EXOTIC CRITICAL PHENOMENA IN CLASSICAL SYSTEMS

Loops and strings on lattices

John Chalker
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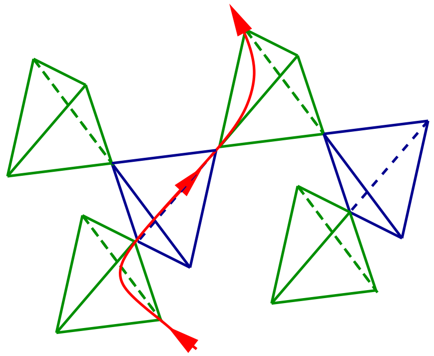
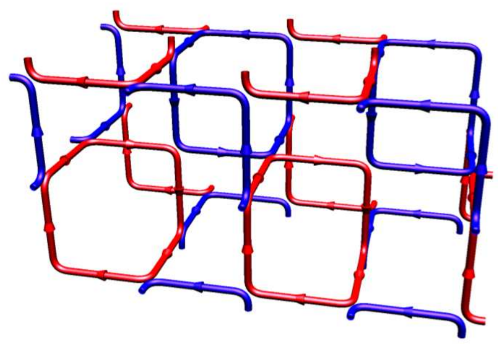




Work with

Ludovic Jaubert & Peter Holdsworth (ENS Lyon), & Roderich Moessner (Dresden)

Adam Nahum (Oxford), Miguel Ortuño, Andres Somoza, & Pedro Serna (Murcia)

Outline

Statistical mechanics with extended degrees of freedom

Coulomb phases

Geometrically frustrated magnets, dimer models

Correlations from constraints

Close-packed loop models

Loop colours as non-local degrees of freedom

See also poster session

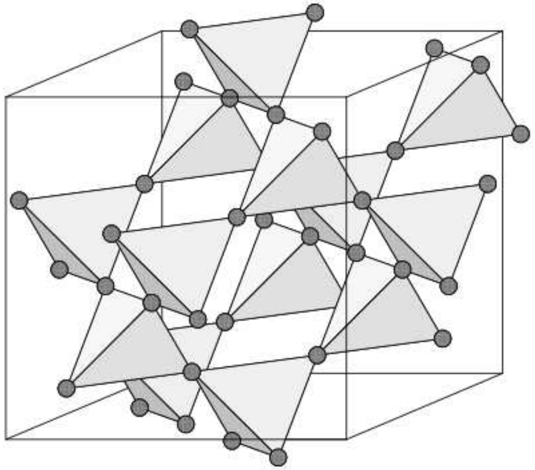
Phase transitions

Ordering transitions from the Coulomb phase

Transitions between extended-loop and short-loop phases

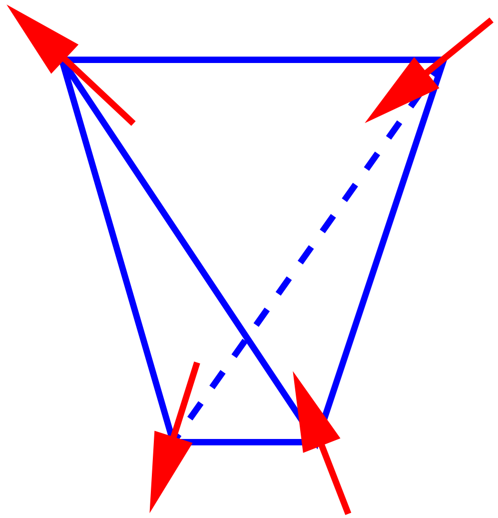
Spin Ice

$Ho_2Ti_2O_7$ and $Dy_2Ti_2O_7$



'Two-in, two-out'

