3. ELECTROMAGNETIC WAVES

A. PLANE WAVES IN DIELECTRICS

1. Maxwell's equations in dielectrics
2. wave equation
3. plane wave solutions

(a) waves are transverse

(b) polarization

(c) dispersion relation

(d) ratio of amplitudes of $\vec{E}$ and $\vec{B}$ fields (impedance)

B. PLANE WAVES IN CONDUCTORS

1. Maxwell's equations
2. good and poor conductors

(a) decay time of charge distribution vs period of em wave

(b) conduction vs displacement current

(c) numbers

3. wave equation
4. plane wave solutions

(a) waves are transverse

(b) polarization

(c) dispersion relation

(d) skin depth

(e) ratio of amplitudes of $\vec{E}$ and $\vec{B}$ fields (impedance)

(f) good and poor conductor limits

3. Electromagnetic Waves

A. Plane Waves in Dielectrics

Maxwell's equations

$$\boxed{\begin{aligned} \text{div } \underline{D} &= \rho \\ \text{div } \underline{B} &= 0 \\ \text{curl } \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \text{curl } \underline{H} &= \underline{J} + \frac{\partial \underline{D}}{\partial t} \end{aligned}}$$

(1) Maxwell's equations in dielectrics (ie. insulators)

for dielectrics: no free current, no free charge; material linear, isotropic, homogeneous

linear ↙ $\underline{B} = \mu\mu_0 \underline{H}$, $\underline{D} = \epsilon\epsilon_0 \underline{E}$

isotropic ↓ μ, ϵ independent of direction

homogeneous ↘ μ, ϵ independent of position

Maxwell's equations become

$$\text{div } \underline{E} = 0 \qquad \text{①}$$

$$\text{div } \underline{B} = 0 \qquad \text{②}$$

$$\text{curl } \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \text{③}$$

$$\text{curl } \underline{B} = \mu\mu_0\epsilon\epsilon_0 \frac{\partial \underline{E}}{\partial t} \qquad \text{④}$$

(2) Wave equation

curl ③

$$\text{curl curl } \underline{E} \equiv \underbrace{\text{grad div } \underline{E}}_{\text{zero using ①}} - \nabla^2 \underline{E} = \underbrace{-\frac{\partial}{\partial t} \text{curl } \underline{B}}_{\text{using ④}\ \ -\mu\mu_0\epsilon\epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}}$$

$$\therefore \boxed{\nabla^2 \underline{E} = \mu\mu_0\epsilon\epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}}$$

(taking curl ④ gives an identical equation for $\underline{B}$)

wave equation: solutions have a speed of propogation

$$v = \frac{1}{(\mu\mu_0\epsilon\epsilon_0)^{1/2}}$$

in free space $v \equiv c = \frac{1}{(\mu_0\epsilon_0)^{1/2}}$

velocity of light, which is part of the e.m spectrum

refractive index n defined by

$$n = \frac{c}{v} = (\mu\epsilon)^{1/2}$$

(NB in general n, ϵ depend on ω)

(3) plane wave solutions

choose $\hat{\underline{z}}$ to define direction of propagation

plane wave means $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \equiv 0$

(a) plane waves are transverse

① $\Longrightarrow \frac{\partial E_z}{\partial z} = 0$ ② $\Longrightarrow \frac{\partial B_z}{\partial z} = 0$

③ $\Longrightarrow \frac{\partial B_z}{\partial t} = 0$ ④ $\Longrightarrow \frac{\partial E_z}{\partial t} = 0$

$\therefore$ all derivatives of E_z, B_z zero $\therefore$ E_z, B_z constant and not part of any wave motion $\therefore$ wave transverse

(b) polarization

x and y components of ③

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad \text{(5a)}, \qquad \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad \text{(5b)}$$

x and y components of ④

$$-\frac{\partial B_y}{\partial z} = \mu\mu_0\epsilon\epsilon_0\frac{\partial E_x}{\partial t} \quad \text{(6a)}, \qquad \frac{\partial B_x}{\partial z} = \mu\mu_0\epsilon\epsilon_0\frac{\partial E_y}{\partial t} \quad \text{(6b)}$$

solving (5a), (6b) will give a sol^{n} for B_x, E_y

" (5b), (6a) " " " " " B_y, E_x

two independent solutions with $\underline{B}$, $\underline{E}$ and the direction of propagation $(\hat{\underline{z}})$, mutually orthogonal

The direction of the $\underline{E}$ - field is taken to define the polarization of the wave. Here we have two plane polarised solutions.

