ELECTROSTATICS

Polarisable Materials

0. vector identities

1. polarization; definition and physical origins
2. bound charge
3. Gauss' law in dielectrics and $\underline{D}$
4. linear dielectrics and ϵ
5. field due to a point charge in a linear dielectric
6. boundary conditions on $\underline{E}$, $\underline{D}$ at boundary between dielectrics
7. examples
   a. dielectric slab, uniform field
   b. dielectric slab, imposed polarization
   c. dielectric sphere, uniform field

$\left.\begin{array}{l} \text{a} \\ \text{b} \\ \text{c} \end{array}\right\}$ not covered in 2010-11 lectures

VECTOR IDENTITIES

V1. $\text{div}\,(f\underline{A}) = f\,\text{div}\,\underline{A} + \underline{A}\cdot\text{grad}\,f$

V2. $\text{div}\,(\underline{A}\wedge\underline{B}) = \underline{B}\cdot\text{curl}\,\underline{A} - \underline{A}\cdot\text{curl}\,\underline{B}$

V3. $(\underline{A}\cdot\text{grad})\,f = \text{div}\,(f\underline{A}) - f\,\text{div}\,\underline{A}$

V4. $\text{curl}\,(\underline{A}\wedge\underline{B}) = \underline{A}\,(\text{div}\,\underline{B}) - (\underline{A}\cdot\text{grad})\,\underline{B} + (\underline{B}\cdot\text{grad})\,\underline{A} - \underline{B}\,(\text{div}\,\underline{A})$

V5. $\text{grad}\,\dfrac{1}{r} = \dfrac{-\hat{\underline{r}}}{r^2}$

$$\text{grad}\,\frac{1}{|\underline{r}-\underline{r}'|} = -\frac{(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}$$

$$\text{grad}'\,\frac{1}{|\underline{r}-\underline{r}'|} = \frac{(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}$$

($'$ means take derivatives w.r.t. x', y', z')

V6. $\text{curl}\,\dfrac{\hat{\underline{r}}}{r^2} = 0$

V7. $\text{div}\,\dfrac{\hat{\underline{r}}}{r^2} = 4\pi\,\delta^3(\underline{r})$

V8. $\oint_C (\underline{a}\cdot\underline{r})\,\underline{dl} = \int_S \underline{dS}\wedge\underline{a}$ for any constant $\underline{a}$

(S is an open surface spanning the closed curve C)

Notes on the vectors identities:

V1 – V4 standard product theorems – prove by writing the vectors as components

V5

$$\text{grad } \frac{1}{r} = -\frac{1}{r^2} \text{ grad } r \equiv -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)$$

rule

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \qquad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\therefore \text{grad } r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \hat{\underline{r}}$$

$$\therefore \text{grad } \frac{1}{r} = -\frac{\hat{\underline{r}}}{r^2}$$

similarly

$$\text{grad } \frac{1}{|\underline{r}-\underline{r}'|} = -\frac{1}{|\underline{r}-\underline{r}'|^2} \text{ grad } |\underline{r}-\underline{r}'|$$

$$|\underline{r}-\underline{r}'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$\therefore 2|\underline{r}-\underline{r}'| \frac{\partial|\underline{r}-\underline{r}'|}{\partial x} = 2(x-x')$$

$$\frac{\partial|\underline{r}-\underline{r}'|}{\partial x} = \frac{(x-x')}{|\underline{r}-\underline{r}'|}$$

$$\therefore \text{grad } |\underline{r}-\underline{r}'| = (\widehat{\underline{r}-\underline{r}'})$$

if I calculate $\text{grad}' \frac{1}{|\underline{r}-\underline{r}'|}$, ie differentiate wrt the $'$ variables, I will pick up an extra – sign here