ground states



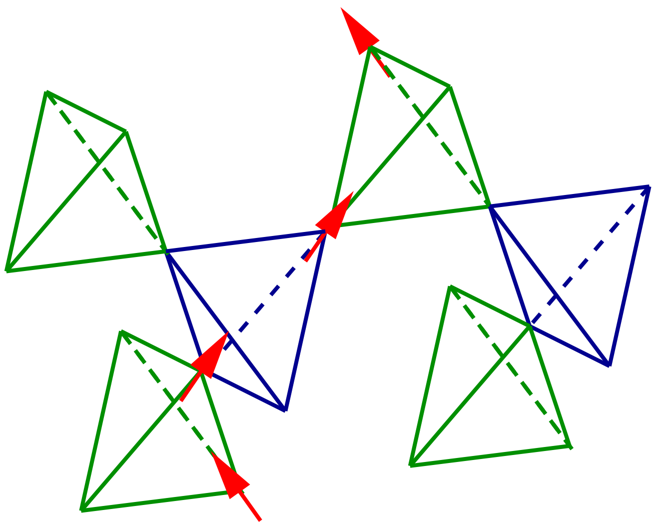
Pyrochlore ferromagnet with single-ion anisotropy

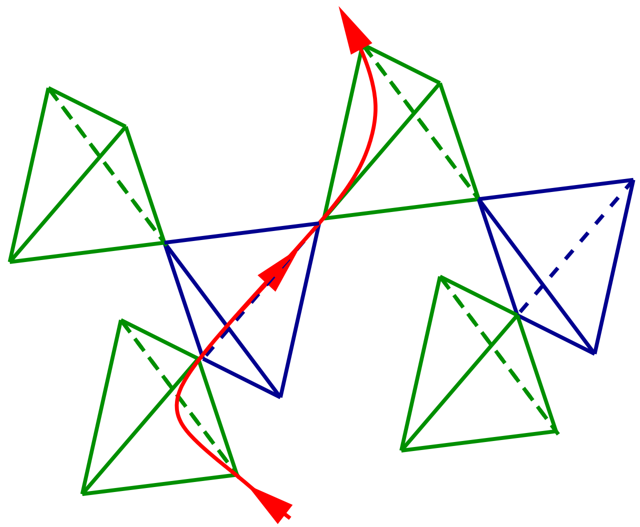
$$\mathcal{H} = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_i (\hat{\mathbf{n}}_i \cdot \mathbf{S}_i)^2 - \mathbf{h} \cdot \sum_i \mathbf{S}_i$$

Gauge theory of ground state correlations

Youngblood *et al* (1980), Huse *et al* (2003), Henley (2004)

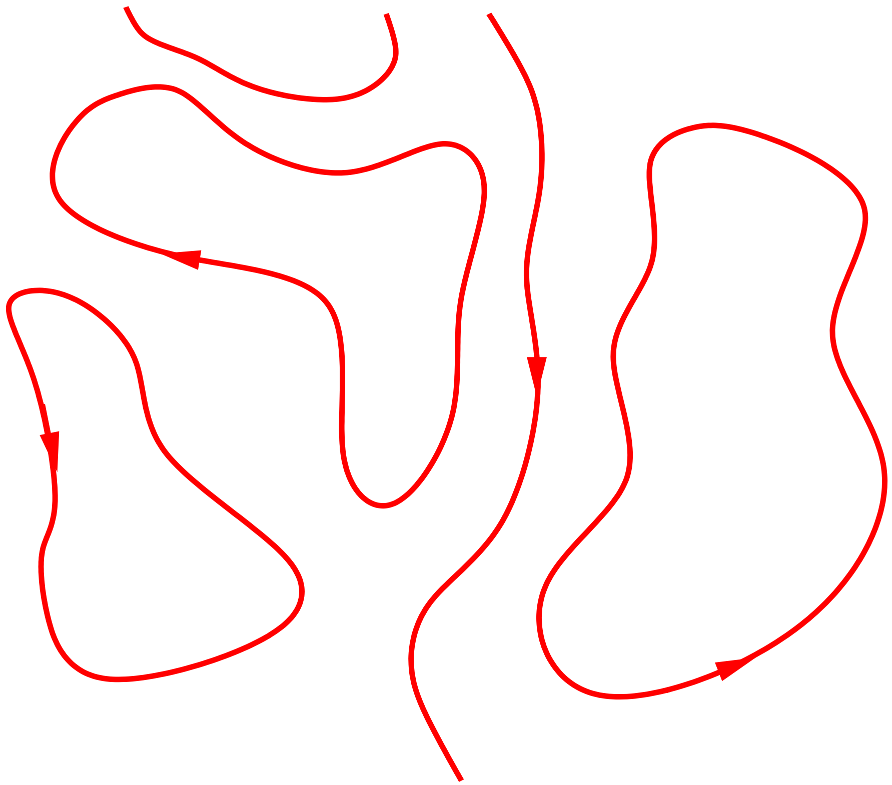
Map spin configurations . . . **. . . to vector fields** $\mathbf{B}(\mathbf{r})$