(c) dispersion relation

For plane waves of a single frequency ω, with $\underline{E}$ polarised along $\hat{\underline{x}}$

wave travelling towards +ve $\hat{\underline{z}}$

$$\underline{E}(z,t) = E_0 e^{j(\omega t \mp kz)} \quad \hat{\underline{x}}$$

wave travelling towards -ve $\hat{\underline{z}}$

$$\underline{B}(z,t) = B_0 e^{j(\omega t \mp kz)} \quad \hat{\underline{y}}$$

(5b) $\Rightarrow$ $\mp k E_0 = -\omega B_0$ ⑦

(6a) $\Rightarrow$ $\pm k B_0 = \mu\mu_0\epsilon\epsilon_0 \omega E_0$ ⑧

these equations are consistent if

$$\frac{\omega^2}{k^2} = \frac{1}{\epsilon\epsilon_0\mu\mu_0} \equiv v^2$$

speed $\therefore \underline{v = \frac{1}{(\mu\mu_0\epsilon\epsilon_0)^{1/2}}}$

(as we already knew; from properties of the wave equation)

(d) ratio of the amplitudes of $\underline{E}$ and $\underline{B}$ fields

from ⑦ or ⑧

$$\frac{E_0}{B_0} = \pm\frac{\omega}{k} = \pm\frac{1}{\sqrt{\mu\mu_0\epsilon\epsilon_0}}$$

Impedance $Z \equiv \frac{E_0}{H_0} = \frac{\mu\mu_0 E_0}{B_0} = \pm\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$

wave moving towards $+\underline{\hat{z}}$ Z +ve
$-\underline{\hat{z}}$ Z −ve

Impedance of free space $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega$

B. Plane Waves in Conductors

(1) Maxwell's equations

no free charge

linear, isotropic, homogeneous material $\therefore D = \epsilon\epsilon_0 E$; $B = \mu\mu_0 \underline{H}$

$\underline{J} = \sigma \underline{E}$ Ohm's law

↖ conductivity (material property)

Maxwell's equations become

$$\text{div } \underline{E} = 0 \qquad ①$$

$$\text{div } \underline{B} = 0 \qquad ②$$

$$\text{curl } \underline{E} = -\frac{\partial B}{\partial t} \qquad ③$$

$$\text{curl } \underline{B} = \mu\mu_0 \sigma \underline{E} + \mu\mu_0 \epsilon\epsilon_0 \frac{\partial E}{\partial t} \qquad ④$$

(2) Good and poor conductors

(a) decay time of an inhomogenous charge distribution vs period of e.m. wave

If there is a charge distribution $\rho_f(0)$ in a conductor at $t=0$, how quickly does it decay to zero?

continuity equation

$$\operatorname{div} \underline{J} = -\frac{\partial \rho}{\partial t}$$

$$\xrightarrow{\text{Ohm's law}}$$

$$\sigma \operatorname{div} \underline{E}$$

$$\Big\downarrow \text{ Gauss' law}$$

$$\frac{\sigma \rho}{\epsilon \epsilon_0} = -\frac{\partial \rho}{\partial t}$$

N.B.
σ conductivity
ρ volume charge density

$$\therefore \rho(t) = \rho(0)\, e^{-\frac{\sigma t}{\epsilon \epsilon_0}}$$

characteristic decay time $\tau = \dfrac{\epsilon \epsilon_0}{\sigma}$

σ large $\Rightarrow$ τ small ✓

ϵ large $\Rightarrow$ field gradients smaller $\Rightarrow$ τ longer ✓

if an e.m. wave of frequency ω is passing through the conductor the other time scale is ω^{-1}

charge distribution can keep up with fields $\omega^{-1} \gg \tau$ $\qquad \dfrac{\sigma}{\omega \epsilon \epsilon_0} \gg 1$ $\qquad$ good conductor

charge distribution can't keep up with fields $\omega^{-1} \ll \tau$ $\qquad \dfrac{\sigma}{\omega \epsilon \epsilon_0} \ll 1$ $\qquad$ poor conductor

(b) conduction vs displacement currents

comparing to Maxwell ④

$$\therefore \text{curl } \underline{B} = \mu\mu_0 \sigma \underline{E} + \mu\mu_0 \epsilon\epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\underset{\text{conduction current}}{\uparrow} \qquad \underset{\text{displacement current}}{\uparrow}$$

for a wave $\sim e^{j\omega t}$

$$\frac{\text{conduction current}}{\text{displacement current}} \sim \frac{\sigma}{\epsilon\epsilon_0\omega}$$

∴ good conductor conduction current >> displacement current

poor conductor " << "

(c) numbers

crossover frequency between good and poor conductor $\omega^* \sim \frac{\sigma}{\epsilon\epsilon_0}$

