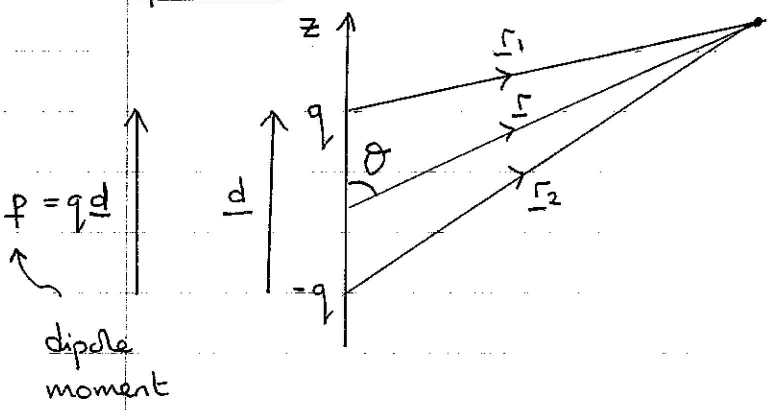
I ELECTROSTATICS

C. MULTIPOLES

1. Electric dipole
   a. potential
   b. electric field
   c. energy in a constant field
   d. force in a constant field
   e. torque in a constant field
   f. force in a field that varies
2. The multipole expansion

Electric Dipole

a. potential



N.B. $\underline{p}$ is along the z-axis of a spherical polar co-ordinate system

$$V(\underline{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$r_{\substack{1\\2}}^{-1} = \left\{ r^2 + \left(\frac{d}{2}\right)^2 \mp \frac{2rd}{2}\cos\theta \right\}^{-\frac{1}{2}}$$

$$= r^{-1}\left\{ 1 + \left(\frac{d}{2r}\right)^2 \mp \frac{d}{r}\cos\theta \right\}^{-\frac{1}{2}}$$

$$= r^{-1}\left(1 \pm \frac{d}{2r}\cos\theta + O\left(\frac{d}{r}\right)^2 \right)$$

$$\therefore V(\underline{r}) = \frac{q}{4\pi\epsilon_0}\, r^{-1} \cdot \frac{2d}{2r}\cos\theta$$

$$V(\underline{r}) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \equiv \frac{\underline{p}\cdot\hat{\underline{r}}}{4\pi\epsilon_0 r^2}$$

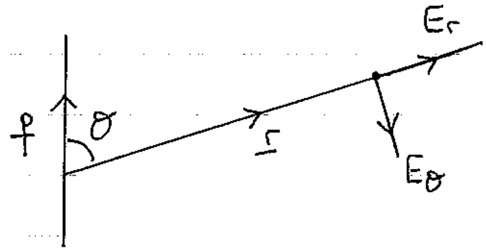
N.B.

b. electric field

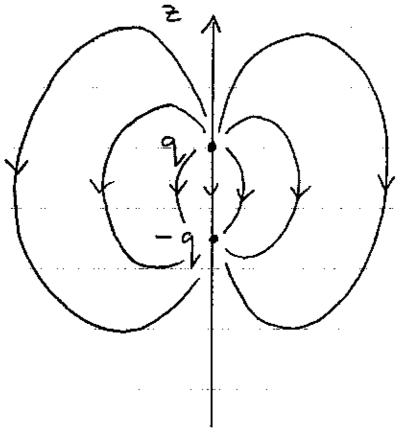
$$\underline{E} = -\,\mathrm{grad}\, V = \left(-\frac{\partial V}{\partial r},\ -\frac{1}{r}\frac{\partial V}{\partial \theta},\ -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} \right)$$

$$= \left(\frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \frac{p\sin\theta}{4\pi\epsilon_0 r^3}, 0 \right)$$