'two-in two out' groundstates . . . **. . . map to divergenceless** $\mathbf{B}(\mathbf{r})$

Ground states as flux loops

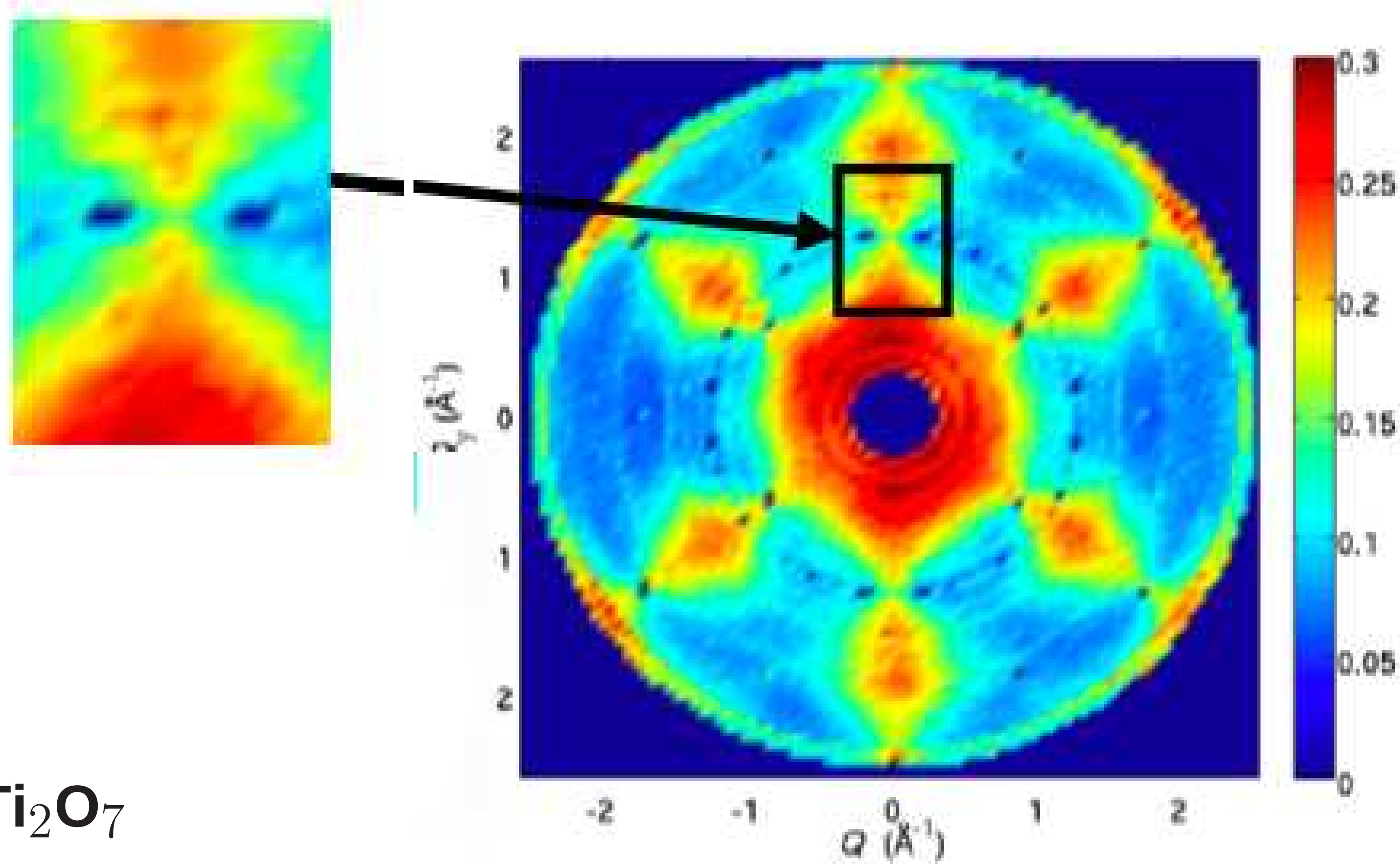


Entropic distribution: $P[\mathbf{B}(\mathbf{r})] \propto \exp(-\kappa \int \mathbf{B}^2(\mathbf{r})\mathrm{d}^3\mathbf{r})$

Power-law correlations: $\langle B_i(\mathbf{r})B_j(\mathbf{0})\rangle \propto r^{-3}$

Low T correlations from neutron diffraction

