Quantum Gravity
and the
Dimension of Space-time

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Why quantum gravity?

For ninety years our understanding of gravitational physics has been based on the general theory of relativity which accurately describes many phenomena occuring at very different distance scales: from the gravitational red-shift of light observed in the laboratory experiment of Pound and Rebka; through the precession of the perihelion of Mercury; to the evolution of the universe and the cosmic microwave background. More recently discrepancies have been found in large distance physics and at present these are the subject of intense activity. We do not know for sure what will have to be modified; whether general relativity itself or just the parameters of the theory. On the other hand if general relativity is used to evolve our observations of the universe back in time we reach the conclusion that there was a Big Bang and that in the era immediately afterwards the universe was incredibly small – in fact much smaller than an atom. This leads to another sort of problem; things the size of atoms cannot be understood properly without using quantum mechanics so we also expect that the extrapolation back to the Big Bang using general relativity, which is a classical field theory, will fail at some point. In other words, in order to understand fully the Big Bang we expect that it is necessary to have a theory of gravitation which is fully quantum mechanical.

From the theoretical point of view there are other motivations for pursuing a quantum theory of gravity. Ever since classical electromagnetism and quantum mechanics were combined into quantum electrodynamics theorists have believed that there must be some way of doing the same with the theory of gravity. That belief has only been strengthened by the successes of electroweak theory and of QCD, the theory of the strong interactions. However gravity has proved extraordinarily resistant to consistent quantization and even today, after many decades of effort, research on the topic continues.

Models of quantum gravity

Many different ways of formulating quantum gravity have been investigated; some, in particular string theory, have attracted more attention than others but none is entirely satisfactory. This article describes some of the work being done by the EN-RAGE Research Training Network within the discretised approach which is explained

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in more detail below [1]. ENRAGE also works on causal set theory [2], and other researchers study loop quantum gravity [3]. We should note that the bibliography in this article is neither intended to be exhaustive nor to record the historical development of the subject; many relevant references can be found in the bibliographies of the works cited here.

In classical general relativity the dynamical degree of freedom is the metric tensor $g_{\mu\nu}$. The metric satisfies Einstein's equations that relate it to the energy-momentum tensor describing the matter which is the source of gravitation. So for any given set of consistent initial conditions there is just one unique evolution of the metric in time, just as a particle follows a particular path in classical physics. When the particle motion is quantised the single classical trajectory from point A at $t = 0$ to B at time t is replaced by an amplitude that the particle starting at A ends up at B. This amplitude is given by a sum over the set $\mathcal{P}$ of all possible paths from $(A, 0)$ to (B, t)

$$\langle B, t| A, t = 0\rangle = \sum_{p \in \mathcal{P}} \exp(iS_p/\hbar), \tag{1}$$

where S_p is the action for the path p. Extending this naively to the gravitational case we expect the metric to evolve from $g_{\mu\nu}^A$ at $t = 0$ to $g_{\mu\nu}^B$ at t with amplitude

$$\langle g_B, t| g_A, t = 0\rangle = \sum_{g \in \mathcal{G}} \exp(iS_g/\hbar), \tag{2}$$

where $\mathcal{G}$ is the set of all possible metrics satisfying $g = g_A$ at $t = 0$ and $g = g_B$ at t, and S_g is the action (the natural choice for which is the Einstein action which in two dimensions consists of the cosmological constant term alone). Note that we have made a number of assumptions here concerning the consistent definition of t.

To evaluate the amplitude (2) requires a systematic way of describing the set $\mathcal{G}$. The discretized random surface is one way of doing this [1]. For simplicity consider a two-dimensional manifold with euclidean metric (so this is not really gravity which should have a lorentzian metric) and spherical topology. Disregarding the metric for a moment such a manifold can be triangulated with $N \geq 4$ triangles; the idea is that by taking $N \to \infty$ in an appropriate way we can recover a continuum space. The trick is then to associate a triangulation with a metric in the following way: we suppose that all triangles are equilateral of side a and define the geodesic distance between any two points as La where L is the number of links in the shortest path connecting them. Then every distinct triangulation T leads to a distinct metric and the vacuum amplitude is given by

$$Z = \sum_{T \in \mathcal{T}} \exp(-S_T) \tag{3}$$

where $\mathcal{T}$ is the set of all distinct triangulations of the sphere and we have now set $\hbar = 1$. As well as the metric we can identify the Ricci scalar R in this construction. It is simply proportional to the deficit angle for parallel transport around a closed loop divided by a^2; since all the triangles are equilateral the deficit angle at site i is simply given by $(6 - n_i)\frac{\pi}{3}$ where n_i is the number of triangles which have i as a

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vertex. Many objects of interest have been calculated in this particular model which is often known as 'two-dimensional euclidean quantum gravity'.

