Gravity, Random Graphs and Spectral Dimension

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This is partly a review and many people have contributed to the subject
Some recent work on spectral dimension with Bergfinnur Durhuus and Thordur Jonsson
is in arXiv: hep-th/0509191, math-ph/0607020, and arXiv:0908.3643

Gravity, Random Graphs & Spectral Dimension

1. From quantum gravity to graphs
2. Large scale structure
3. Some graph ensembles
   - Combs
   - Trees
   - Triangulations
4. Open questions

1. From quantum gravity to graphs

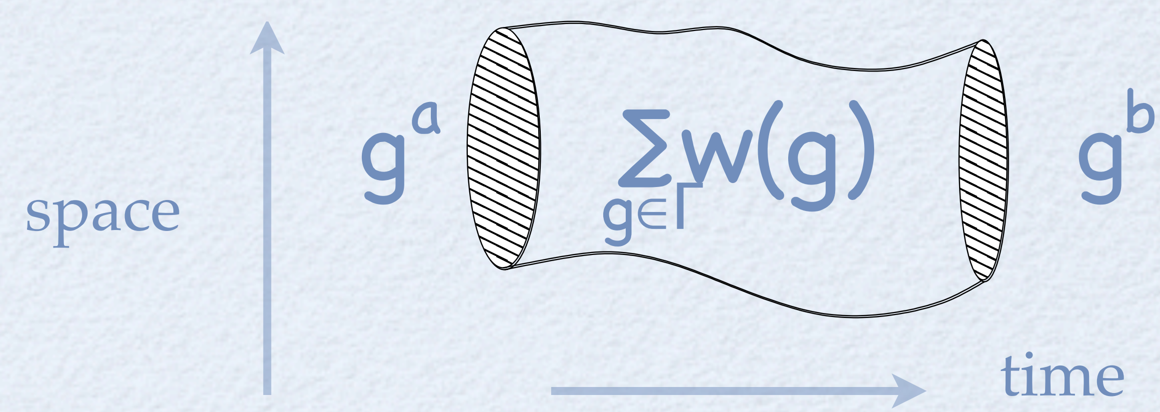
Gravity's dynamical degree of freedom is the metric $g_{\mu\nu}(x,t)$

Classically $g_{\mu\nu}(x,t)$ obeys Einstein's equations:

$$g_{\mu\nu}(x,0) \longrightarrow g_{\mu\nu}(x,t)$$

Quantum mechanics is different:

$$\langle g^b(x),\ t=T \mid g^a(x),\ t=0 \rangle \sim$$



Probability amplitude for evolution from g^a to g^b

How is Γ defined ?

In the discretized approach by triangulation, in 2d...

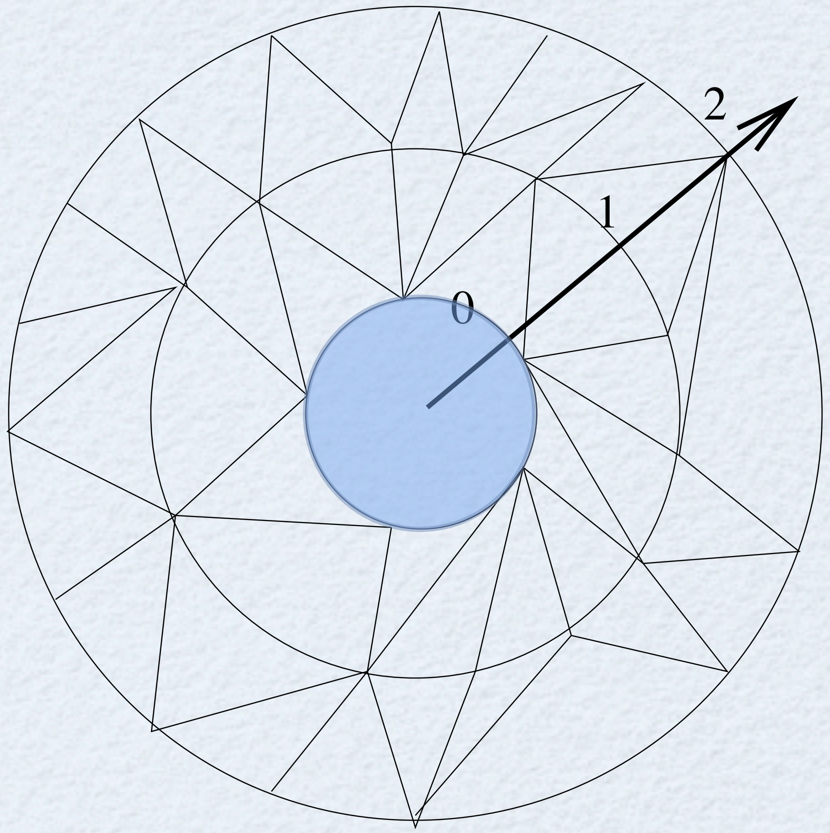
1. Unconstrained -- Planar Random Graphs

2. Constrained -- Causal Triangulations

$g_{\mu\nu}(x,t) \rightarrow$ geodesic distance

$\sim a \times$ graph distance R

continuum $R \rightarrow \infty$, $a \rightarrow 0$



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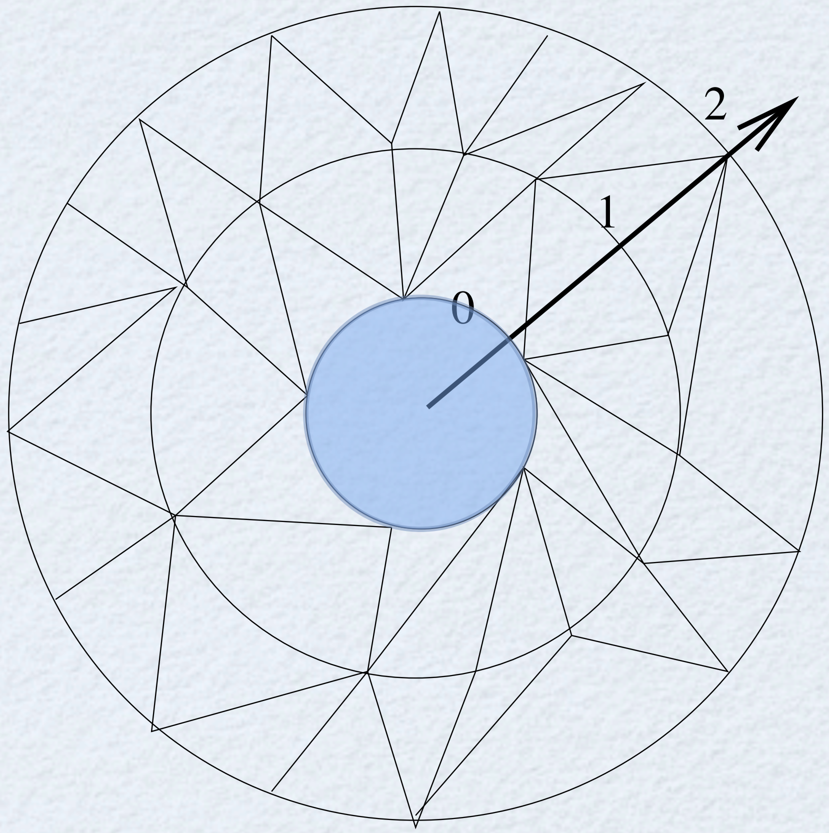


continuum $R \rightarrow \infty,\ a \rightarrow 0$

Physics depends on large scale properties

2. Large scale structure

In fact many interesting physical systems can be expressed in terms of ensembles of graphs generated by local rules eg

- Percolation clusters
- Generic random trees
- Planar random graphs
- Causal dynamical triangulations

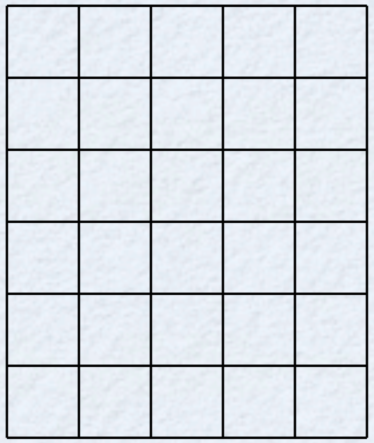
A simple way to characterize the typical large scale properties of graphs in these ensembles is through the notion of dimension

Hausdorff dimension d_H -- we assume ∞ graphs

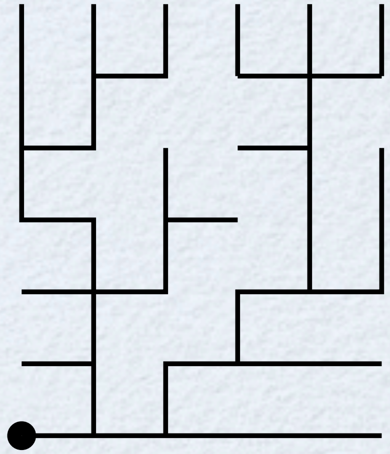
1. Choose a point r_0
2. Find all points $B_R(r_0)$ within graph distance R of r_0
3. $| \ B_R(r_0) \ | \sim R^{d_H}$ as $R \rightarrow \infty$, independent of r_0

d_H tells us about the volume distribution but is blind to some sorts of connectivity eg

