2 Find a normalisation constant N (depending on the energy E) such that the free particle positive energy solutions to the Dirac equation

$$\psi = N \begin{pmatrix} \phi \\ \sigma.\mathbf{p}/E & +m \end{pmatrix} e^{-ip.x}, \; E = \sqrt{m^2 + \mathbf{p}^2} \; (\hbar = c = 1)$$

are normalised to $\psi^\dagger\psi = E/m$. Why should one want $\psi^\dagger\psi$ to behave under a Lorentz transformation as an energy|?

Show that for the negative energy solutions it is also possible to choose the normalisation such that $\psi^\dagger\psi$ is possible.

3 A rotation through an angle α about the x-axis is specified by a change in coordinates

$$x' = x, \quad y' = \cos\alpha \; y + \sin\alpha \; z, \quad z' = -\sin\alpha \; y + \cos\alpha \; z.$$

Under this transformation of coordinate axes, a Dirac spinor ψ transforms by

$$\psi' = \exp(i\Sigma_x\alpha/2)\psi$$

where

$$\Sigma_x = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}.$$

Verify that

$$\psi^\dagger \alpha_y \psi$$

transforms as the y-component of a vector under this transformation, where α_y is the Dirac matrix $\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$.

(ii) Verify that $\overline{\psi}\psi$ (defined as $\psi^\dagger\beta\psi$) is invariant under the above transformation, and also under the 'boost' transformation

$$\omega' = \exp(\alpha_x\vartheta/2)$$

where α_x is the Dirac matrix $\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}$.