Fennell *et al*   Science 326, 415 (2009)



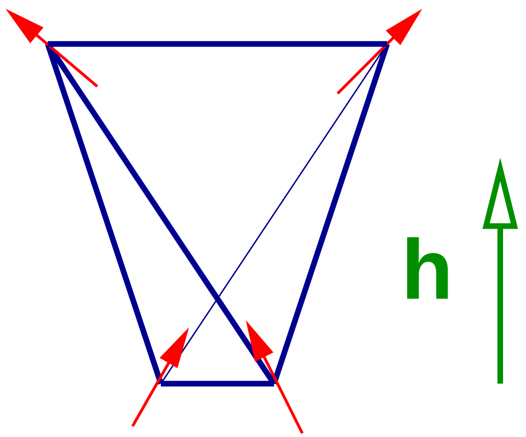
$Ho_2Ti_2O_7$

Engineering transitions in spin ice

Select ordered state with Zeeman field or strain

Kasteleyn transition

in staggered field



Magnetisation

vs h^{eff}/T

for $h^{\text{eff}}, T \ll J$

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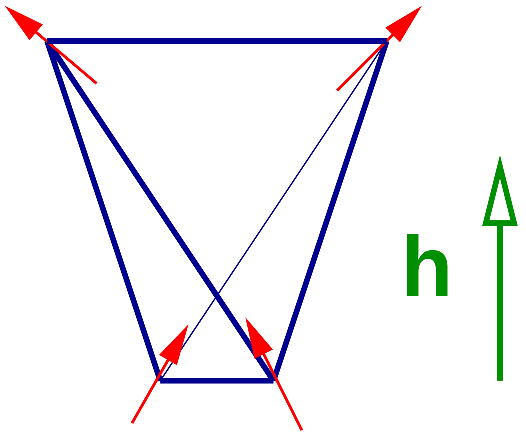