V7

$$\operatorname{div} \frac{\hat{\underline{r}}}{r^2} = \underbrace{\frac{1}{r^2} \frac{\partial}{\partial r}}_{\substack{\text{def}^{\underline{n}} \text{ of div in polars} \\ \text{if no } \theta, \varphi \text{ dependence}}} \left(r^2 \cdot \frac{1}{r^2} \right) = 0$$

unless $r = 0$

to find the value at $r=0$ consider

$$\int_{\text{sphere}} \operatorname{div} \frac{\hat{\underline{r}}}{r^2} \, d\tau = \int_{\substack{\text{surface} \\ \text{of sphere}}} \frac{\hat{\underline{r}}}{r^2} \cdot d\underline{S}$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{\hat{\underline{r}} \cdot \hat{\underline{r}}}{r^2} \cdot r^2 \sin\theta \, d\theta \, d\varphi = 4\pi$$

when integrate $\operatorname{div} \frac{\hat{\underline{r}}}{r^2}$ over a volume containing the origin

get 4π

$$\therefore \operatorname{div} \frac{\hat{\underline{r}}}{r^2} = 4\pi \, \delta^3(\underline{r})$$

V8

Stokes' thm.

$$\oint_c \underline{A} \cdot d\underline{l} = \int_S \operatorname{curl} \underline{A} \cdot d\underline{S}$$

let $\underline{A} = f \, \underline{b}$

scalar

constant vector

$$\therefore \oint_c f\,\underline{b}\cdot d\underline{l} = \int_s \operatorname{curl}(f\underline{b})\cdot d\underline{S}$$

$$= \int_s \{ f \underbrace{\operatorname{curl}\,\underline{b}}_{0 \text{ as } \underline{b} \text{ constant}} - \underline{b}\wedge \operatorname{grad} f\}\cdot d\underline{S}$$

$$\therefore \underline{b}\cdot \oint_c f\, d\underline{l} = \underline{b}\cdot \int_s d\underline{S}\wedge \operatorname{grad} f$$

true $\forall$ constant $\underline{b}$

$$\therefore \oint_c f\, d\underline{l} = \int_s d\underline{S}\wedge \operatorname{grad} f$$

put $f = \underline{a}\cdot\underline{r}$

$\operatorname{grad} f = \underline{a}$

$$\therefore \oint_c (\underline{a}\cdot\underline{r})\, d\underline{l} = \int_s d\underline{S}\wedge\underline{a}$$

D. Polarizable Materials

1. polarization: definition and physical origin

> When an insulator (dielectric) is put in an electric field $\underline{E}$ the field induces a dipole moment. This is measured by the polarization $\underline{P}$, defined as the dipole moment per unit volume.

An insulator (dielectric) has electrons 'held down' to their nuclei – cf conductors where electrons are free to move through the material.

Why does the field induce a dipole moment?

(i) neutral atoms

electrons are displaced relative to the nucleus

no field

with field

field pulls +ve charges →

Coulomb force pulls +ve charges ←

(field $\sim 10^4\,V\,m^{-1}$; displacement $\sim 10^{-18}\,m$)
∴ tiny effect

as long as $\underline{E}$ not too big, induced dipole moment $\propto \underline{E}$

$$\underline{p} = \alpha\, \underline{E}$$

atomic polarizability

(ii) polar molecules

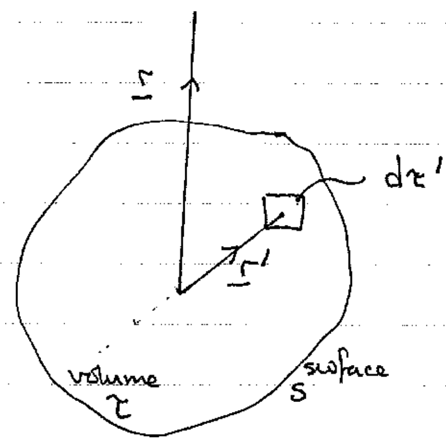
already have a dipole moment
in a field the moments will tend to point along the field
competing thermal effects will randomise the directions
small excess pointing along field and a polarisation $\propto \underline{E}$
(cf Curie's law for paramagnets)

