7 Time-dependent perturbations; transitions; selection rules; magnetic resonance

1. A particle is in the ground state of a simple box (infinite square potential well) of side L. Describe qualitatively the evolution of the system
a. If the length of the side of the box is suddenly doubled.
b. If the length of the side of the box is very slowly increased to $2L$.
c. What would be the result of an attempt to decrease the box's diameter suddenly to $L/2$?

2. A harmonic oscillator starts in its ground state ($n = 0$) at $t = -\infty$. A perturbation $\delta H = -e\mathcal{E}x \exp(-t^2/\tau^2)$ is applied between $t = -\infty$ and $+\infty$.
a. What is the probability that the oscillator makes a transition to the state $n = 2$?
b. Show that the probability the oscillator makes a transition to the state $n = 1$ is
$$P_{0\to 1} = \frac{e^2\mathcal{E}^2\pi\tau^2}{2m\omega\hbar} e^{-\omega^2\tau^2/2}$$
[there are two integrals to do here, one for the matrix element, and an integral over time. For the matrix element, you can either perform the integral in the x basis, or use raising/lowering operators.]
c. Plot the dependence of $P_{0\to 1}$ on the timescale τ of the perturbation, and comment on the behaviour in the limits $\tau \to 0$ and $\omega\tau \gg 1$.

3. **Sudden approximation** In the β decay H^3 (1 proton + 2 neutrons in the nucleus) $\to$ $(He^3)^+$ (2 protons + 1 neutron in the nucleus), the emitted electron has a kinetic energy of 16 keV. We will consider the effects on the motion of the atomic electron, i.e. the one orbiting the nucleus, which we assume is initially in the ground state of H^3.
a. Show by a brief justification that the perturbation is sudden. What is the state of the atomic electron immediately after the perturbation (e.g. at a time around 5×10^{-17} s after the electron was emitted)?
b. Give the mean kinetic energy and mean potential energy of the atomic electron before and after the perturbation, expressing your results as multiples of the Rydberg energy (ignore the very slight change in reduced mass.) [Use your knowledge of hydrogen to avoid doing integrals]. Hence find the new mean energy.
c. Show that the probability for the electron to be left in the ground state of $(He^3)^+$ is $2^3(2/3)^6 \simeq 0.7$, and comment on how this result relates to the information you obtained in part (b). [Comment: the integral here is one you have done before, but to save time you might like to change variable to $u = 3r/2$ to convert it exactly into a standard one for hydrogen, for which you then know the value].
d. Describe the main features of the subsequent behaviour of the $(He^3)^+$ ion when the electron is not left in the ground state.

4. **Dipole radiation** For a single atom, in the process called *electric dipole radiation*, what two physical entities are interacting together? Briefly discuss whether or not an isolated atom possesses an electric dipole moment.

5. **Selection rules** This question illustrates the way selection rules can come about, by treating a system in a state of well-defined orbital angular momentum, such as the hydrogen atom when spin-orbit coupling is ignored. For such a system (i.e. one whose energy eigenfunctions have the form $R(r)Y_{l,m_l}(\theta, \phi)$) find the electric dipole selection rules on m_l, by considering the components of the electric dipole matrix element $\langle e\mathbf{r}\rangle$ in the form of three integrals. [Hint: consider z and $x \pm iy$, expressing them in polars.]

6. **Magnetic resonance** A beam of hydrogen atoms is formed by effusion into vacuum from a small hole in an oven and a collimating aperture. A velocity-selector blocks all atoms except those in a narrow range of velocities centred around v. Let the x axis of a coordinate

system be along the beam. The atomic beam passes through a region of large magnetic field gradient dB/dz (as in a Stern-Gerlach apparatus) perpendicular to x, followed by a region of constant uniform magnetic field 1 mT directed along z, followed by a third region like the first but with reversed field gradient. Atoms which emerge from the third magnet in the same direction they entered the first are detected.

a. Briefly describe the motion of an atom in an arbitrary spin-state through this apparatus.

b. A magnetic field $B_a \cos(\omega t)$ directed along y and oscillating at angular frequency ω with amplitude $B_a \ll 1$ mT is applied in the second region (in addition to the constant uniform field there). This oscillating field can have a large effect on the atoms. Sketch rough graphs and describe in qualitative terms the detected signal as a function of ω at fixed B_a, and as a function of B_a when ω is tuned to resonance.

c. Calculate the resonant frequency ω_0.

d. If the atomic beam is velocity-selected to have speed 800 m/s, and the length of the region where the oscillating magnetic field is applied is 10 cm, then what is the smallest value of B_a required to reduce the detected number of atoms to a minimum?

7. **Derivation** Let $\psi(t)$ be the state of a system subject to a Hamiltonian $H = H^0 + \delta H(t)$ where H^0 is time-independent and has eigenfunctions ψ_n^0. The basic perturbation theory result, accurate to first order in δH, is

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t e^{i\omega_{fi}t'} \left\langle \psi_f^0 \right| \delta H(t') \left| \psi_i^0 \right\rangle dt'. \tag{1}$$

for the amplitude of the state ψ_f^0 in $\psi(t)$ when a system starts in the state ψ_i^0 and is subject to perturbation $\delta H(t)$ between times 0 and t. After first checking your notes, see if you can perform the derivation of this result without returning to them for help.

8. **Calculate a matrix element** Here is an example of an electric dipole matrix element. It is for one of the Zeeman components of the 1s–2p transition in hydrogen, ignoring fine structure.

$$|\left\langle n=2, l=1, m=0 \right| z \left| n=1, l=0, m=0 \right\rangle|^2 = \frac{2^{15}}{3^{10}}\, a_0^2$$

Use the hydrogen wavefunctions to verify this result. [Use spherical polar coordinates. The radial integral can be looked up, or alternatively by a change of variable to $u = 3r/4$ it can be converted into the same form as $\langle r^2 \rangle$ for hydrogen ground state, for which you can use your previous study of such integrals in problem set 1.]