The idea can be extended to space-times of three and four dimensions and to lorentzian space-times; unfortunately it becomes progressively more difficult with increasing dimension to obtain analytic results in these models although much has been learned by doing numerical simulations [4, 5]. Our interest is in obtaining analytic results so the rest of this article concerns low dimensional models.

Dimension

A moment's thought will show that many of the triangulations contributing to $\mathcal{T}$ are highly irregular. So although they are locally two-dimensional they might have global properties that are quite different, as do fractal graphs, and on the average the properties of $\mathcal{T}$ may show little resemblance to a smooth flat two-dimensional space. These differences can be characterised to some extent through 'dimensions' which probe different aspects of the space; usually these quantities are defined so that they all give the same result for smooth flat spaces (eg 2 for R^2). We will be concerned with just two of them: the Hausdorff dimension d_h and the spectral dimension d_s.

The Hausdorff dimension d_h is a measure of the volume growth. Let $V_0(r)$ be the number of points within a distance r (defined as in the previous section) of the point 0; then

$$\langle V_0(r)\rangle_\mu \sim const \; r^{d_h}. \tag{4}$$

Note that this definition leads to $d_h = 1$ for a line, $d_h = 2$ for R^2 etc. In the case of the triangulations of the sphere it was proved by Chassaing and Schaeffer [6] and by Angel and Schramm [7, 8] that $d_h = 4$, a result that had been known, albeit less rigorously, to physicists for many years [9]. This value requires some explanation. The point is that the structure of the typical graph with an infinite number of points is highly branched – it consists of an infinite tube whose diameter goes roughly like r^2 with many finite outgrowths branching out of it:
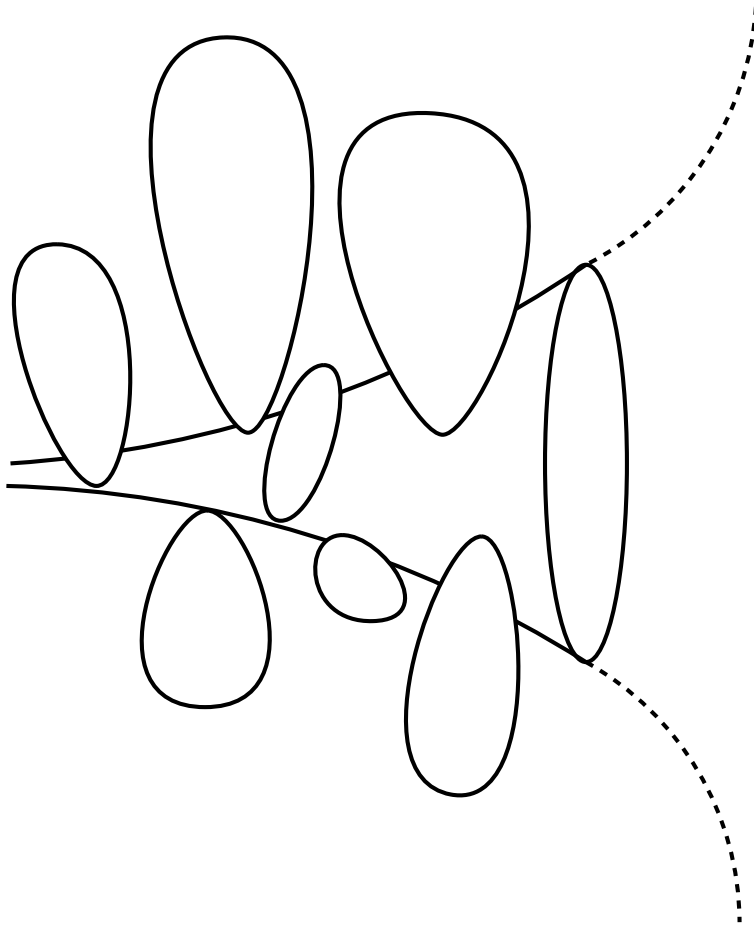
As was proved by Angel most of the volume is in the outgrowths and this is what makes $d_h = 4$. This branching structure rather suggests that it might be worthwhile studying simpler sets of graphs with an infinite spine such as trees:
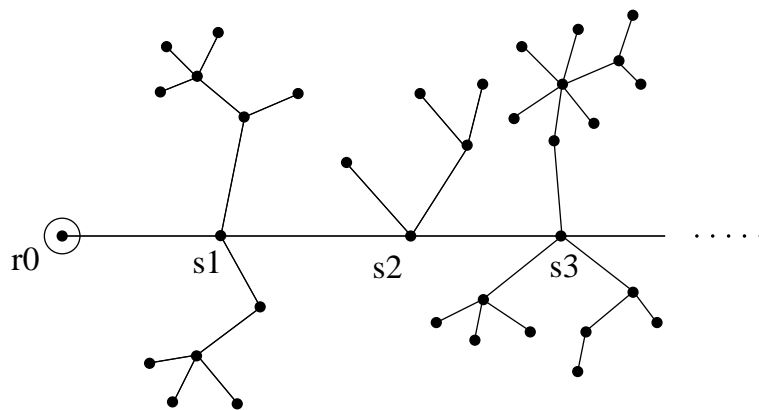
and combs:
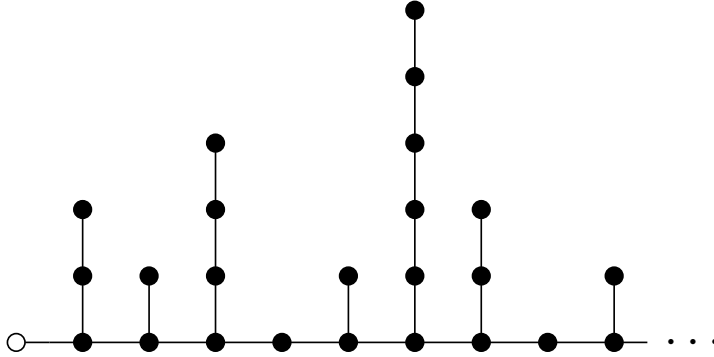
These sets of graphs are simple enough that, as well as d_h, the spectral dimension d_s can also be computed exactly. They still have a metric defined by the geodesic distance between points and they still have scalar curvature defined by the deficit angle: vertices of order 3 have $R < 0$, vertices of order 2 have $R = 0$ and vertices of order 1 have $R > 0$.