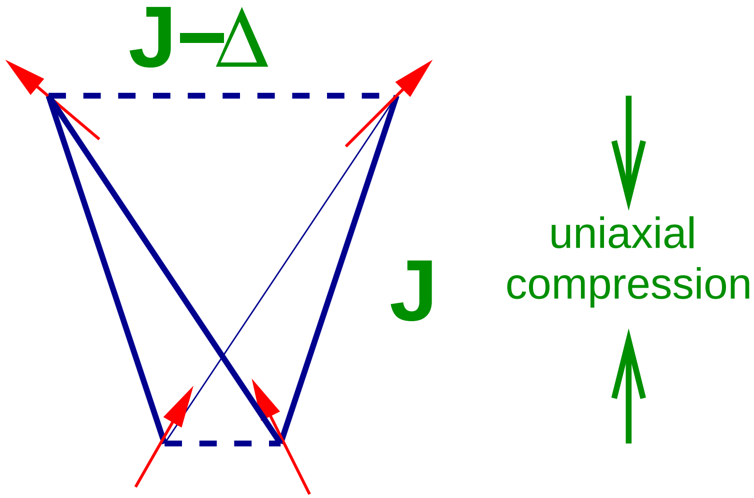
Magnetisation

vs h^{eff}/T

for $h^{\text{eff}}, T \ll J$

Ferromagnetic ordering

strain + magnetoelastic coupling



Magnetic order for

$T \ll \Delta$

Coulomb phase for

$\Delta \ll T \ll J$

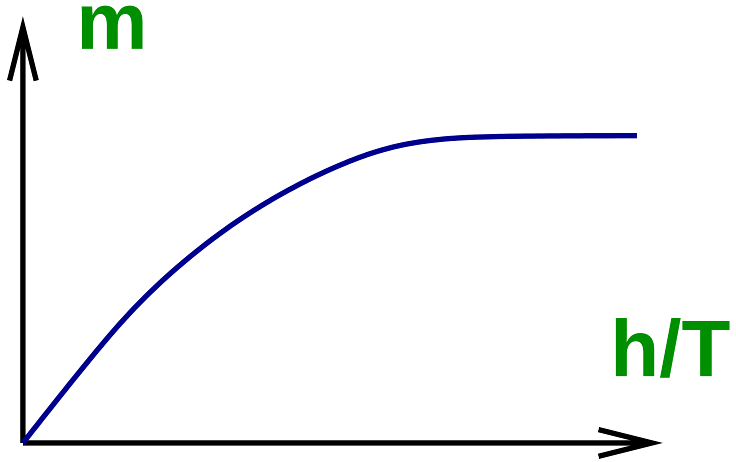
Jaubert, JTC, Holdsworth + Moessner, PRL (2008) + (2010)