2. bound charge

The potential (and $\therefore$ field) of an object with polarization $\underline{P}$ is the same as the potential produced by
a volume charge density $\rho_b = -\operatorname{div}\underline{P}$
plus a surface charge density $\sigma_b = \underline{P}\cdot\hat{\underline{n}}$

unit normal to the surface

proof



potential at $\underline{r}$ due to a dipole $\underline{p}$ at $\underline{r}'$

$$V_1(\underline{r}) = \frac{\underline{p}\cdot(\widehat{\underline{r}-\underline{r}'})}{4\pi\varepsilon_0|\underline{r}-\underline{r}'|^2}$$

$$\underline{p} = \underline{P}(\underline{r}')\,d\tau'$$

dipole moment per unit volume

$\therefore$ potential at $\underline{r}$ due to polarised object occupying τ is

$$V(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\underline{P}(\underline{r}')\cdot(\widehat{\underline{r}-\underline{r}'})}{|\underline{r}-\underline{r}'|^2}\,d\tau'$$

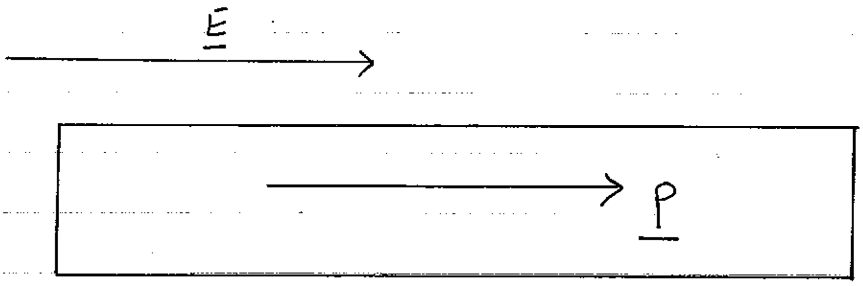
$$= \frac{1}{4\pi\varepsilon_0}\int_\tau \underline{P}(\underline{r}')\cdot \operatorname{grad}'\frac{1}{|\underline{r}-\underline{r}'|}\,d\tau' \qquad (V5)$$

$$= \frac{1}{4\pi\varepsilon_0}\int_\tau \operatorname{div}'\left\{\frac{\underline{P}(\underline{r}')}{|\underline{r}-\underline{r}'|}\right\}d\tau' - \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\operatorname{div}'\underline{P}(\underline{r}')}{|\underline{r}-\underline{r}'|}\,d\tau' \qquad (V1)$$

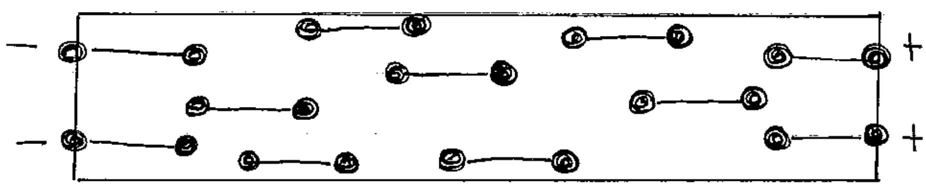
$$= \frac{1}{4\pi\varepsilon_0}\int_S \frac{\underline{P}(\underline{r}')\cdot\hat{\underline{n}}}{|\underline{r}-\underline{r}'|}\,dS' - \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\operatorname{div}'\underline{P}(\underline{r}')}{|\underline{r}-\underline{r}'|}\,d\tau'$$

potential of a surface charge density $\sigma_b = \underline{P}\cdot\hat{\underline{n}}$

potential of a volume charge density $\rho_b = -\operatorname{div}\underline{P}$



macroscopic



microscopic

$\rho_b = -\operatorname{div} \underline{P}$: only get a contribution if $\underline{P}$ varies with position otherwise +ve & −ve ends of dipoles cancel.