$d_H = 2$ for Z^2

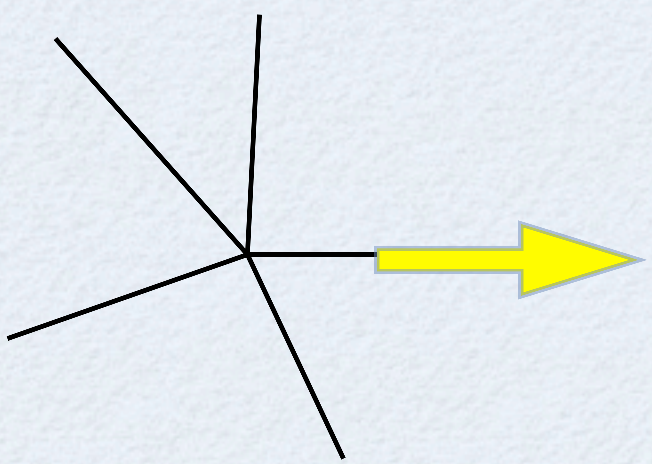


and GRT



Spectral dimension d_s

1. Choose a point r_0
2. Random walker leaves r_0 at time 0 and returns at time t with probability $q_G(t;r_0)$

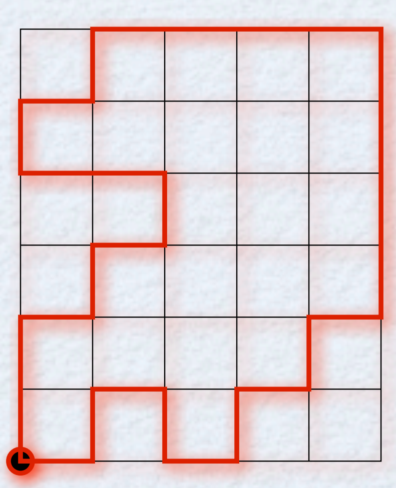


$\text{prob} = \sigma^{-1}$

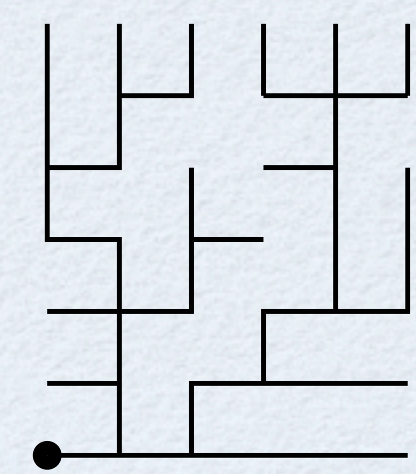
$q_G(t;r_0) \sim t^{-d_s/2}$ as $t \to \infty$

Random walk sees connectivity:

$d_s = 2$ for Z^2



but $4/3$ for GRT



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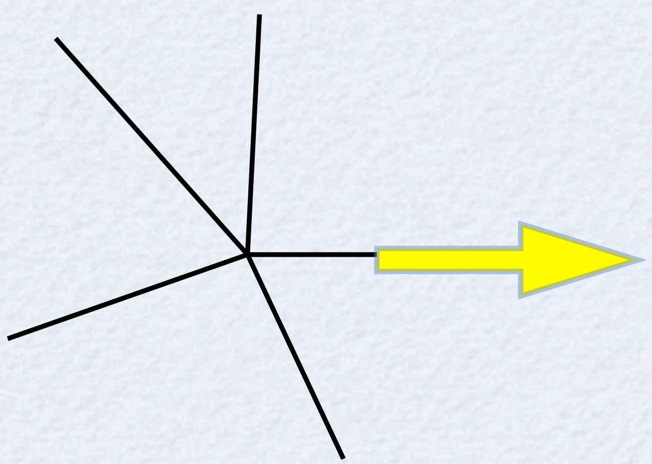

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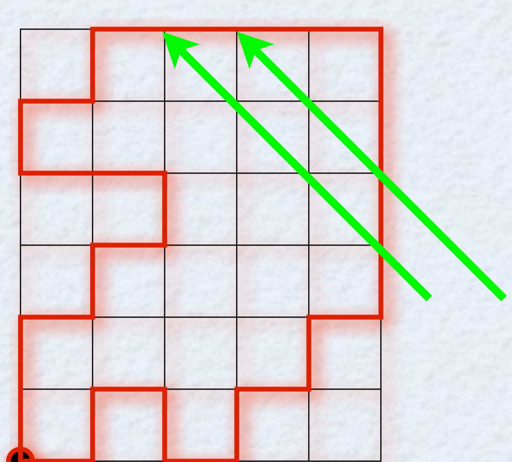

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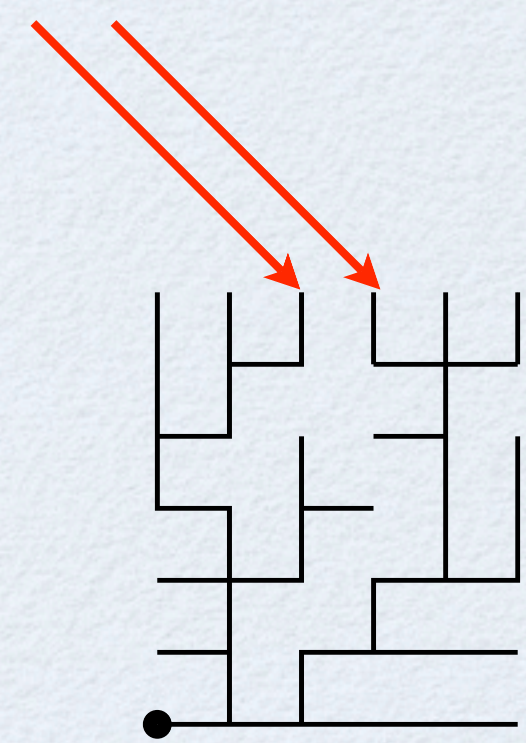
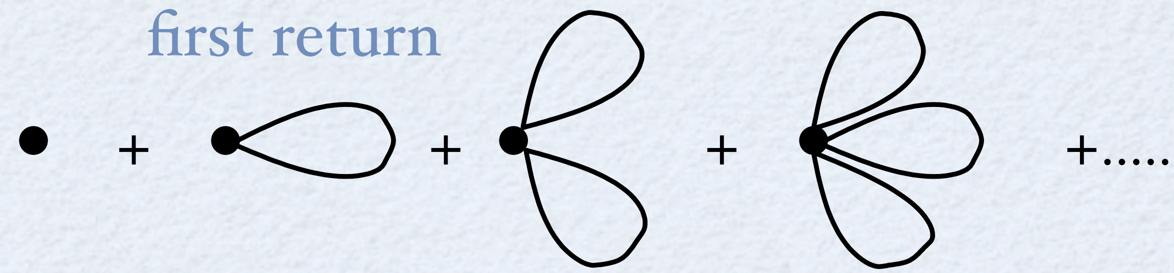


Recurrence

$q_G(t;r_0) =$



$$Q_G(x) = 1+\sum_{t=2}^{\infty} q_G(t;r_0)\,(1-x)^{t/2}$$

$$= \frac{1}{1 - P_G(x)}$$

first return

1. If $d_S > 2$ then $Q_G(0)$ finite $\Rightarrow 1-P_G(0) > 0$, walker can escape, graph is *non-recurrent*
2. If $Q_G(0)$ infinite $\Rightarrow 1-P_G(0) = 0$, walker always comes back, graph is *recurrent* and $d_S \leq 2$

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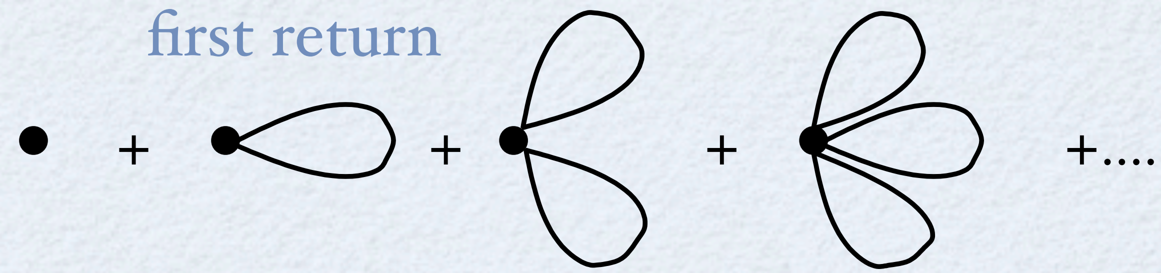
$q_G(t;r_0) =$ 

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2. If $Q_G(0)$ infinite $\Rightarrow 1-P_G(0) = 0$, walker always comes back, graph is *recurrent* and $d_s \leq 2$