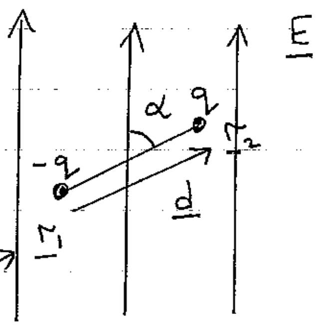
N.B.



careful with co-ordinates



energy in a constant field



position vector of $-q$

α is the angle between $\underline{d}$ and $\underline{E}$

$$U = q\,V(\underline{r}_2) - q\,V(\underline{r}_1)$$

$$V(\underline{r}_2) - V(\underline{r}_1) = -\int_{\underline{r}_1}^{\underline{r}_2} \underline{E} \cdot d\underline{l}$$

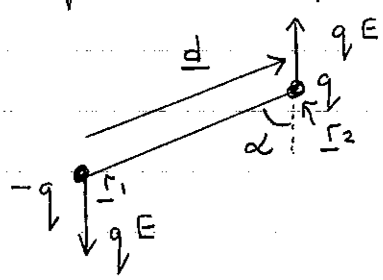
$$= -\underline{E} \cdot (\underline{r}_2 - \underline{r}_1)$$

$$= -\underline{E} \cdot \underline{d}$$

$$\therefore U = -q\,\underline{d} \cdot \underline{E} = -\underline{p} \cdot \underline{E}$$

d <u>force on a dipole in a constant field</u> is <u>zero</u>

e <u>torque on a dipole in a constant field</u>



$$\Gamma = qE\, d \sin\alpha = pE\sin\alpha$$

$$\underline{\Gamma} = \underline{p} \wedge \underline{E}$$

out of paper — right magnitude and out of paper

f <u>force on dipole in field that is not constant</u> (on the scale of d)

$$\underline{F} = q\left(\underline{E}(\underline{r}_2) - \underline{E}(\underline{r}_1)\right)$$

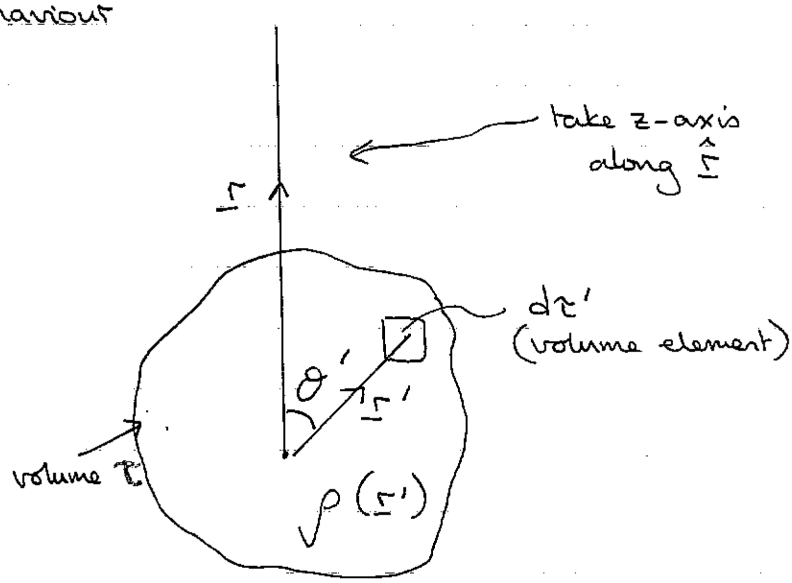
$$\underline{E}(\underline{r}_2) - \underline{E}(\underline{r}_1) = (\underbrace{\underline{d}}_{(\underline{r}_2 - \underline{r}_1)} \cdot \text{grad})\, \underline{E} = (\underline{d}\cdot\text{grad})\,\underline{E}$$