$\sigma_b = \underline{P} \cdot \hat{\underline{n}}$: charge builds up on the surfaces

3. Gauss' law in dielectrics and the definition of $\underline{D}$

inside a dielectric

$$\boxed{\mathrm{div}\,\underline{E} = \frac{\rho_f + \rho_b}{\epsilon_0}}$$

$$= \frac{\rho_f - \mathrm{div}\,\underline{P}}{\epsilon_0}$$

$$\therefore\ \mathrm{div}\,(\epsilon_0 \underline{E} + \underline{P}) = \rho_f$$

define $\boxed{\underline{D} = \epsilon_0 \underline{E} + \underline{P}}$ ① electric displacement, usually just called '$\underline{D}$'

$$\boxed{\mathrm{div}\,\underline{D} = \rho_f}$$

N.B. integral form is $\boxed{\int \underline{D}\cdot d\underline{S} = q_f}$

useful because formulae for $\underline{D}$ involve only free charge
formula for $\underline{E}$ must account for all (free and bound) charge

4. Linear dielectrics and the relative permittivity, ϵ

Linear: i.e. $\underline{P} \propto \underline{E}$ $\quad (\therefore \underline{D} \propto \underline{E})$

write $\underline{P} \equiv \epsilon_0 \chi_e \underline{E}$

χ_e: electric susceptibility

using ① $\boxed{\therefore \underline{D} = \epsilon_0 (1 + \chi_e)\underline{E} \equiv \epsilon_0 \epsilon \underline{E}}$ ②

ϵ: relative permittivity

from ① and ② $\boxed{\underline{P} = (\epsilon - 1)\epsilon_0 \underline{E}}$ ③

(N.B. Griffiths uses 'ϵ' where I use '$\epsilon\epsilon_0$')

5. Field due to a point charge in a linear dielectric

in dielectric

rel. permittivity

q_f $\quad$ ϵ

there will be bound charges, but writing Gauss' law for $\underline{D}$ we can automatically include them

$$\int_S \underline{D}.d\underline{S} = q_f \qquad \therefore \underline{D} = \frac{q_f}{4\pi r^2}\hat{\underline{r}}$$

sphere

$$\underline{E} = \frac{\underline{D}}{\epsilon\epsilon_0} = \frac{q_f \hat{\underline{r}}}{4\pi\epsilon\epsilon_0 r^2}$$

$\therefore$ the field and potential of a charge distribution in a linear dielectric is related to that in free space by writing $\epsilon\epsilon_0$ instead of ϵ_0 in the relevant formulas.

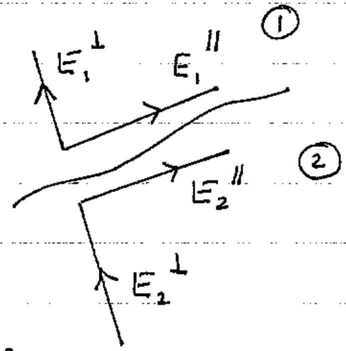
N.B.

$$\left.\begin{aligned} \text{div } \underline{E} &= \frac{\rho_f + \rho_b}{\epsilon_0} \\ \text{div } \underline{D} &= \rho_f \end{aligned}\right\} \text{always}$$

$$\left.\begin{aligned} \text{but } \underline{D} &= \epsilon\epsilon_0 \underline{E} \\ \therefore \text{div } \underline{E} &= \frac{\rho_f}{\epsilon\epsilon_0} \end{aligned}\right\} \text{linear dielectric}$$

6. Boundary conditions at ~~[illegible]~~ boundary between dielectrics

from § IA9



(i) $\underline{E_1^{\parallel} = E_2^{\parallel}}$

followed from $\int \underline{E} \cdot \underline{dl} = 0$; always true

(ii) $\underline{E_1^{\perp} - E_2^{\perp} = \dfrac{\sigma}{\varepsilon_0}}$

followed from Gauss' thm ; always true but need to remember that $\sigma = \sigma_f + \sigma_b$

so at a boundary between dielectrics usually easier to work with $\underline{D}$. Gauss' thm for $\underline{D}$ gives

