I ELECTROSTATICS

B. Poisson and Laplace Equations

1. The equations and uniqueness
2. Poisson 1D
3. Laplace 3D, Cartesians
4. Laplace 3D, spherical polars (spherical conductors, uniform $\underline{E}$ - field)
5. " (specify $\sigma(\theta)$ on surface of sphere)

B POISSON + LAPLACE EQUATIONS

1. The equations and uniqueness

Poisson: $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$

Laplace: $\nabla^2 V = 0$

Given suitable boundary conditions the solution to these equations is unique $\therefore$ if you find a sol$^{\underline{n}}$ that obeys the boundary conditions it must be the right one.

simple example of suitable boundary conditions – V specified at all points on boundary of given region

to demonstrate the uniqueness thm suppose there are 2 sol$^{\underline{ns}}$ to the Laplace eq$^{\underline{n}}$ V_1, V_2 each of which obeys the given boundary conditions

$$\nabla^2 V_1 = 0 \qquad \nabla^2 V_2 = 0$$

define $V_3 = V_1 - V_2$

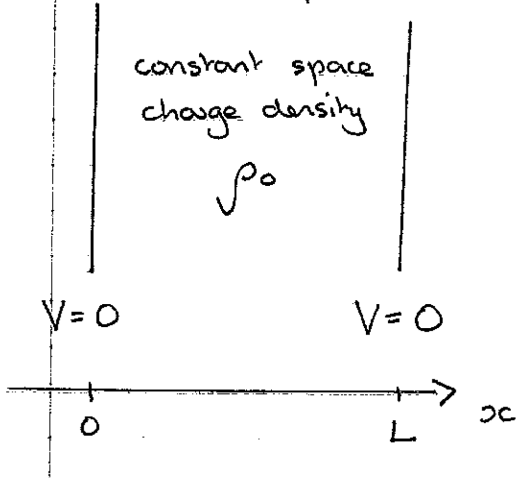
$$\therefore \nabla^2 V_3 = 0$$

and $V_3 = 0$ on all boundaries $\Rightarrow$ $V_3 = 0$ everywhere

physically sensible, see e.g. Griffiths for math. justification

$$\therefore V_1 = V_2$$

2. Poisson 1D



equation $\dfrac{d^2 V}{dx^2} = -\dfrac{\rho_0}{\varepsilon_0}$

boundary conditions

$V = 0$ at $x = 0$, $x = L$

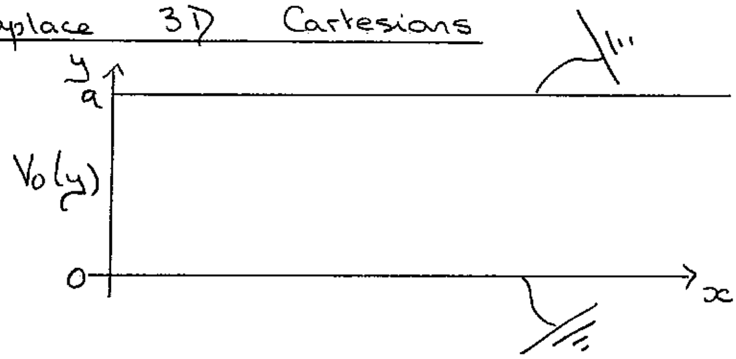
$$\therefore \frac{dV}{dx} = -\frac{\rho_0 x}{\epsilon_0} + C_1$$

$$V = -\frac{\rho_0 x^2}{2\epsilon_0} + C_1 x + C_2$$

$\Downarrow$ use boundary conditions to get C_1, C_2

$$\underline{V(x) = \frac{\rho_0}{2\epsilon_0} x (L-x)}$$

3. Laplace 3D Cartesians



translationally invariant in z-direction $\therefore$ sol^n independent of z.

equation $\nabla^2 V = 0$

boundary conditions $V(x,0) = V(x,a) = 0$ ① ②

as $x \to \infty$ $V \to 0$ ③

$V(0,y) = V_0(y)$ ④

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

assume $V = X(x)\,Y(y)$

$\therefore X''Y + XY'' = 0$ where $X'' = \frac{d^2X}{dx^2}$, etc.