Two questions about the dimension of ensembles of ∞ graphs:

1. Average quantities eg for the GRT

$$\langle \, | B_R(r_0) | \, \rangle_{\mu_\infty} \sim R^{d_H} \quad \text{with} \quad d_H = 2$$

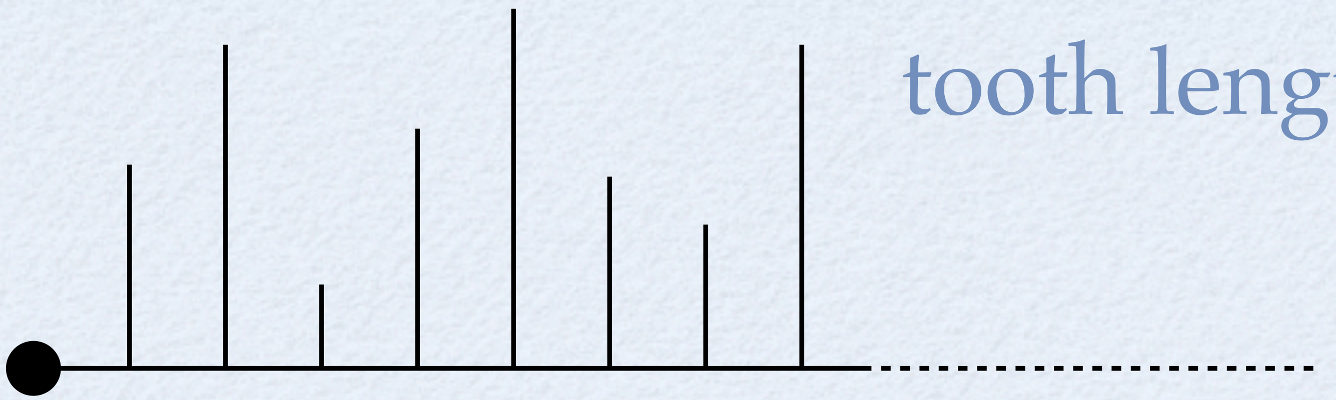
2. There may be a subset of graphs which appear with measure 1 and all have the same property eg for the GRT

$$| B_R(r_0) | \overset{a.s.}{\sim} R^2 \quad \text{up to log } R \text{ factors}$$

Clearly there are infinite trees for which $d_H \neq 2$ but they are rare -- they have measure 0

3. Some graph ensembles: Combs

tooth length i.i.d. $\pi(n) \sim n^{-a}$, $n \to \infty$



∞ spine

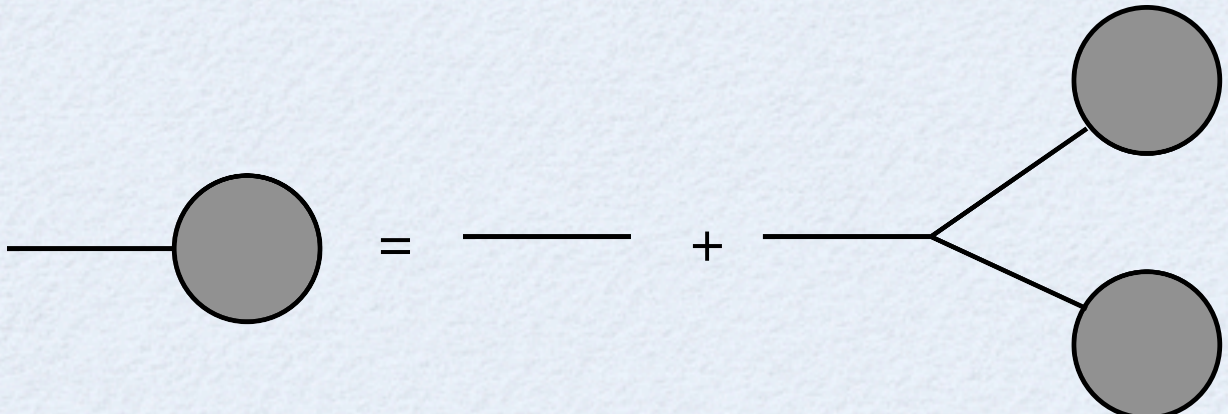
$\langle \, | B_R(r_0) | \, \rangle \sim R^{d_H}$ with $d_H = 3 - a$, $1 < a \leq 2$
and 1 if $a > 2$

$\langle \, q_G(t;r_0) \, \rangle \sim t^{-d_s/2}$ with $d_s = 2 - a/2$, $1 < a \leq 2$
and 1 if $a > 2$

Intuition? It is the very long teeth which matter....

3. Some graph ensembles: Trees

Generic Random Tree eg binary tree

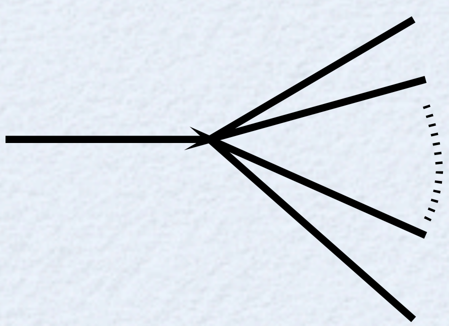


$Z = g + g\,Z^2$

so $\quad Z = \dfrac{1-(1-4g^2)^{1/2}}{2g}$

At $g=1/2$ we get a Critical Galton Watson ensemble

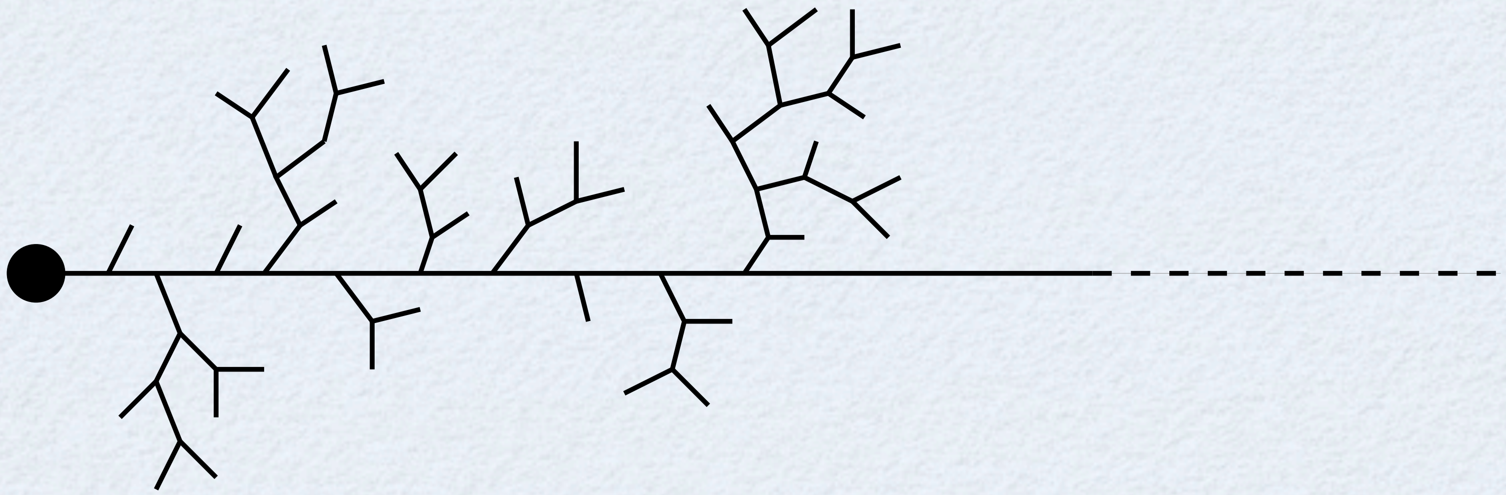
Special case of



p_n probability of n offspring

$f(x)=\Sigma\; p_n\, x^n$ CGW if $f(1) = f'(1) = 1, \quad f''(1) < \infty$

Generic Random Trees are the ∞ trees, measure μ_∞



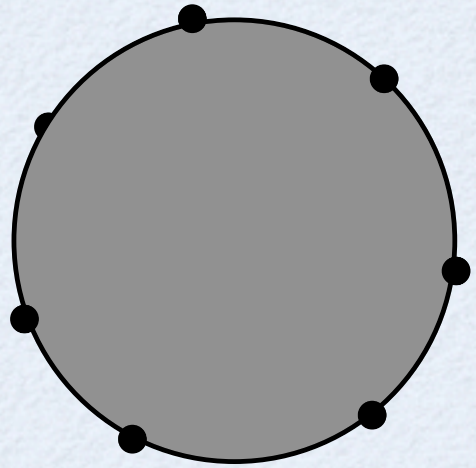
single ∞ branch

$$\langle\, | B_R(r_0) | \,\rangle_{\mu_\infty} \sim R^{d_H} \quad \text{with} \quad d_H = 2$$

$$\langle\, q_G(t;r_0) \,\rangle_{\mu_\infty} \sim t^{-d_S/2} \quad \text{with} \quad d_S = 4/3$$

- $d_H = 2$ a.s.
- $d_S = 4/3$ a.s.