$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$

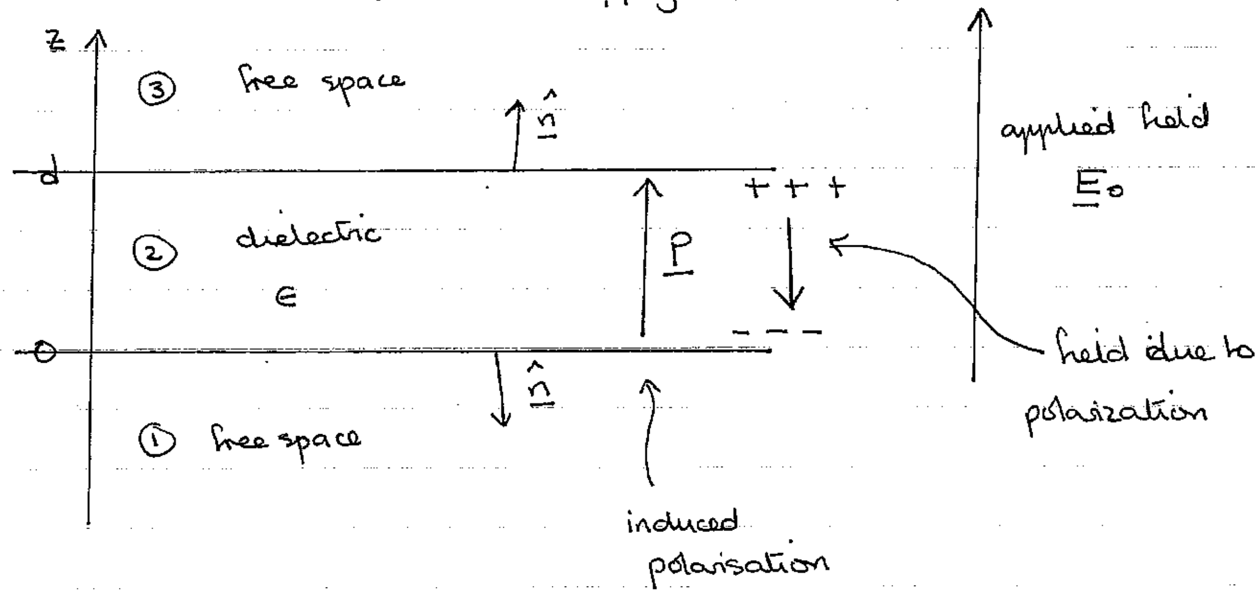
but, for dielectrics, in general, $\sigma_f = 0$ $\quad \therefore \underline{D_1^{\perp} = D_2^{\perp}}$

At a boundary between dielectrics the convenient boundary conditions are

$E^{\parallel}$	('E tangential')	continuous
$D^{\perp}$	('D normal')	continuous

7a <u>example</u>

<u>linear dielectric slab</u>   <u>apply field $\underline{E}_0$</u>



what are $\underline{D}_1$, $\underline{D}_2$, $\underline{D}_3$ $\underline{E}_1$, $\underline{E}_2$, $\underline{E}_3$ and in ② $\underline{P}$ ρ_b σ_b ?

①, ③ free space $\therefore \underline{E}_1 = \underline{E}_0$, $\underline{E}_3 = \underline{E}_0$

$$\underline{D}_1 = \epsilon_0 \underline{E}_0 , \quad \underline{D}_3 = \epsilon_0 \underline{E}_0$$

$D^\perp$ continuous $\therefore \underline{D}_2 = \varepsilon_0 \underline{E}_0$ (α)

but $\underbrace{\underline{D}_2 = \epsilon\epsilon_0 \underline{E}_2}_{(\beta 1)} = \underbrace{\varepsilon_0 \underline{E}_2 + \underline{P}}_{(\beta 2)}$