	σ (Ωm)	ϵ	ω^* (s^{-1})
typical metal	5×10^7	~ 1	10^{19}
graphite	7×10^4	~ 1	10^{16}
salt water	20	80	10^{10}
silicon	4×10^{-4}	12	10^7
pure water	4×10^{-6}	80	10^4
typical insulator (e.g. wood, glass)	10^{-10}	5	10

(3) wave equation

curl ③

$$\text{curl curl } \underline{E} = \underset{\uparrow}{\text{grad div } \underline{E}} - \nabla^2 \underline{E} = -\frac{\partial}{\partial t} \text{curl } \underline{B}$$

zero using ①

using ④

$$-\mu\mu_0 \sigma \frac{\partial E}{\partial t} - \mu\mu_0 \epsilon\epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\therefore \boxed{\nabla^2 \underline{E} = \mu\mu_0 \sigma \frac{\partial \underline{E}}{\partial t} + \mu\mu_0 \epsilon\epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}} \qquad ⑤$$

plane wave sol^ns still exist but the refractive index / wavevector } is complex ....

(4) plane wave solutions

(a) waves transverse

(b) two independent polarizations with $\underline{E}$, $\underline{B}$ and the direction of propagation mutually perpendicular.

→ proof ≈ identical to dielectrics

(c) dispersion relation

Consider waves moving in +ve z-direction with $\underline{E}$ polarised along $\hat{\underline{x}}$

$$\underline{E}(z,t) = E_0 e^{j(\omega t - \tilde{k} z)} \hat{\underline{x}}$$

In general $\tilde{k}$ is complex $\therefore$ write $\tilde{k} = k - j\kappa$

$$\therefore \underline{E}(z,t) = E_0 e^{-\kappa z} e^{j(\omega t - kz)} \hat{x} \qquad ⑥$$

$$\underline{B}(z,t) = B_0 e^{-\kappa z} e^{j(\omega t - kz)} \hat{y}$$

definitions: $v = \frac{\omega}{k}$

refractive index $n = \frac{ck}{\omega}$

complex refractive index $\tilde{n} = \frac{c\tilde{k}}{\omega}$

skin depth $\delta = \kappa^{-1}$ (length over which amplitude decays by a factor e^{-1})

subst. ⑥ into the wave eq$^{\underline{n}}$ ⑤

$$(\kappa + jk)^2 = j\mu\mu_0\sigma\omega - \mu\mu_0\epsilon\epsilon_0\omega^2$$

$$\therefore \kappa^2 - k^2 = -\mu\mu_0\epsilon\epsilon_0\omega^2 \quad ⑦ \qquad 2k\kappa = \mu\mu_0\sigma\omega \quad ⑧$$

subst. ⑧ into ⑦ eliminating k to get a quadratic for κ^2; solve and take $\sqrt{}$ to give

$$\boxed{\begin{aligned} \kappa &= \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0\omega^2}{2}} \left\{ \sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0\omega}\right)^2} - 1 \right\}^{\frac{1}{2}} \qquad ⑨ \\ k &= \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0\omega^2}{2}} \left\{ \sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0\omega}\right)^2} + 1 \right\}^{1/2} \end{aligned}}$$

N.B. always take $\sqrt{}$ so that κ +ve

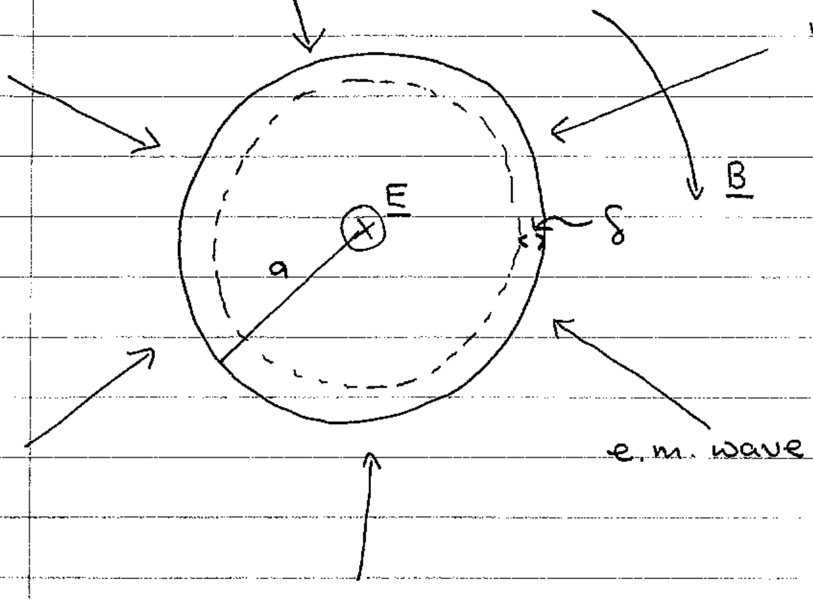
N.B. take $+\sqrt{}$ for wave travelling to $+\hat{\underline{z}}$