3. Some graph ensembles: Causal Triangulations



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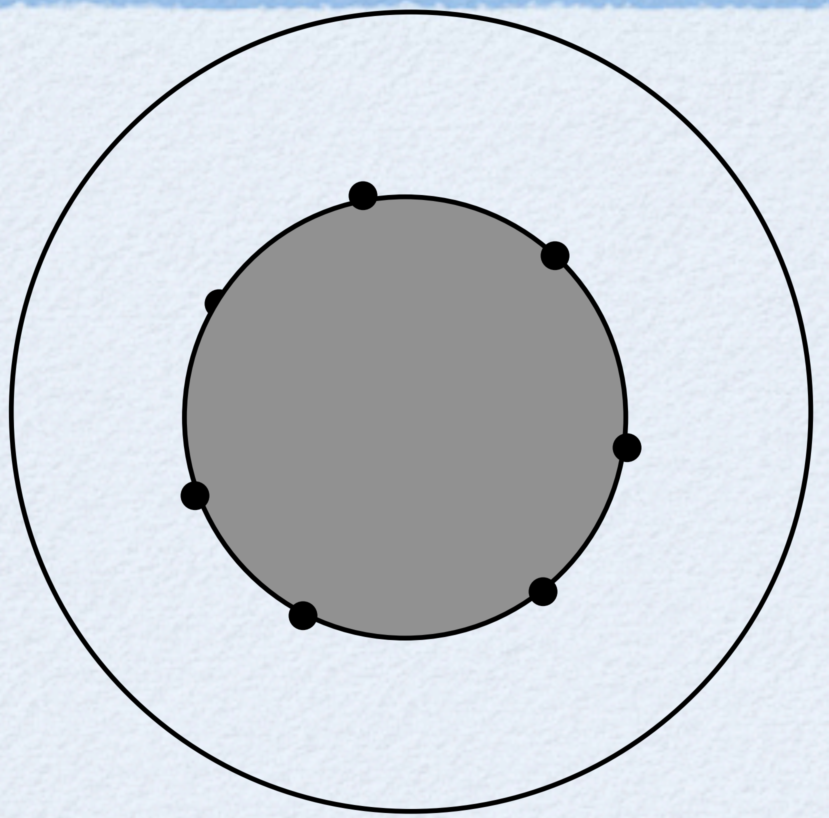

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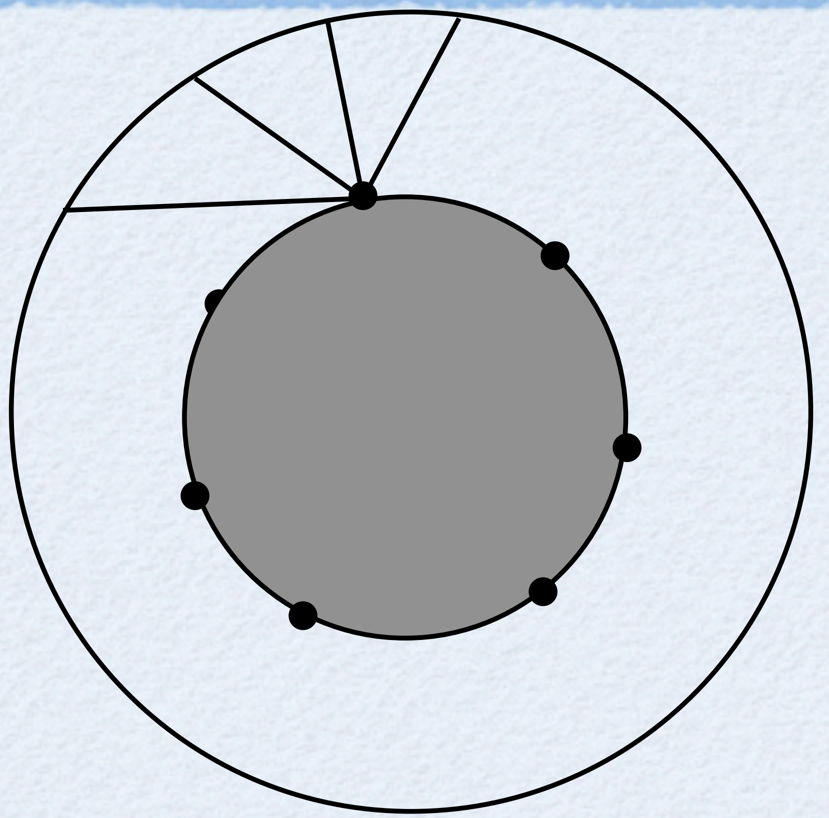

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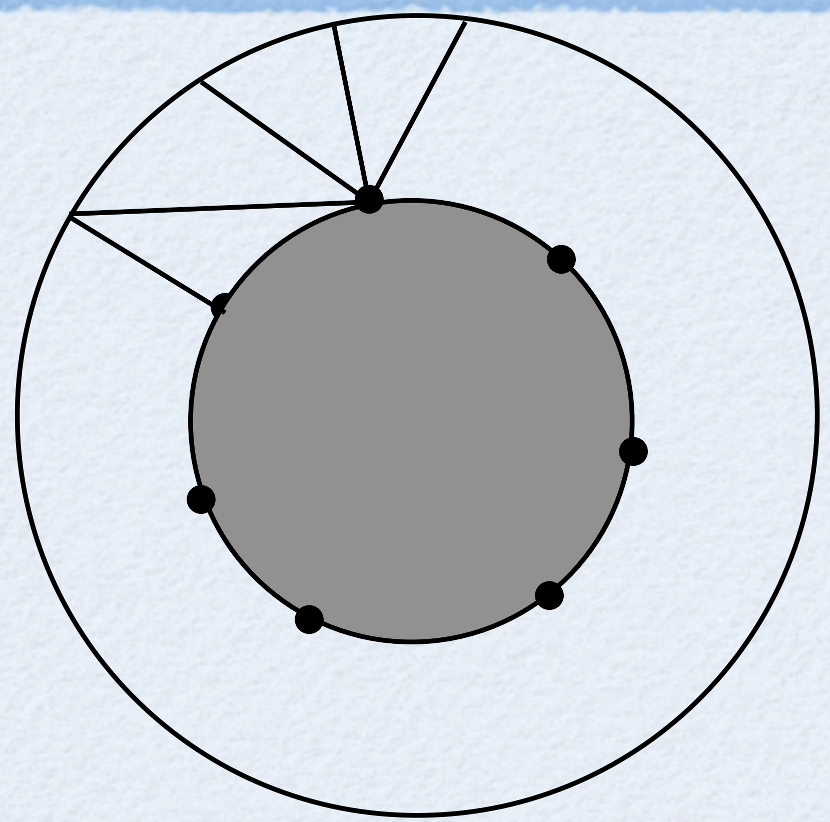

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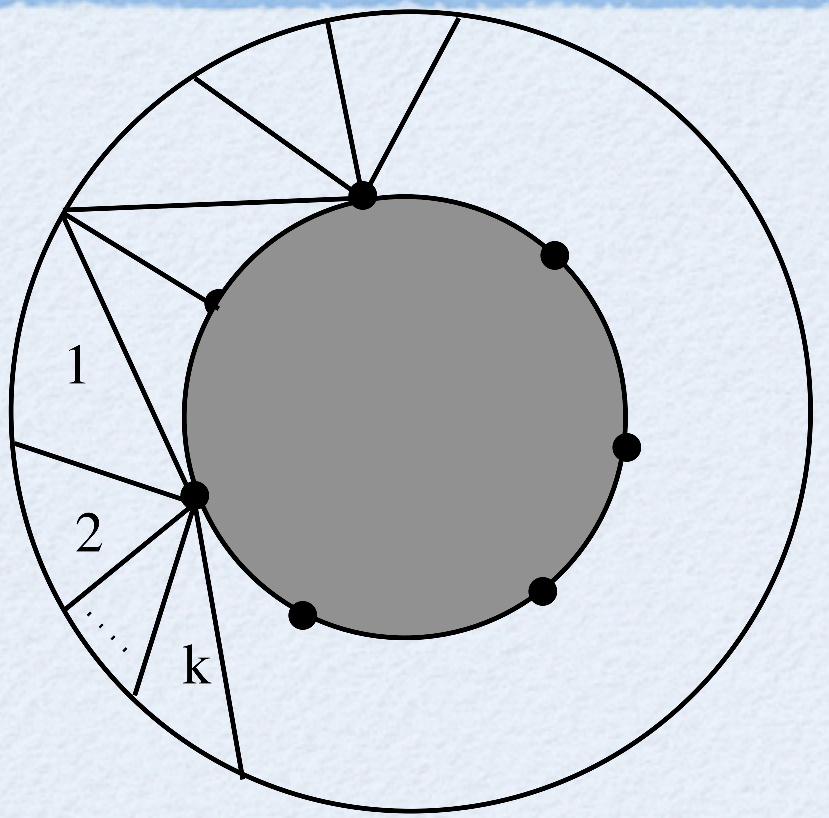

