II STEADY CURRENTS AND MAGNETISM

B. MAGNETIC MULTIPOLES

1. magnetic vector potential $\underline{A}$
2. summary of formulae for $\underline{B}$, $\underline{A}$, $\underline{J}$
3. multipole expansion of $\underline{A}$ and the magnetic dipole

B. MAGNETIC MULTIPOLES

1. Magnetic vector potential $\underline{A}$

$$\text{div } \underline{B} = 0$$

$\therefore$ can write $\underline{B} = \text{curl } \underline{A}$

$\underline{A}$ is not uniquely defined – can add any function with zero curl to it

one convenient choice is to ensure $\text{div } \underline{A} = 0$ 'Coulomb gauge'

How do I calculate $\underline{A}$ from $\underline{J}$?

$$\text{curl } \underline{B} = \mu_0 \underline{J} = \text{curl curl } \underline{A} = \text{grad div } \underline{A} - \nabla^2 \underline{A}$$

Ampère's law

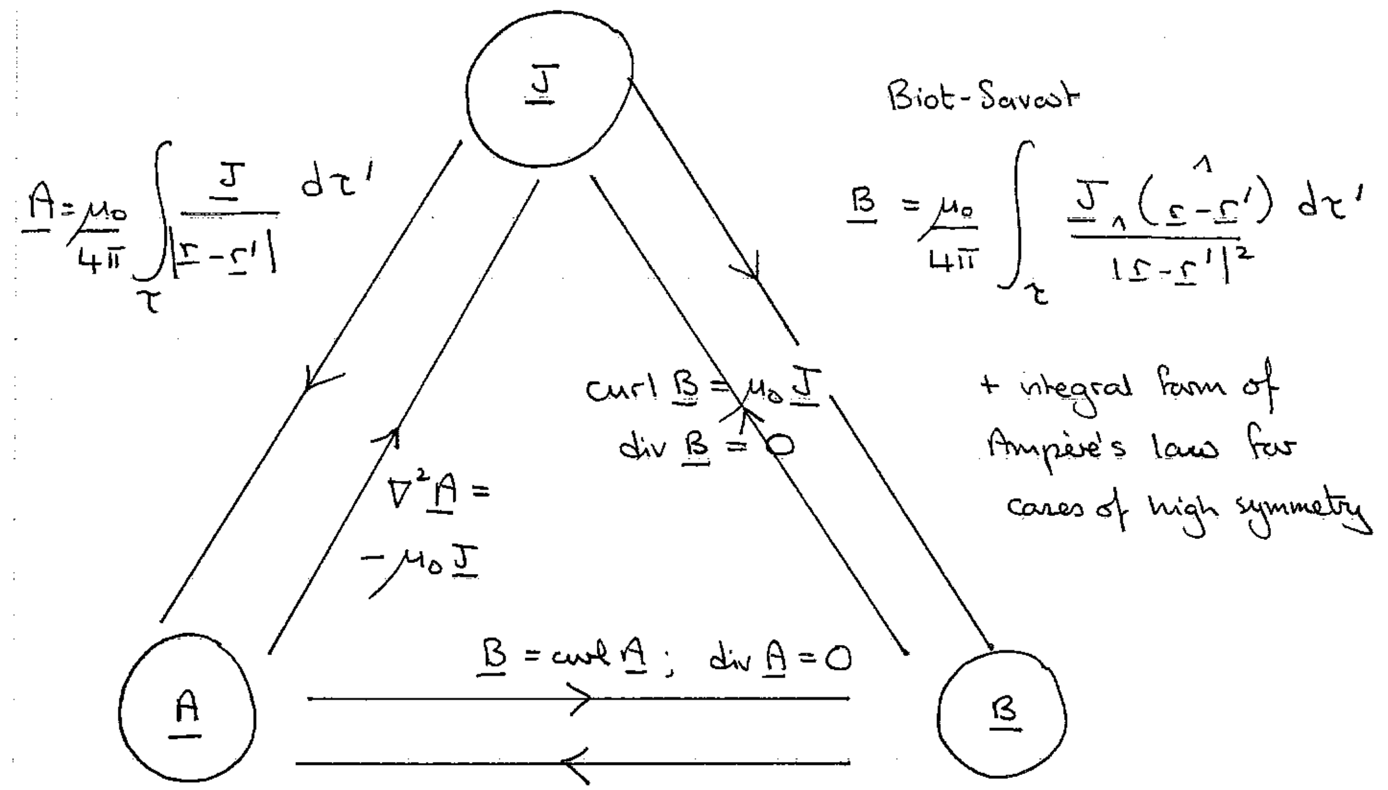
zero by construction

$$\therefore \nabla^2 \underline{A} = -\mu_0 \underline{J} \qquad \text{compare} \qquad \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