A Kasteleyn transition

Magnetisation induced by applied field

Magnetisation vs temperature

In a paramagnet



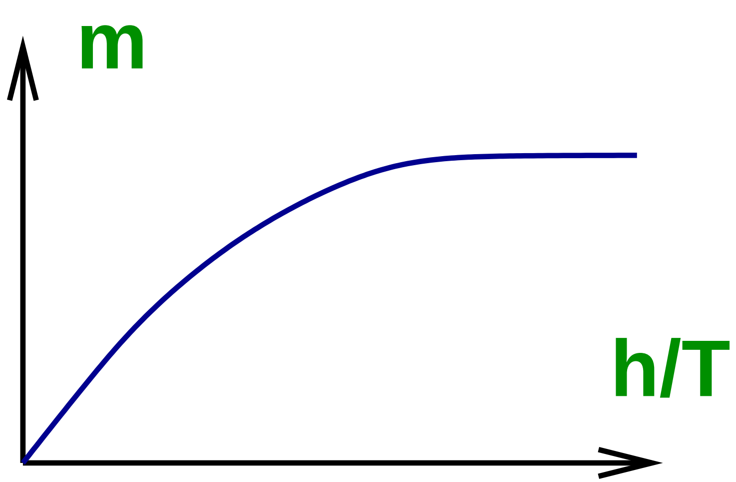
No transition

A Kasteleyn transition

Magnetisation induced by applied field

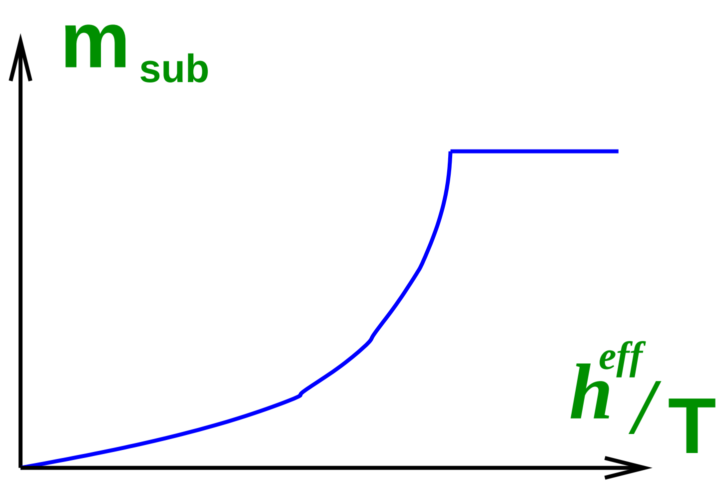
Magnetisation vs temperature

In a paramagnet



No transition

From the Coulomb phase

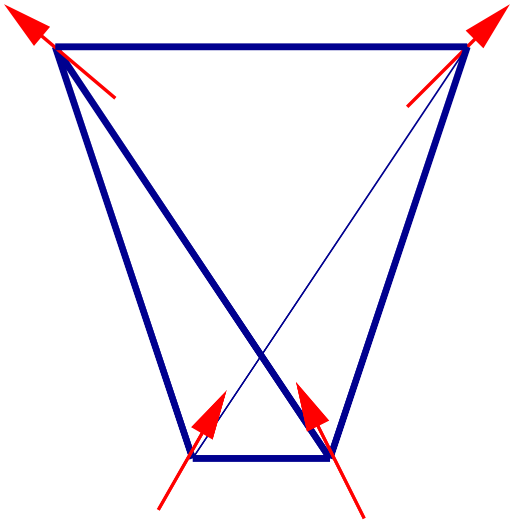


One-sided transition

- **Continuous from low-field side**
- **First-order from high-field side**

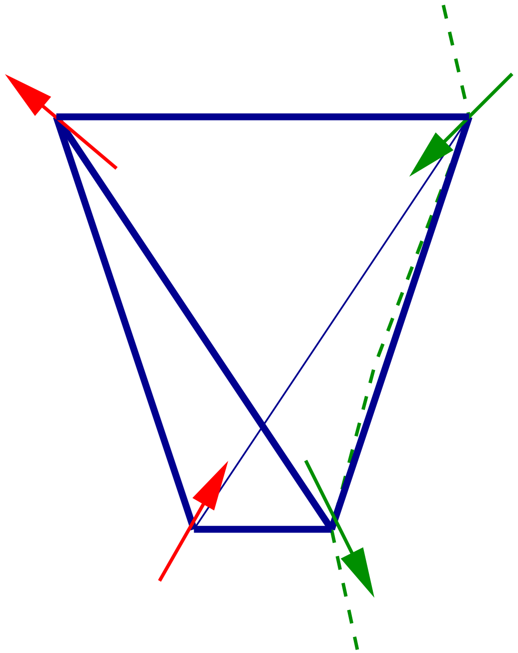
Description of the transition

Reference state: fully polarised



‘Vacuum’

Excitations: spin reversals



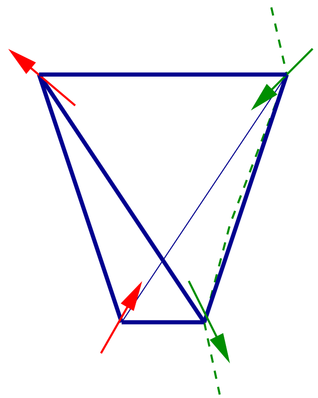
String excitation

Thermodynamics of isolated string, length L:

Energy $L \cdot h$ **Entropy** $L \cdot k_{\mathrm{B}} \ln(2)$ **Free energy** $L \cdot [h - k_{\mathrm{B}} T \ln(2)]$

String density: finite for $h/k_{\mathrm{B}}T < \ln(2)$ **zero for** $h/k_{\mathrm{B}}T > \ln(2)$