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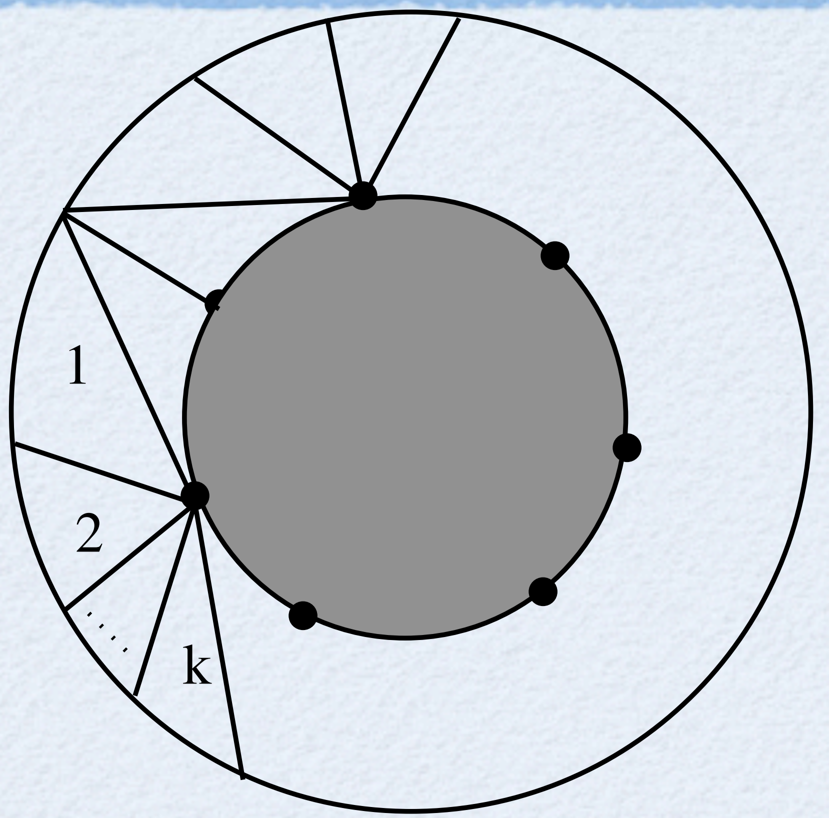

$$w_G = \prod_{v \in G} g^{k_v + 1}$$

$$\mathcal{Z}(g) = \sum_G w_G$$

Critical at $g_c = 1/2$

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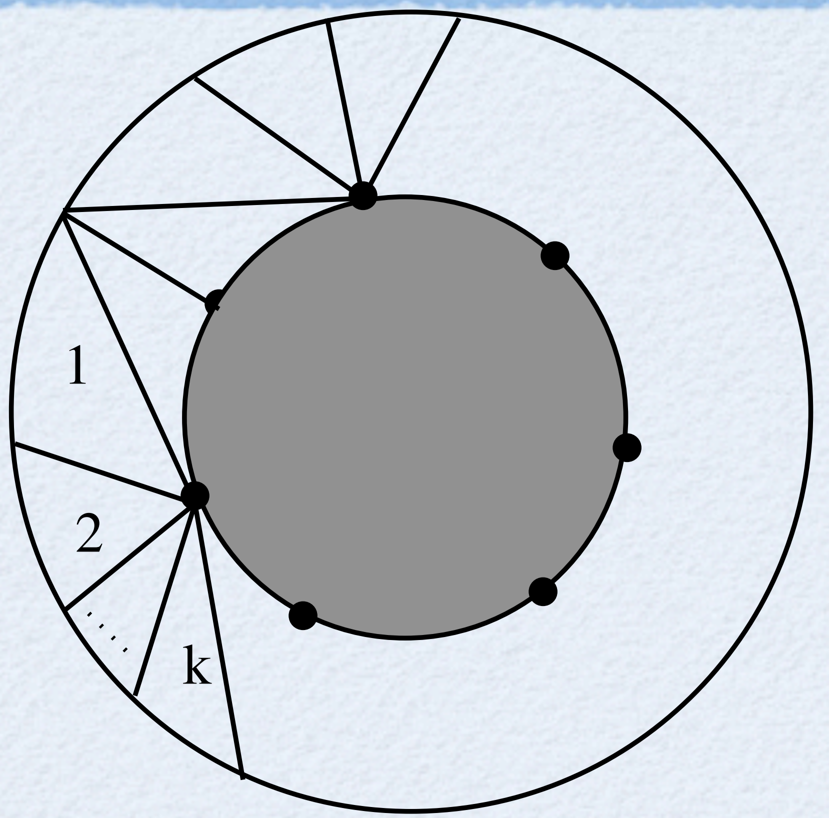

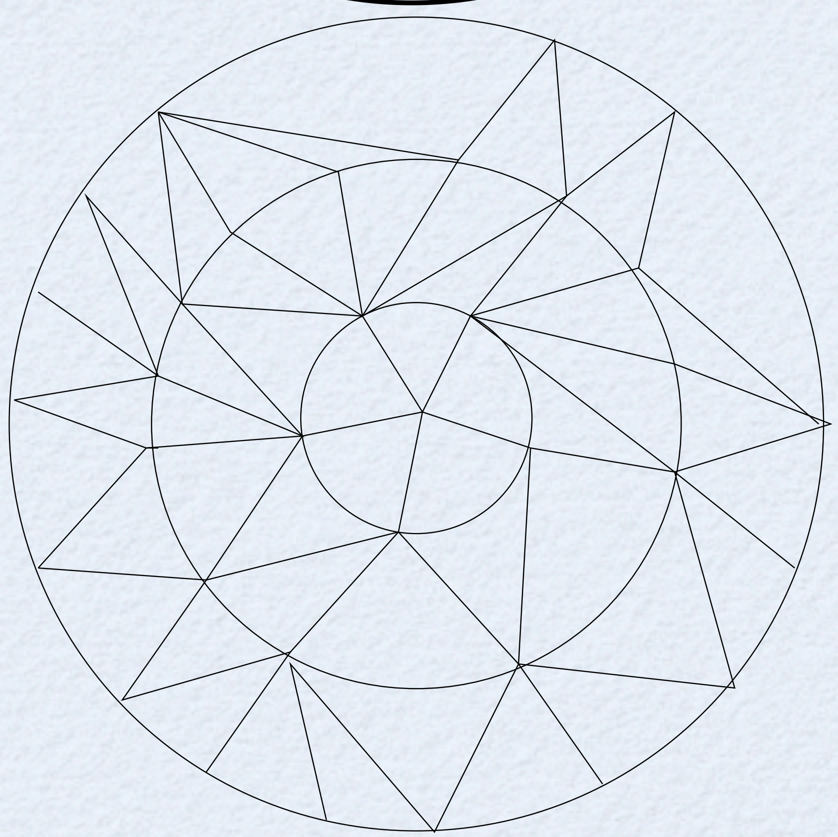

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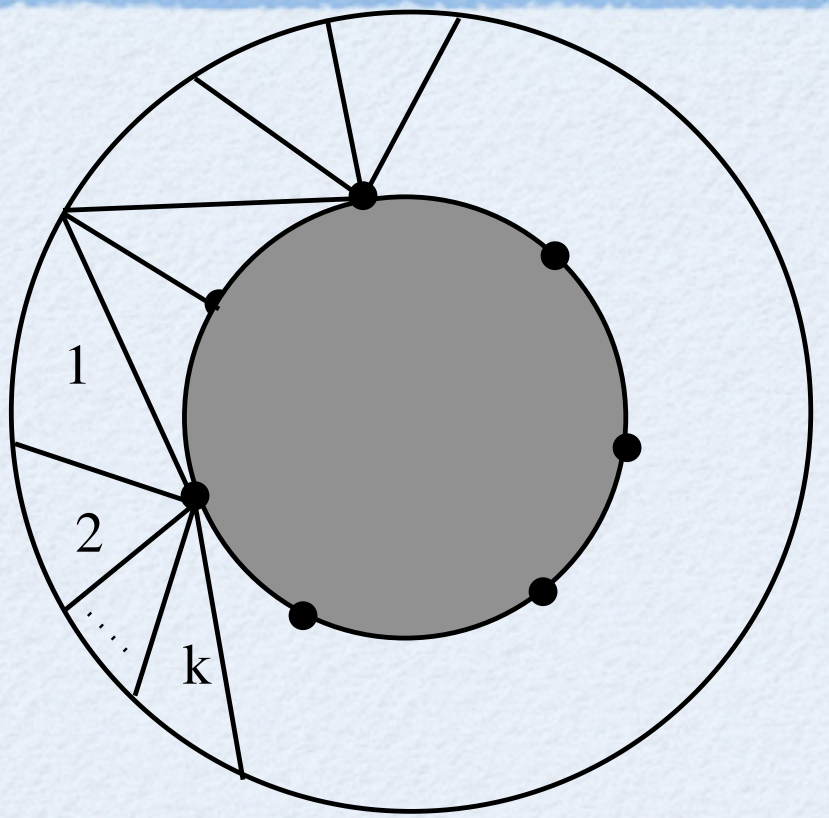