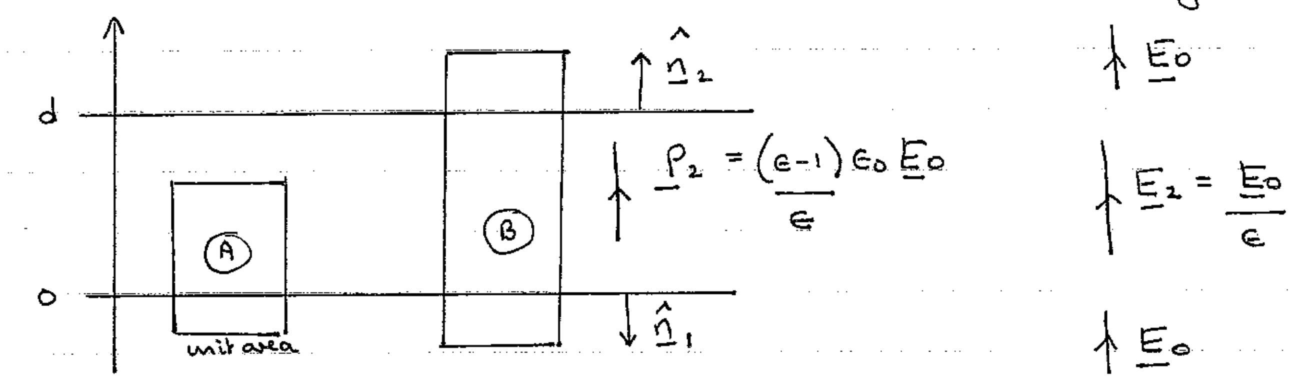
field actually in the dielectric $\neq \underline{E}_0$

from (α) and ($\beta 1$) $\quad \underline{E}_2 = \dfrac{\underline{E}_0}{\varepsilon} = \underline{E}_0 + \dfrac{(1-\epsilon)}{\epsilon}\underline{E}_0$

external field   field due to bound charge

from ($\beta 2$) $\quad \underline{P} = (\epsilon-1)\epsilon_0 \underline{E}_2 = \dfrac{(\epsilon-1)}{\epsilon}\varepsilon_0 \underline{E}_0$

Does this all fit with what we know about bound charge?



$\underline{P}$ constant $\therefore \rho_b = 0$

at $z = 0$ $\quad \sigma_b(0) = \underline{P}_2 \cdot \hat{\underline{n}}_1 = -\frac{(\epsilon-1)}{\epsilon}\epsilon_0 \underline{E}_0$

at $z = d$ $\quad \sigma_b(d) = \underline{P}_2 \cdot \hat{\underline{n}}_2 = \frac{(\epsilon-1)}{\epsilon}\epsilon_0 \underline{E}_0$

Gaussian surface A

$$\int \underline{E} \cdot d\underline{S} = \int \frac{\rho}{\epsilon_0} dV$$

$$\frac{\underline{E}_0}{\epsilon} - \underline{E}_0 = -\frac{(\epsilon-1)}{\epsilon}\frac{\epsilon_0}{\epsilon_0}\underline{E}_0 \quad \checkmark$$

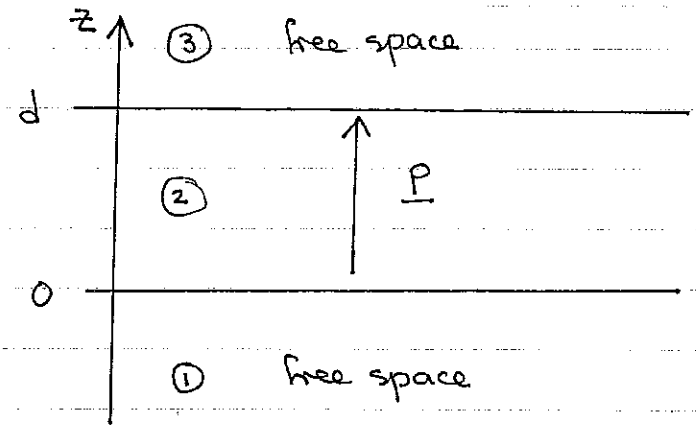
Gaussian surface B

$$0 = -\frac{(\epsilon-1)}{\epsilon}\epsilon_0 \underline{E}_0 + \frac{(\epsilon-1)\epsilon_0 \underline{E}_0}{\epsilon} = 0 \quad \checkmark$$

7b example (not covered in 2010-11 lectures)

How do $\underline{D}$, $\underline{E}$, $\underline{P}$ and the bound charges all fit together?

