Advanced Quantum Mechanics MT 2007

Problem Set 2: Relativistic Quantum Mechanics

1 A 4-D Green function for the scalar wave equation is defined (with $c = 1$) by

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t').$$

(i) Show that the Fourier transform of G with respect to the variable $t - t'$, namely

$$\text{(A)} \qquad \overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = \int G(\boldsymbol{r}-\boldsymbol{r}', t-t') e^{i\omega(t-t')} d(t-t'),$$

satisfies the equation

$$(\nabla^2 + \omega^2)\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = -\delta(\boldsymbol{r}-\boldsymbol{r}').$$

Show further that the Fourier transform of $\overline{G}$ with respect to the variable $\boldsymbol{r}-\boldsymbol{r}'$, namely

$$\text{(B)} \qquad \overline{\overline{G}}(\boldsymbol{p}, \omega) = \int \overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) e^{-i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')} d^3(\boldsymbol{r}-\boldsymbol{r}'),$$

is given by $\overline{\overline{G}}(\boldsymbol{p}, \omega) = \frac{1}{\boldsymbol{p}^2 - \omega^2}$.

(ii) Show from the inverse of (B) that

$$\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = \frac{1}{16\pi^2 i|\boldsymbol{r}-\boldsymbol{r}'|} \int_{-\infty}^{\infty} dp \left[e^{ip|\boldsymbol{r}-\boldsymbol{r}'|} - e^{-ip|\boldsymbol{r}-\boldsymbol{r}'|}\right] \left(\frac{1}{p-\omega} + \frac{1}{p+\omega}\right),$$

where $p = |\boldsymbol{p}|$.

(iii) Evaluate $\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega)$ by contour integration, *assuming* that $\omega \to \omega + i\epsilon$ is the correct way to deal with the singularities. Show that this leads to

$$\overline{G}(\boldsymbol{r}-\boldsymbol{r}', \omega) = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} e^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|}.$$

Hence show that

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\delta(t' - t_{\text{ret}})$$

where the "retarded time" t_{ret} is given by

$$t_{\text{ret}} = t - |\boldsymbol{r}-\boldsymbol{r}'|.$$

(iv) Solve the equation

$$\Box\phi(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, t)/\epsilon_0$$

and explain the answer physically; in particular, why should the choice $\omega \to \omega - i\epsilon$ be ruled out in part (iii)?

4 Show that an alternative choice for the matrices $\boldsymbol{\alpha}$ and β in the Dirac equation is

$$\boldsymbol{\alpha} = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Writing the wavefunction as

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} e^{ip.x}, \ (\hbar = c = 1),$$

obtain the equations satisfied by ϕ and χ. Verify that for consistency one requires that $E^2 = m^2 + \mathbf{p}^2$.

Show that the ϕ and χ equations decouple if $m = 0$. Interpret the equations in this case *i.e.* what are the properties (in particular their helicities) of the particles described by them?

Find the explicit forms for ϕ and χ in the case $p = p(\sin\theta, 0, \cos\theta)$ and $m = 0$ satisfying $\phi^\dagger\phi = \chi^\dagger\chi = 1$.

[Hint : show that $\phi_1/\phi_2 = -\tan(\theta/2)$]

5 A positive energy spin-1/2 particle of mass m is incident from the left ($z < 0$) on a one-dimensional barrier of height V which exists in the region $z \geq 0$. What boundary condition should you impose on the wavefunction at $z = 0$? [Hint: consider ρ and $\boldsymbol{j}$.]

Write down the appropriate Dirac equation and solve it, using your boundary condition. Show that if V is *large* enough the reflected current is larger than the incident current. Suggest a physical interpretation of this result.