(d) **skin depth** $\delta = \kappa^{-1}$

For a typical metal in the visible $\delta \sim$ few nm

microwave $\sim \mu m$

radio $\sim$ mm



current carried very near surface of wire

N.B. 1. $\delta \sim \omega^{-1/2}$ for a good conductor ∴ d.c. limit sensible

2. resistance $R = \frac{l}{A\sigma}$ ∴ $\frac{R_{ac}}{R_{dc}} \sim \frac{A_{dc}}{A_{ac}} \sim \frac{\pi a^2}{2\pi a \delta} = \frac{a}{2\delta}$

∴ a.c. resistance >> d.c. resistance

3. for insulators ; δ independent of ω and >> dimensions of any sensible wire

(e) ratio of amplitudes of $\underline{E}$ and $\underline{B}$-fields

Maxwell ③, taking y-component of curl,

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

substituting in the plane wave sol^n ⑥

$$-(\kappa + jk)E_0 = -j\omega B_0$$

$$\therefore B_0 = \frac{(\kappa + jk)E_0}{j\omega}$$

$$Z = \frac{\mu\mu_0 E_0}{B_0} = \frac{\mu\mu_0\omega}{k - j\kappa} \equiv Z_0 e^{+j\phi}$$

complex impedance

where $Z_0 = \mu\mu_0 \left|\frac{E_0}{B_0}\right| = \frac{\mu\mu_0\omega}{(\kappa^2 + k^2)^{1/2}}$

$$\therefore \tan\varphi = \frac{\kappa}{k} = \left\{ \frac{\sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0\omega}\right)^2} - 1}{\sqrt{1 + \left(\frac{\sigma}{\epsilon\epsilon_0\omega}\right)^2} + 1} \right\}^{1/2}$$

(f) Limits

(i) 'good' conductor $\frac{\sigma}{\epsilon\epsilon_0\omega} \gg 1$

$$\therefore k = \kappa = \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0\omega^2\sigma}{2\ \epsilon\epsilon_0\omega}} = \sqrt{\frac{\mu\mu_0\omega\sigma}{2}}$$

IMPORTANT N.B.

to work out the dispersion relation or skin depth of a good conductor can derive the general formula ⑨ for κ, k and then approximate

but unless you need the general formula for some other reason it is much easier to approximate earlier e.g. in ⑤

$$⑤ \Rightarrow \nabla^2\underline{E} = \mu\mu_0\sigma\frac{\partial\underline{E}}{\partial t} + \cancel{\mu\mu_0\epsilon\epsilon_0\frac{\partial^2\underline{E}}{\partial t^2}}$$

↑ negligible for a good conductor

substituting in the plane wave solution ⑥

$$\underline{E}(z,t) = E_0 e^{-\kappa z} e^{j(\omega t - kz)}\ \hat{\underline{x}},$$

gives the good conductor dispersion relation

$$(\kappa + jk)^2 = j\mu\mu_0\sigma\omega$$

equating real and imaginary parts

$$\therefore \kappa^2 - k^2 = 0 \qquad (\text{cf } ⑦)$$

$$2k\kappa = \mu\mu_0\sigma\omega \qquad (\text{cf } ⑧)$$

$$\therefore \kappa = k = \sqrt{\frac{\mu\mu_0\sigma\omega}{2}} \qquad (\text{as before})$$

(ii) **'poor' conductor** $\frac{\sigma}{\epsilon\epsilon_0\omega} \ll 1$

$$K = \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0\omega^2}{2}}\left(1 + \frac{\sigma^2}{2(\epsilon\epsilon_0\omega)^2} - 1\right)^{1/2}$$

$$= \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0}{2 . 2}} . \omega . \frac{\sigma}{\epsilon\epsilon_0\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$$

skin depth of a poor conductor is independent of ω

$$k = \sqrt{\frac{\mu\mu_0\epsilon\epsilon_0}{2}} . \omega\sqrt{2}\left(1 + \frac{\sigma^2}{4(\epsilon\epsilon_0\omega)^2}\right)^{\frac{1}{2}}$$

$$= \omega\sqrt{\mu\mu_0\epsilon\epsilon_0}\left(1 + \underset{\uparrow}{\frac{\sigma^2}{8(\epsilon\epsilon_0\omega)^2}}\right)$$

can ignore cf

(iii) **dielectric** $\sigma = 0$ $K = 0$ ✓

$k = \omega . \frac{1}{v}$ ✓