eg 1 dilectric slab impose polarization $\underline{P} = (0, 0, kz)$



What are	$\underline{D}_1$,	$\underline{D}_2$,	$\underline{D}_3$
	$\underline{E}_1$,	$\underline{E}_2$,	$\underline{E}_3$
and in ②	$\underline{P}$ ↑ given	ρ_b	σ_b

Three arguments that $\underline{D} = 0$:

(i) $\underline{D} = 0$ at ∞ and, by symmetry is along $\hat{\underline{z}}$. $D^{\perp}$ is continuous
$\therefore \underline{D} = 0$ everywhere

(ii) $\underline{D} = 0$ at ∞ and is along $\hat{\underline{z}}$. There are no free charges $\therefore$ using Gauss' law, $\underline{D} = 0$ everywhere

(iii) $\operatorname{div} \underline{D} = \rho_f = 0$. $\underline{D}$ is along $\hat{\underline{z}}$ $\therefore \dfrac{\partial D_z}{\partial z} = 0$, $\underline{D}$ is constant

but $\underline{D} = 0$ at ∞ $\therefore \underline{D} = 0$ everywhere.

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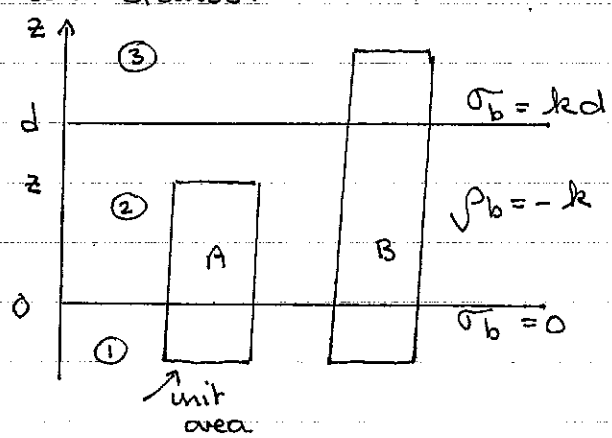
①, ③ free space $\therefore \underline{D} = \varepsilon_0 \underline{E}$ $\therefore \underline{E}_1 = 0$, $\underline{E}_3 = 0$

in ② $\underline{D}_2 = \varepsilon_0 \underline{E}_2 + \underline{P} = 0$ $\therefore \underline{E}_2 = -\dfrac{\underline{P}}{\varepsilon_0} = -\dfrac{kz}{\varepsilon_0}\hat{\underline{z}}$

$$\rho_b = -\operatorname{div} \underline{P} = -k$$
$$\sigma_b(0) = \underline{P}(0) \cdot \hat{\underline{n}} = 0$$
$$\sigma_b(d) = \underline{P}(d) \cdot \hat{\underline{n}} = kd$$

check Gauss:



$\underline{E}_3 = 0$

$\underline{E}_2 = -\dfrac{kz}{\varepsilon_0}\hat{\underline{z}}$

$\underline{E}_1 = 0$

Gaussian cylinder A

$$\int \underline{E} \cdot d\underline{S} = \frac{1}{\varepsilon_0}\int \rho \, dV$$

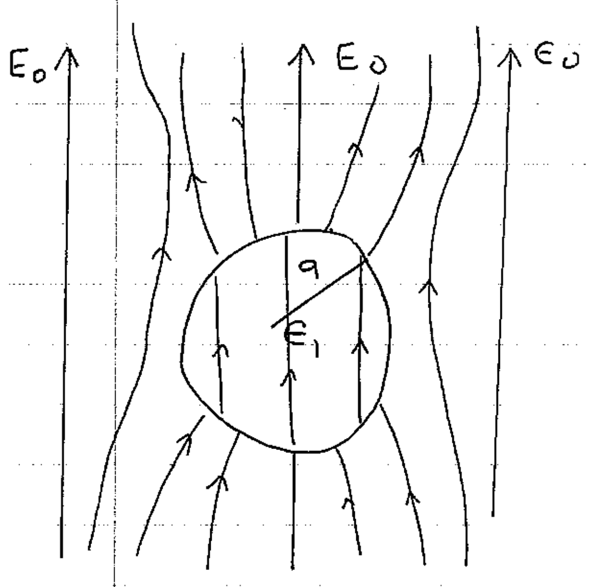
$$-\frac{kz}{\varepsilon_0} = \frac{1}{\varepsilon_0}\underbrace{(-kz)}_{\text{from } \rho_b}$$