The spectral dimension d_s is a measure of how likely it is that a random walk returns to the point of origin. Walks on graphs are defined so that at each time step the walker chooses to hop to one of the σ neighbouring points with uniform probability σ^{-1}. After a long time t has elapsed the probability that the walker is back at the starting point is given by

$$p(t) \sim \frac{const}{t^{d_s/2}}. \tag{5}$$

It is known that for spaces with scalar curvature $R \geq 0$ everywhere $d_s = d_h$. So for example we have the famous $\frac{1}{\sqrt{t}}$ return probability for random walk on a line. However spaces with branching, such as combs and trees, necessarily have $R < 0$ at the branching points and d_s can be different from d_h. The work we have been doing recently has been to determine d_s for some of these models with branching.

Combs

The spectral dimension of several ensembles of combs is investigated in detail in [12] where proofs of all the statements made here can be found. In that paper we made a particular point of finding methods that were as elementary as possible and avoided the use of heavy mathematical machinery.

The important point to appreciate is that, from the quantum gravity motivation where we started, we are not interested in the properties of any one graph but in the average properties of the whole ensemble. The models are characterised by the assumption that the length ℓ of each tooth is distributed identically and independently with probability $\pi(\ell)$.

If $\pi(\ell) = 0$ for $\ell > \ell_0$, all teeth are of finite length and it turns out that they do not affect the large t behaviour so $d_s = 1$. This can be understood by noting that although there is negative curvature at the branching points the compensating regions of positive curvature at the ends of the teeth are nearby; so averaging over

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a finite region produces zero net curvature – on large enough scales the combs just look like a line. At the other extreme $\pi(\infty) = 1$; now all teeth are infinite and the positive curvature is pushed off to infinity. A simple calculation shows that $d_s = \frac{3}{2}$; clearly $d_h = 2$ so we see the effect of the branching in reducing d_s below d_h. In fact to get $d_s = \frac{3}{2}$ it does not matter that *every* tooth is infinite, just that teeth have a finite probability of being infinite. To interpolate between these regimes it is useful to consider $\pi(\ell) = b\ell^{-a}$, $a > 1$; the smaller a the longer the typical tooth and so the larger we might expect the shift away from $d_s = 1$ to be. Our results are shown in the following table:

	d_H	d_s	$\frac{2d_H}{1+d_H}$
Random tooth spacing	2	$\frac{3}{2}$	$\frac{4}{3}$
Random tooth length $a \geq 2$	1	1	1
Random tooth length $1 < a < 2$	$3 - a$	$\frac{4-a}{2}$	$\frac{6-2a}{4-a}$
Random trees	2	$\frac{4}{3}$	$\frac{4}{3}$