(e.g. x component
$$E_x(\underline{r}_2) - E_x(\underline{r}_1) = d_x \frac{\partial E_x}{\partial x} + d_y \frac{\partial E_x}{\partial y} + d_z \frac{\partial E_x}{\partial z}$$
)

$$\therefore \underline{F} = (\underline{p}\cdot\text{grad})\,\underline{E}$$

2. Multipole Expansion

expansion of the potential of a localised charge distribution in powers of r^{-1} – so we can look at 'far field' behaviour



$$V(\underline{r}) = \int_\tau \frac{\rho(\underline{r}')\,d\tau'}{4\pi\varepsilon_0 |\underline{r}-\underline{r}'|}$$

$$|\underline{r}-\underline{r}'|^{-1} = \left\{r^2 + r'^2 - 2rr'\cos\theta'\right\}^{-1/2}$$

$$= r^{-1}\Big\{1 + \underbrace{\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'}_{\text{small}}\Big\}^{-1/2}$$

small $\therefore \Downarrow$ binomial expansion

$$= \frac{1}{r}\left\{1 - \frac{1}{2}\left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right) + \frac{3}{8}\left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right)^3 + \dots\right\}$$

$\Downarrow$ collect terms

$$= \frac{1}{r}\left\{1 + \frac{r'}{r}\cos\theta' + \frac{r'^2}{r^2}\frac{1}{2}(3\cos^2\theta' - 1) + \frac{r'^3}{r^3}\frac{1}{2}(5\cos^3\theta' - 3\cos\theta') + \dots\right\}$$

$$\equiv \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$\therefore V(\underline{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_v (r')^n P_n(\cos\theta') \rho(\underline{r}') \, d\tau'$$

we have rewritten V as an expansion in powers of r^{-1}

<u>monopole term</u> $\quad n = 0$

$$V_0(\underline{r}) = \frac{1}{4\pi\epsilon_0 r} \underbrace{\int_v \rho(\underline{r}') \, d\tau'}_{\text{total charge}}$$

if total charge is zero the dominant term will be the

<u>dipole term</u> $\quad n = 1$

$$V_1(\underline{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int_v r' \cos\theta' \rho(\underline{r}') \, d\tau'$$

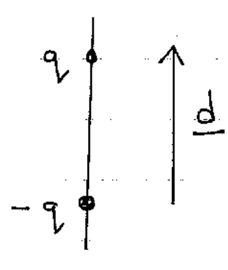
$$r'\cos\theta' = \underline{r}' \cdot \hat{\underline{r}}$$

$$\therefore V_1(\underline{r}) = \frac{1}{4\pi\epsilon_0 r^2} \hat{\underline{r}} \cdot \underbrace{\int_v \underline{r}' \rho(\underline{r}') \, d\tau'}_{\text{dipole moment } \underline{p} \quad \text{(definition)}}$$

$$V_1(\underline{r}) = \frac{\underline{p} \cdot \hat{\underline{r}}}{4\pi\epsilon_0 r^2} \quad \text{, as before}$$

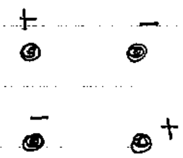
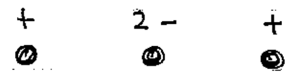
the dipole we are used to is a special case of a localised charge distribution with total charge zero and

$$\underline{p} = \int_v \underline{r}' \left\{ q\,\delta\left(\underline{r}' - \frac{\underline{d}}{2}\right) - q\,\delta\left(\underline{r}' + \frac{\underline{d}}{2}\right) \right\} d\tau'$$

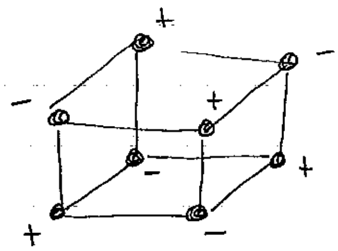


$$= q\underline{d} \quad \text{as expected}$$

if total charge 0 and $\underline{p} = 0$ quadrupolar term dominates

e.g.  or  $V \sim \frac{1}{r^3}$

next term is the octopole

e.g.  with $V \sim \frac{1}{r^4}$