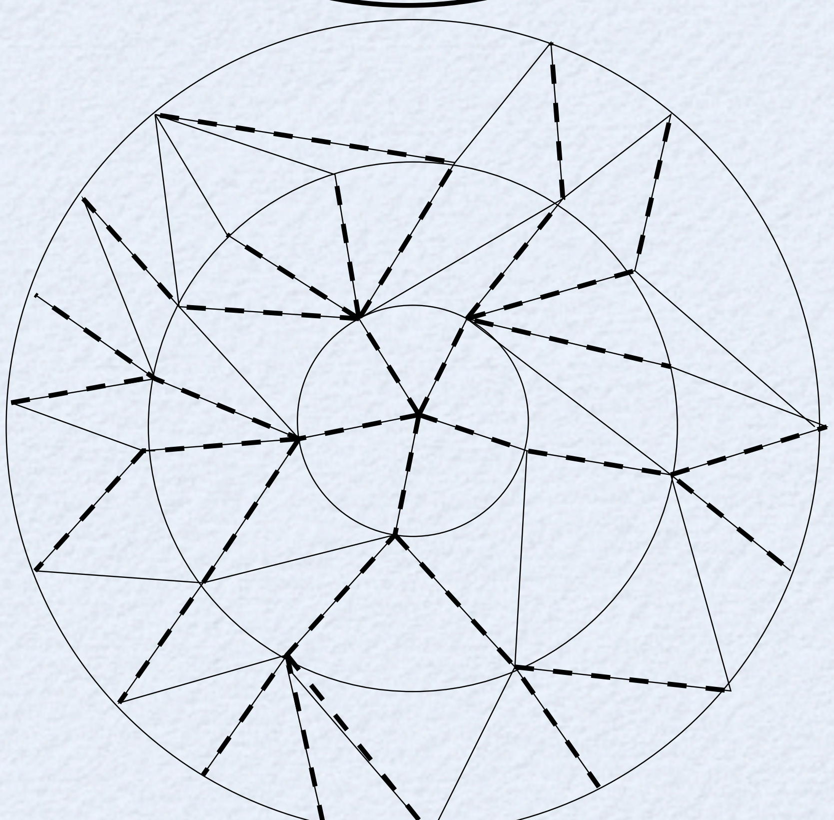

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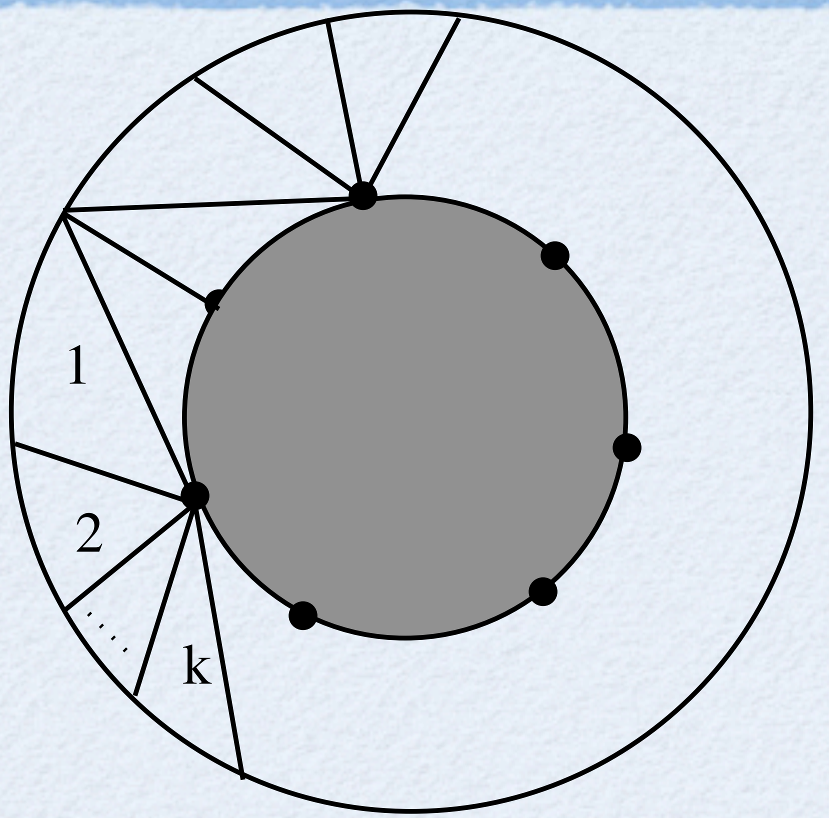

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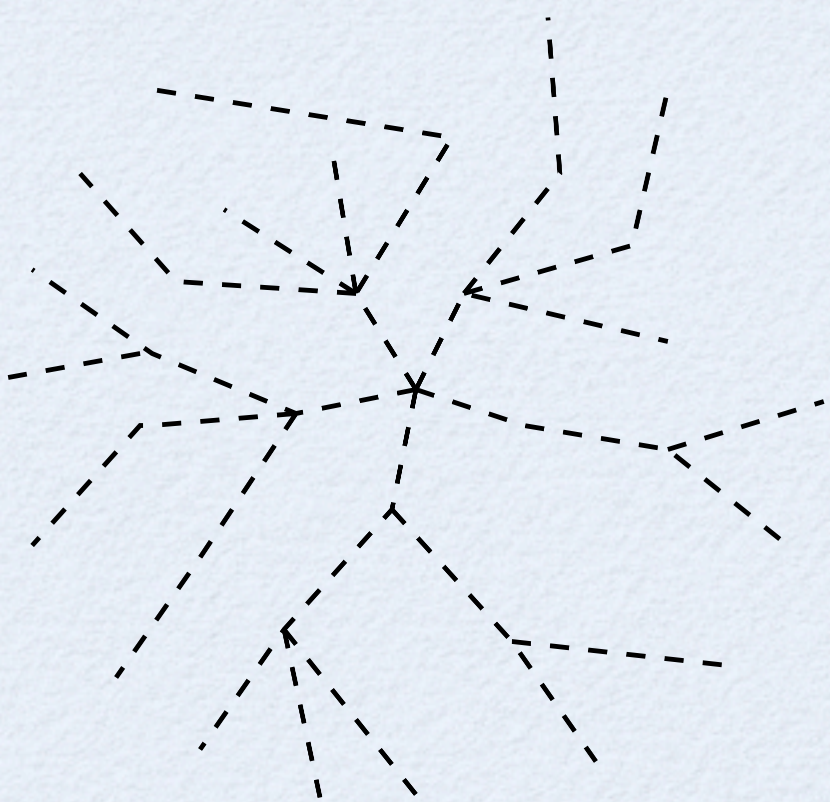

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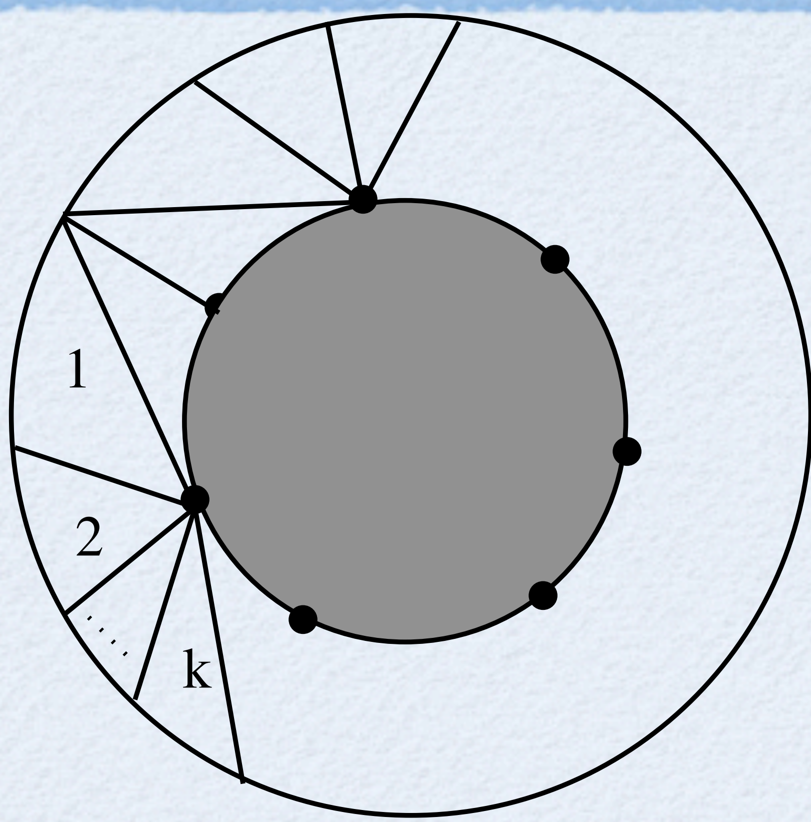