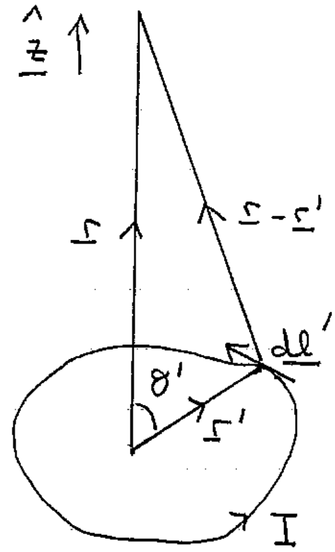
$$\Downarrow$$

$$\underline{A} = \frac{\mu_0}{4\pi} \int_\tau \frac{\underline{J}}{|\underline{r}-\underline{r}'|} \, d\tau' \Leftarrow \qquad V = \frac{1}{4\pi\varepsilon_0} \int_\tau \frac{\rho}{|\underline{r}-\underline{r}'|} \, d\tau'$$

2. Summary of links between $\underline{A}$, $\underline{J}$, $\underline{B}$



3. Multipole expansion of $\mathbf{A}$



recall

$$(\mathbf{r}-\mathbf{r}')^{-1} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

vector potential of a current loop

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{\mathbf{dl}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\,\mathbf{dl}'$$

<u>monopole term</u> $n=0$

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mu_0 I}{4\pi r}\oint \mathbf{dl}' = \underline{0} \qquad \text{as expected}$$

<u>dipole term</u> $n=1$

(z-axis along $\mathbf{r}$)

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta'\,\mathbf{dl}' = \frac{\mu_0 I}{4\pi r^2}\oint \mathbf{r}'\cdot\hat{\mathbf{r}}\;\mathbf{dl}'$$

V8 : with $\mathbf{a} = \hat{\mathbf{r}}$

$$= \frac{\mu_0 I}{4\pi r^2}\int_S \mathbf{dS}'\wedge\hat{\mathbf{r}} \equiv \frac{\mu_0}{4\pi r^2}\,\mathbf{m}\wedge\hat{\mathbf{r}}$$

where $\mathbf{m} = I\int_S \mathbf{dS}' = I\,\mathbf{S}$

magnetic dipole moment

current

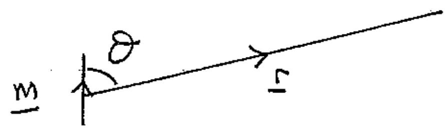
vector area of loop

to find the dipolar field $\underline{B}_1(\underline{r})$

(N.B. new problem, new co-ordinate system)

$\underline{m}$ along $\hat{\underline{z}}$; spherical polars

$$\underline{A}_1(\underline{r}) = \frac{\mu_0}{4\pi r^2} \underline{m} \wedge \hat{\underline{r}}$$



$$= \frac{\mu_0 m \sin\theta}{4\pi r^2} \hat{\underline{\varphi}}$$

$$\underline{B}_1(\underline{r}) = \text{curl } \underline{A}_1(\underline{r})$$

$$= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} (r\sin\theta A_\phi) \hat{\underline{r}} - \frac{1}{r\sin\theta} \frac{\partial}{\partial r} (r\sin\theta A_\phi) \hat{\underline{\theta}}$$

curl in sph. polars

(if $A_r, A_\theta = 0$)

$$= \frac{\mu_0 m}{4\pi} \left(\frac{2\cos\theta}{r^3} \hat{\underline{r}} + \frac{\sin\theta}{r^3} \hat{\underline{\theta}} \right)$$

same form as electric dipole

(N.B. for a real pair of charges | current loop fields only same at sufficiently large r)