{2}}$

2. a)
(i) $-2+3i$
(ii) $4-i$
(iii) $-5-i$
(iv) $-(1+5i)/13$
(v) $\sqrt{2}$
(vi) $1-i$

b)
(i) $(1+i)/\sqrt{2}$
(ii) $(3+3i)/\sqrt{2}$
(iii) $-2i$
(iv) -2
(v) 2
(vi) $\sqrt{2}(1-i)$

3. (i) $x^2 - y^2 + 2ixy$, (ii) $(x-iy)/(x^2+y^2)$, (iii) $-i$, (iv) $(20-9i)/37$, (v) i.

4.(i) point: (3,-2)
(ii) point with polar coordinates $r = 4, \theta = -\pi/6$
(iii) circle: centre $(1, 0)$, radius 1
(iv) rectangular hyperbola: $x^2 - y^2 = 4$
(v) line: $y = x - 3, \quad x > 0$
(vi) ellipse: $x^2/16 + y^2/15 = 1$
(vii) anticlockwise spiral starting from origin
(viii) semicircle: centre $(5/2, 0)$, radius $3/2, \quad y < 0$

6. $(10 - 15\cos 2x + 6\cos 4x - \cos 6x)/32$

7. (i) $2^4(1+i)$ (ii) $-i$

8. (i) $e^{i(\frac{\pi}{3}+\frac{n\pi}{2})} \quad n = 0, 1, 2, 3$
(ii) $4e^{i(\pi+\frac{4}{3}n\pi)} \quad n = 0, 1, 2$
(iii) $\pm\sqrt{2}, \pm\sqrt{2}i, \pm1 \pm i$

10. $z = i\cot[(2k+1)\pi/2n] \quad (k = 0, \ldots, n-1)$,
$x = -\cot^2[(2k+1)\pi/12] \quad (k = 0, 1, 2)$.

13. (i) -8, (ii) $\frac{4n+1}{2}\pi i$, (iii) $1 + i(2n+1)\pi$,
(iv) $\{\cos(y\ln\sqrt{2}) + i\sin(y\ln\sqrt{2})\}e^{-y(\frac{\pi}{4}+2n\pi)}$,
(v) $i\sinh 1$, (vi) $\frac{-41}{9}$,
(vii) $\frac{\tanh x\sec^2 y + i\tan y sech^2 x}{1+\tanh^2 x\tan^2 y}$,
(viii) $\frac{(2n+1)\pi}{2} + \frac{i}{2}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$, (ix) $2n\pi i + \ln(\sqrt{2}-1); (2n+1)\pi i + \ln(\sqrt{2}+1)$