Gaussian cylinder B

$$0 = \frac{1}{\varepsilon_0}\big(\underbrace{-kd}_{\text{from } \rho_b} + \underbrace{kd}_{\text{from } \sigma_s}\big)$$

(not covered in 2009-10 lectures)
or 2010-11

7. Laplace revisited: dielectric sphere, relative permittivity ϵ_1, radius a in a uniform field. Find V everywhere.



Laplace, spherical polars, azimuthal symmetry

$\Downarrow$

$$V(r,\theta) = \sum_l \left\{ A_l r^l + \frac{B_l}{r^{l+1}} \right\} P_l(\cos\theta)$$

boundary conditions

V_{in} finite at origin ①

$V_{out} \to -E_0 r\cos\theta$ as $r \to \infty$ ②

at $r = a$ $\quad E^{\parallel}$ continuous ③

$D^{\perp}$ continuous ④

need separate solutions for V_{in}, V_{out}

need to match '$\cos\theta$' term $\therefore$ guess $l = 1$ terms needed.

$$V_{in} = A_1 r\cos\theta$$

① no $\frac{1}{r^2}$ term

$$V_{out} = \overset{②}{-E_0 r\cos\theta} + \frac{B_1}{r^2}\cos\theta$$

$$\underline{E} = -\text{grad}\, V = \left(-\frac{\partial V}{\partial r}, -\frac{1}{r}\frac{\partial V}{\partial \theta}, -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} \right)$$

$$\therefore \underline{E}_{in} = \left(-A_1\cos\theta,\ A_1\sin\theta,\ 0 \right)$$

$E^{\perp}$ $\qquad E^{\parallel}$

$$\underline{E}_{out} = \left(E_0\cos\theta + \frac{2B_1}{r^3}\cos\theta,\ -E_0 r\sin\theta + \frac{B_1}{r^3}\sin\theta,\ 0 \right)$$

$$\underline{D}_{in} = \epsilon \epsilon_0 \underline{E}_{in} \qquad \underline{D}_{out} = \epsilon_0 \underline{E}_{out}$$

$$③ \Rightarrow \quad A_1 = -E_0 a + \frac{B_1}{a^3}$$

$$④ \Rightarrow \quad -\epsilon \epsilon_0 A_1 = \epsilon_0 E_0 + \frac{2\epsilon_0 B_1}{a^3}$$

solve for A_1, B_1 and sub. into expressions for V:

$$⑤ \quad V_{in} = \frac{-3E_0}{\epsilon_1 + 2} r\cos\theta \equiv -E_0 r\cos\theta + E_0 r\cos\theta \left(\frac{\epsilon - 1}{\epsilon + 2}\right)$$

contribution due to external field

$$⑥ \quad V_{out} = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\left(\frac{\epsilon_1 - 1}{\epsilon_1 + 2}\right)\cos\theta$$

contribution due to polarization of sphere

dipolar outside
constant field inside

Two questions

(i) what is $\underline{P}$ in the sphere?

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon_1 \epsilon_0 \underline{E}$$

$$\therefore \underline{P} = (\epsilon_1 - 1)\epsilon_0 \underline{E}_{in} = \frac{(\epsilon_1 - 1)\epsilon_0 \cdot 3E_0}{\epsilon_1 + 2}\hat{\underline{z}}$$

(ii) What is the bound charge on the surface?

$$\sigma_b = \underline{P}\cdot\hat{\underline{n}} = P\cos\theta = \frac{3\epsilon_0(\epsilon_1 - 1)E_0\cos\theta}{\epsilon_1 + 2}$$