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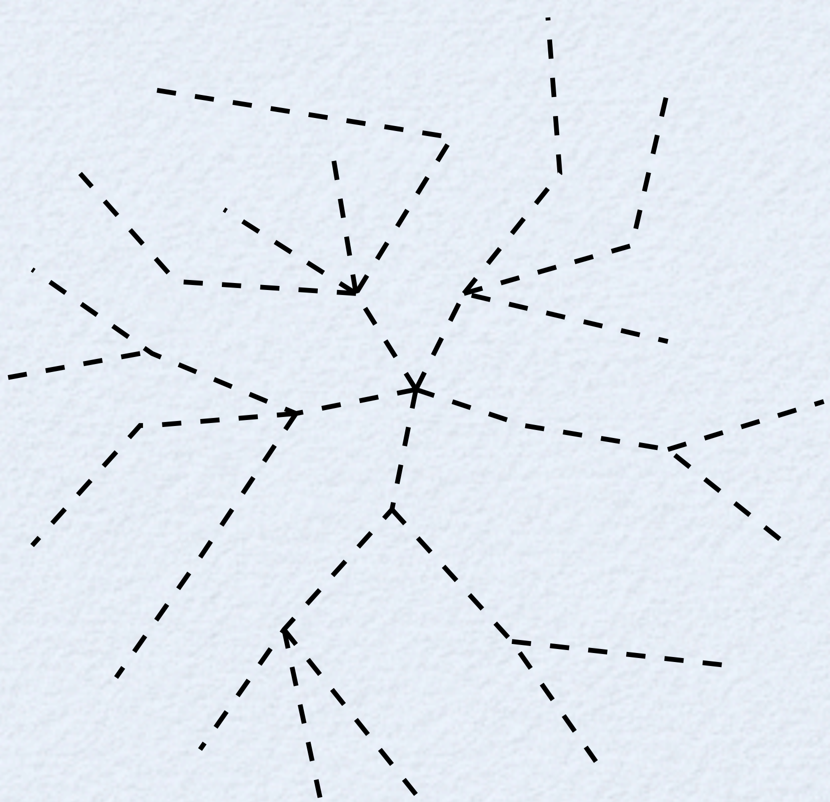


at g_c the trees are CGW with offspring probability

$$p_n = (1/2)^{n+1}$$

$$\mu(\infty \text{ CDT}) \Leftrightarrow \mu(\text{URT})$$

Uniform RT is a particular GRT

- Every vertex in a CT appears in the associated URT so

$$d_H = 2 \text{ a.s.}$$

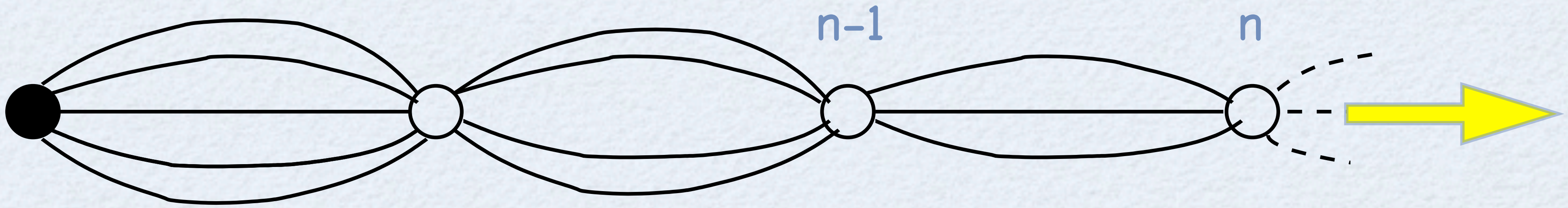
- First return probability $P_G(0) = 1$ a.s. so recurrent and

$$d_S \leq 2 \text{ a.s}$$

- Very weak lower bound from deleting links until only the URT remains

$$d_S \geq 4/3 \text{ a.s.}$$

-- but expect loops to be important so consider .....

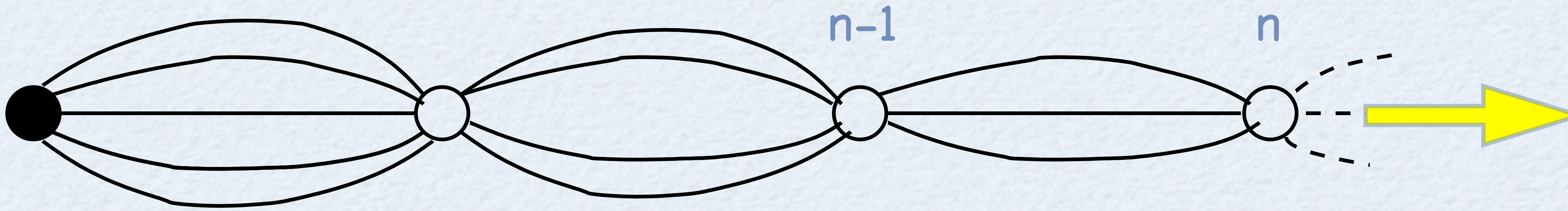


$$p_n = \frac{L_n}{L_n + L_{n-1}}$$

L_n distribution determined by μ_∞

This has a chain structure and (trivial) loops. It is recurrent a.s. and has

$$d_S = 2 \text{ a.s}$$



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CT results don't depend on URT --

for every GRT law there is a *local* action for the CT

$$w_G = \prod_{v \in G} \tau_v$$

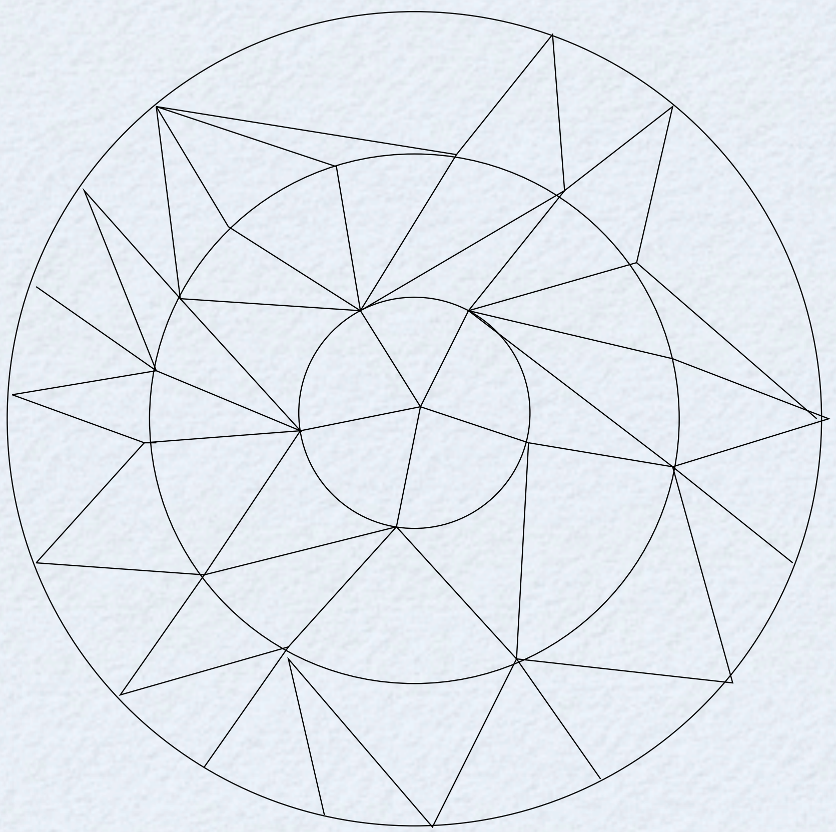
eg CT+dimer model of Di Francesco et al

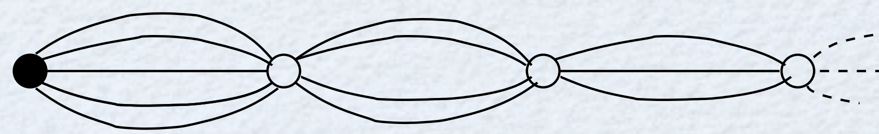
4. Open questions

- Do CTs have $d_S=2$ a.s. ?
- Are PRGs recurrent a.s., what is d_S ?
- What do other probes eg Ising spins show ?
- Can the corresponding annealed systems be controlled ?
- What can be said about higher dimensional CTs ?