$$\therefore \frac{X''}{X} + \frac{Y''}{Y} = 0$$

x, y are independent variables

$$\therefore \frac{X''}{X} = k^2 ; \qquad \frac{Y''}{Y} = -k^2$$

constant, chosen to match boundary conditions as easily as possible

$$\therefore V(x,y) = \sum_k \left(A_k e^{kx} + B_k e^{-kx}\right)\left(C_k \sin ky + D_k \cos ky\right)$$

boundary conditions: ① $\Rightarrow D_k = 0$

② $\Rightarrow k = \frac{n\pi}{a}$

③ $\Rightarrow A_k = 0$

$$\therefore V(x,y) = \sum_n A_n \sin \frac{n\pi y}{a} e^{-kx}$$

④ $\Rightarrow$ $V(0,y) \equiv V_0(y) = \sum_n A_n \sin \frac{n\pi y}{a}$

$$\therefore A_n = \frac{2}{a}\int_0^a V_0(y) \sin \frac{n\pi y}{a}\, dy$$

e.g. if $V_0(y) = V_0 y(a-y)$

$$A_n = \frac{8V_0a^2}{n^3\pi^3}, \quad n \text{ odd} \quad ; \quad 0, \quad n \text{ even}$$

$$\therefore V(x,y) = \sum_{n\,\text{odd}} \frac{8V_0a^2}{n^3\pi^3} \sin \frac{n\pi y}{a} e^{-kx}$$

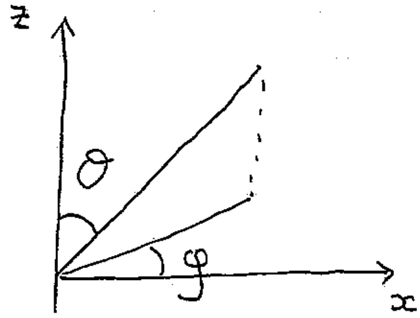
4. <u>Laplace 3D spherical polars</u>

conductor in uniform $\underline{E}$ - field

↑

spherical, radius a

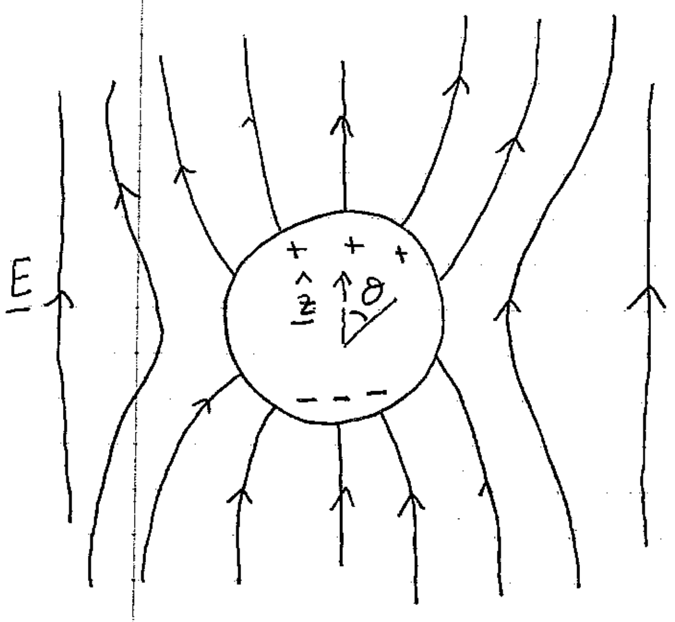
no net charge



use spherical polars

take z-axis along $\underline{E}$

azimuthal symmetry (no φ dependence)



equation: Laplace, spherical polars

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\cancel{\frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial\varphi^2}}=0$$

no φ dependence

boundary conditions

sphere is an equipotential ; choose $V=0$ on surface of sphere

as $r\to\infty$ $\quad V\to -Er\cos\theta$

why ?

$$V=-\int\underline{E}.\underline{dl} = -\int E\,dz = -Ez+C = -Er\cos\theta + C$$

we have chosen $V=0$ on surface of sphere $\therefore$ by symmetry $V=0$ in equatorial plane ($z=0$ or $\theta=\pi/2$)
$\therefore C=0$

summary of boundary conditions

① at $r=a$ $\quad V=0$

② as $r\to\infty$ $\quad V\to -Er\cos\theta$

sol^n to Laplace's eq^n in spherical polars

(by separation of variables + series sol^n; see Math Phys course)

$$V(r,\theta) = \sum_{l=0}^{\infty}\left\{A_l r^l + \frac{B_l}{r^{l+1}}\right\} P_l(\cos\theta) \qquad ③$$

Legendre polynomials

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\vdots$$

putting in boundary conditions i.e. finding the A_l and B_l that match the b.c.'s

we have to match a '$\cos\theta$' boundary condition

$\therefore$ we are likely to need the '$\cos\theta$' terms in ③ ($l=1$)

$\therefore$ <u>try</u> a solution (if it matches the boundary conditions it must be the correct solution due to uniqueness thm.)