It turns out that the crucial point is whether $\langle \ell \rangle_\mu$ is finite; if it is, ie $a > 2$, then the teeth are short enough that $d_s = 1$, but if it diverges then d_s depends on a. The last column is shown in the table because it is known that $d_s \geq \frac{2d_h}{d_h+1}$ for *any* graph in much more general circumstances and therefore it ought to apply to the comb ensemble, as indeed it does.

Trees

Earlier we suggested looking at trees as an example of spaces which share some of the characteristics of the triangulations but which are substantially simpler. In fact certain tree ensembles are particularly interesting to us because they occur naturally when conformal matter fields of central charge $c > 1$ are coupled to the 2D euclidean quantum gravity model. The back reaction is so strong that the space collapses to one made up of tubes whose diameter is $O(a)$ linked up in a tree-like structure:
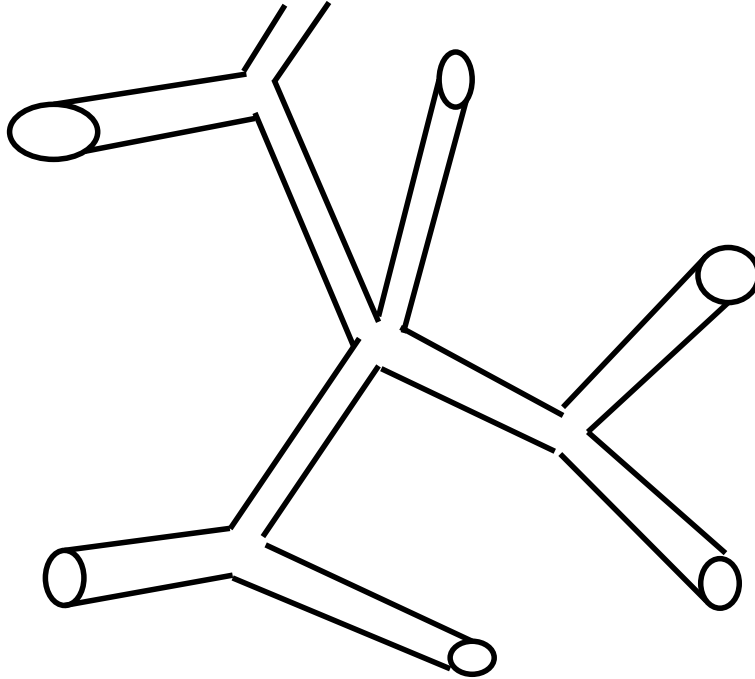
The finite tube diameter can have no effect on the long time properties of a random walk so we replace the tube structures by pure trees with a branching law for the vertices inherited from the original structures. The ensemble of trees obtained in this way is called the generic tree ensemble. It is defined by the condition that the variance of the distribution of vertex order is finite; all such ensembles have the same d_h and d_s regardless of the details of the distribution. The typical infinite tree consists of an infinite spine with finite trees branching off [10, 13]. So these trees have similar properties to the triangulations but are much easier to analyse. It has been known for a long time that they have $d_h = 2$ and a simple proof that the ensemble average has $d_s = \frac{4}{3}$ is given in [13] – note that this saturates the lower bound $2d_h/(d_h + 1)$. The trees have much more negative curvature than the combs; almost every vertex has negative curvature so perhaps it is not surprising that d_s is as low as possible. For some choices of the vertex distribution these trees can also be regarded as percolation clusters at criticality in high dimension and in this context have been investigated extensively by probability theorists who have proved the stronger result that almost all members of these ensembles individually have $d_s = \frac{4}{3}$ [11, 14].

Where next?

The actual results we have discussed apply mostly to toy problems rather than to the quantum gravity models. So now the interesting question is whether the techniques employed so far can be generalised to these. The main difference is that the graphs involved now contain loops rather than just being pure branching objects; of course the presence of loops *per se* has no effect on d_h but the extra connectivity can drastically affect the spectral dimension.

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