Theorem: 2d CDTs are a.s. recurrent

Nash-Williams criterion: if electrical resistance to infinity is infinite, G is recurrent

G = 



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$$\text{Resistance of } G \geq \sum_n \frac{1}{L_n}$$

$$\mu(L_n > K) = \frac{K+2n-1}{2n-1}\left(1 - \frac{1}{2n}\right)^K$$

so if $K >> n$, then μ very small

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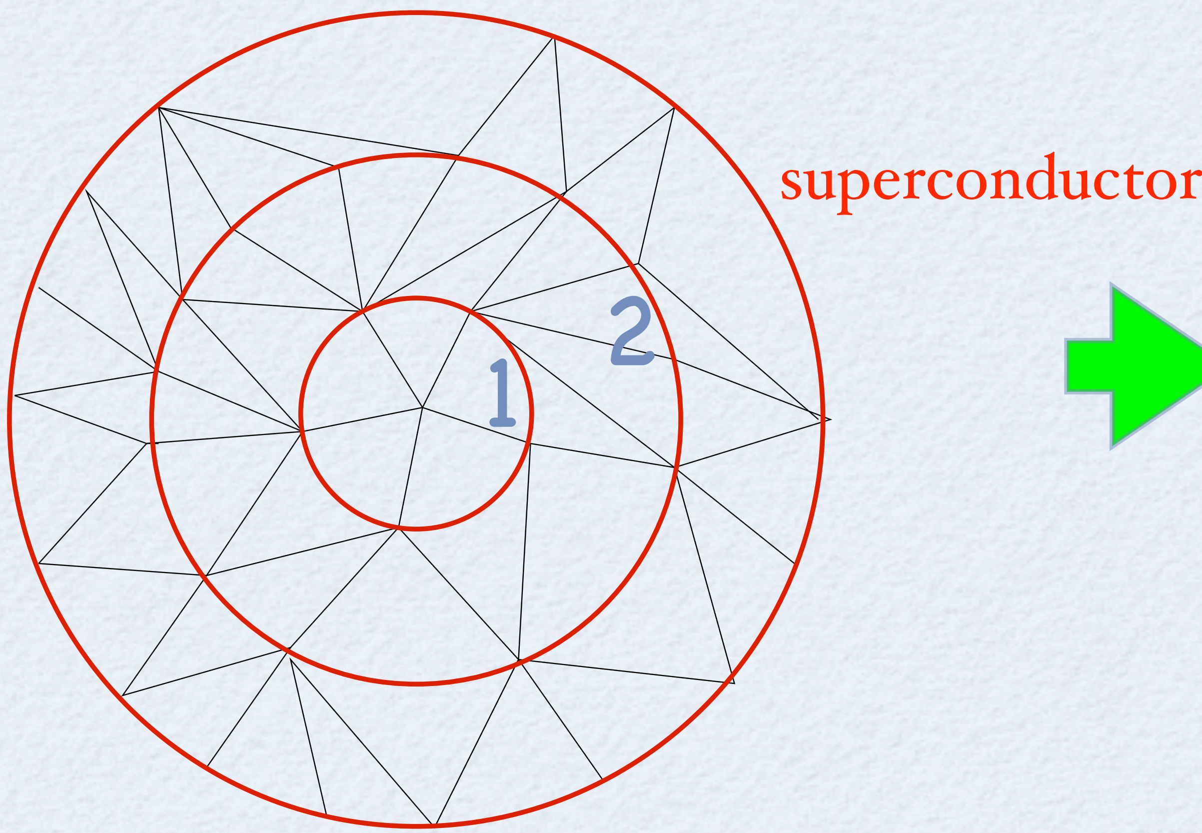

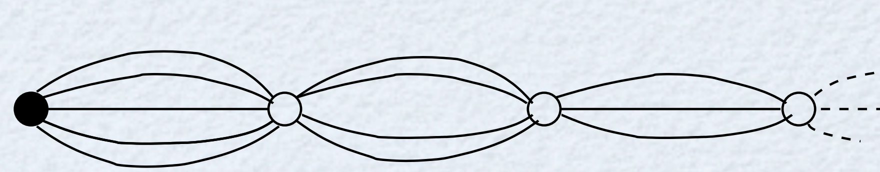

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$$\mathrm{Prob}\big(L_n > 2a\, n \log(n)\big) \le (1 + 2a \log(n))n^{-a}$$

$$\mathrm{Prob}\big(L_n > 2a\, n \log(n) \text{ for at least one } n > N\big)$$

$$\le \sum (1 + 2a \log(n))n^{-a}$$

$$\le C\, N^{1-a} \log(N)$$

Let q_n be the probability that n is the last point where $L_n > 2a\, n \log(n)$ then

$$q_{never} + \sum_{n=N+1}^{\infty} q_n \le C\, N^{1-a} \log(N)$$

- $q_{never} = 0$
- $\langle n \rangle$ is finite if $a>2$

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$\Longrightarrow$

with measure 1 $\exists$ N: $n>N$

$L_n < 2a\ n \log(n)$

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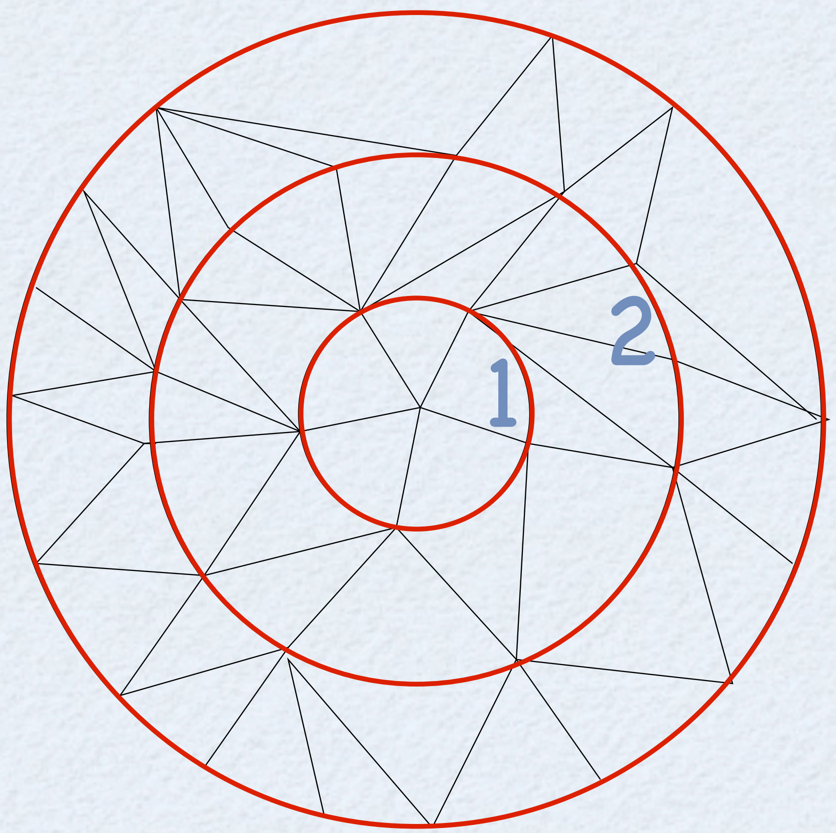
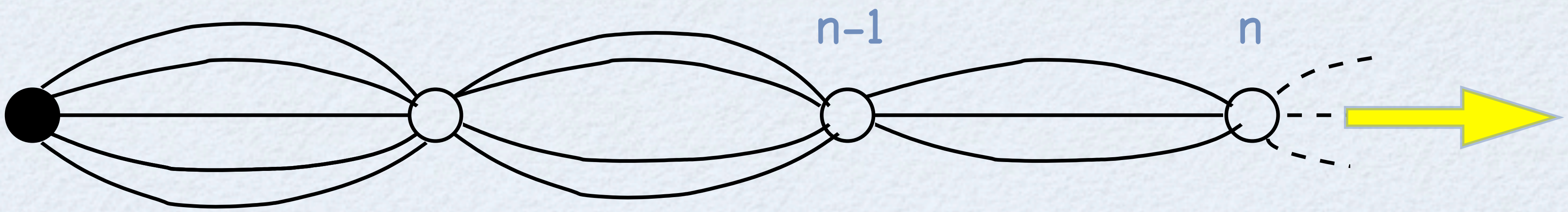


L_n distribution determined by μ_∞

$$\text{Resistance of } G \geq \sum_n \frac{1}{L_n} \overset{a.s}{>} \sum_{n=N}^{\infty} \frac{1}{2a\ n\ \log(n)} = \infty$$

Theorem: 2d RCDT has $d_S=2$ a.s.



$$p_n = \frac{L_n}{L_n + L_{n-1}}$$

$$P_G(x;n-1) = \frac{(1-x)\,(1-p_n)}{1 - p_n\, P_G(x;n)}$$

iterating out to $n=N$ gives

$$Q_G(x;1) \le L_1 \left(\frac{1}{xL_N} + \sum_{k=1}^{N} \frac{1}{L_k} \right)$$

we only need

$$\langle Q_G(x;1) \rangle \le c \left(\frac{1}{xN} + \sum_{k=1}^{N} \frac{1}{k} \right)$$

choosing $N=x^{-1}$ $\sim c \,|\log x\,|$

- Recurrence $Q_G(x;1)$ a.s. diverges as $x \to 0$
- $\langle Q_G(x;1) \rangle$ diverges only as $\log x$
- So $\nexists$ a subset of graphs with non-zero measure: $Q_G(x;1)$ diverges faster than $\log x$ as $x \to 0$
- So $Q_G(x;1)$ a.s. diverges logarithmically as $x \to 0$
- $d_S=2$ a.s.