$$V(r,\theta) = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

② $\Rightarrow$ $A_1 = -E$

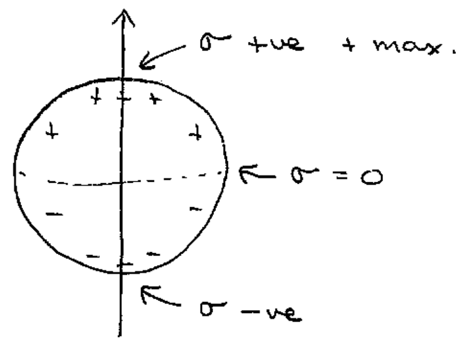
① $\Rightarrow$ $B_1 = a^3 E$

$$\therefore V(r,\theta) = -E_0\Big(\underbrace{r}_{\text{term due to external field}} - \underbrace{\frac{a^3}{r^2}}_{\text{dipole term due to charge induced on conductor}}\Big)\cos\theta$$

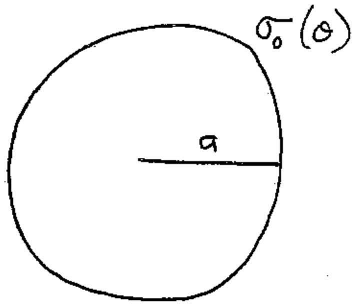
what is the induced surface charge density?

$$E^{\perp}\Big|_{r=a} = \frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial r}\Big|_{r=a}$$

$$\therefore \sigma = 3\epsilon_0 E_0 \cos\theta$$



5. Laplace 3D; spherical polars; specify $\sigma(\theta)$ on surface of sphere (radius a)



charge density $\sigma_0(\theta)$ at $r = a$

find $V(r, \theta, \varphi)$

equation: Laplace, spherical polars
no dependence on φ

solution: $$V(r,\theta) = \sum_l \left\{ A_l r^l + \frac{B_l}{r^{l+1}} \right\} P_l(\cos\theta)$$

boundary conditions:

① at $r = a$ V continuous ($E^{\parallel}$ continuous)

② $$E^{\perp}_{out} - E^{\perp}_{in} = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

③ V_{in} must be finite at $r = 0$

④ $V_{out} \to 0$ as $r \to \infty$

as a concrete example choose $\sigma_0(\theta) = P\cos\theta$

$\therefore$ expect $l = 1$ terms in the sol$^{\underline{n}}$

'guess'

$$V_{in} = \left(A_1 r + \frac{B_1}{r^2} \right) \cos\theta$$

zero due to b.c. ③

$$V_{out} = \left(C_1 r + \frac{D_1}{r^2} \right) \cos\theta$$

zero due to b.c. ④

$$\therefore \; V_{in} = A_1 r \cos\theta$$

$$V_{out} = \frac{D_1}{r^2}\cos\theta$$

$$E = -\operatorname{grad} V = -\Big(\underset{\substack{\uparrow \\ E^{\perp}}}{\frac{\partial V}{\partial r}} \;,\; \frac{1}{r}\underset{\substack{\uparrow \\ E^{\parallel}}}{\frac{\partial V}{\partial \theta}} \;,\; 0 \Big)$$

(3) $$A_1 a = \frac{D_1}{a^2}$$

(4) $$\frac{2D_1}{a^3}\cancel{\cos\theta} + A_1 \cancel{\cos\theta} = \frac{P\cancel{\cos\theta}}{\varepsilon_0}$$

$$\therefore \; A_1 = \frac{P}{3\varepsilon_0} \quad ; \quad B_1 = \frac{Pa^3}{3\varepsilon_0}$$

$$\boxed{V_{in} = \frac{P}{3\varepsilon_0} r\cos\theta \; ; \qquad V_{out} = \frac{Pa^3}{3\varepsilon_0 r^2}\cos\theta}$$ (1)

NB I $\quad \sigma_0(\theta) = P\cos\theta$ is of the same form as the charge density induced on a conductor in a uniform $\underline{E}$ - field (see §4) if we choose $P = 3\varepsilon_0 E_0$.

$\therefore$ the sol$^{\underline{ns}}$ we found in §4 and §5 should be related:

putting $P = 3\varepsilon_0 E_0$ in ① and adding potential due to constant $\underline{E}$ - field
ie potential due to induced charge

$$V_{in} = E_0 r\cos\theta \qquad\qquad - E_0 r\cos\theta$$

$$V_{out} = \frac{E_0 a^3 \cos\theta}{r^2} \qquad\qquad - E_0 r\cos\theta$$

gives

$$V_{in} = 0$$

$$V_{out} = -E_0 r\cos\theta\left(1 - \frac{a^3}{r^3}\right)$$

which were the answers we got in §4 for potential of uncharged conductor in a field.

NB II $\quad$ a nice problem is to take $\sigma_0(\theta) = $ constant, solve using this machinery + check using Gauss' thm.