Classical to quantum mapping



View strings as boson world lines

3D classical $\equiv (2+1)D$ **quantum**

$$Z = \mathrm{Tr}\left(T^L\right) \qquad T \equiv \mathrm{e}^{\mathcal{H}}$$

$\mathcal{H}$ **hard core bosons hopping on** $\langle 100 \rangle$ **plane**

magnetic field $\Leftrightarrow$ **boson chemical potential**

Coulomb phase correlations $\Leftrightarrow$ **Goldstone fluctuations of condensate**

monopole deconfinement $\Leftrightarrow$ **off-diagonal long range order**

Quantum Description as XY ferromagnet

Kasteleyn transition

$$\mathcal{H} = -\mathcal{J}\sum_{\langle ij\rangle}[S_i^+S_j^- + S_i^-S_j^+] - \mathcal{B}\sum_i S_i^z$$

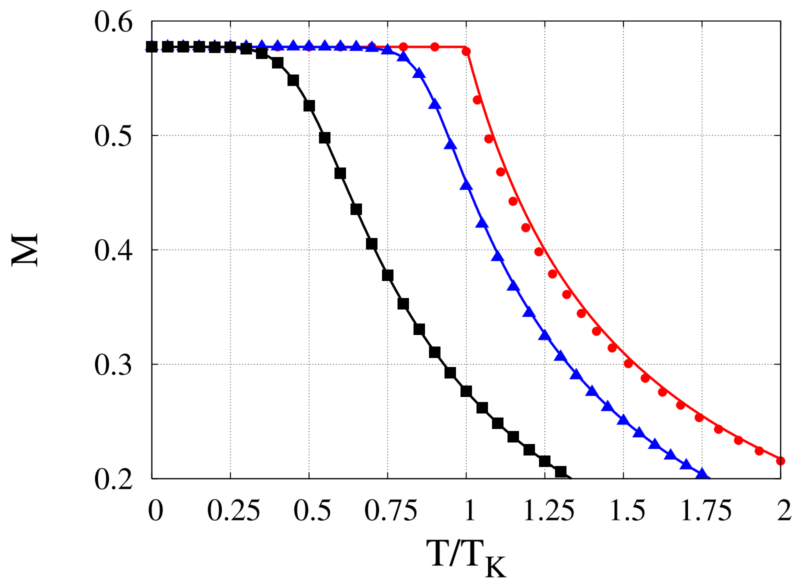
Correspondence with classical description: $\mathcal{B} \equiv h/T$

$\mathcal{B} > \mathcal{B}_{\mathrm{c}}$ Quantum spins polarised along z

$\mathcal{B} < \mathcal{B}_{\mathrm{c}}$ Quantum spins have xy order

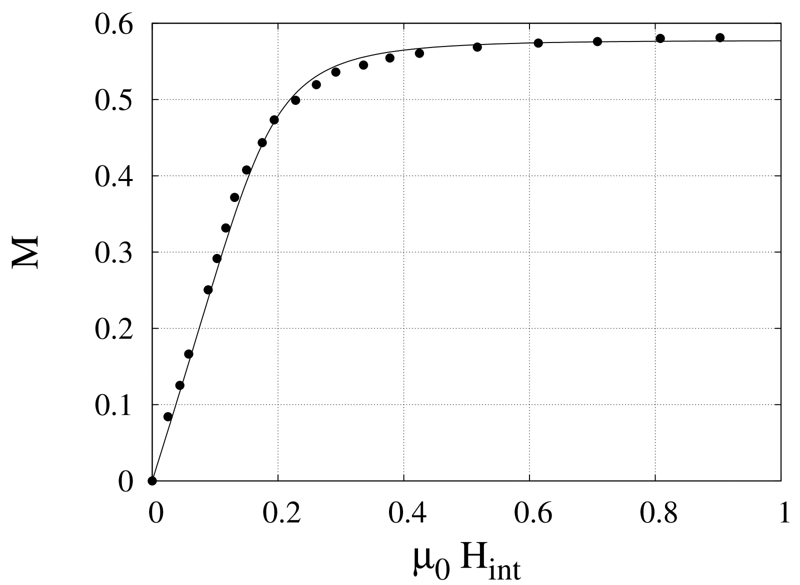
Kasteleyn: Simulation and Experiment

Magnetisation vs T



$$h/J = 0.58,\ 0.13,\ 10^{-3}$$

Magnetisation vs H



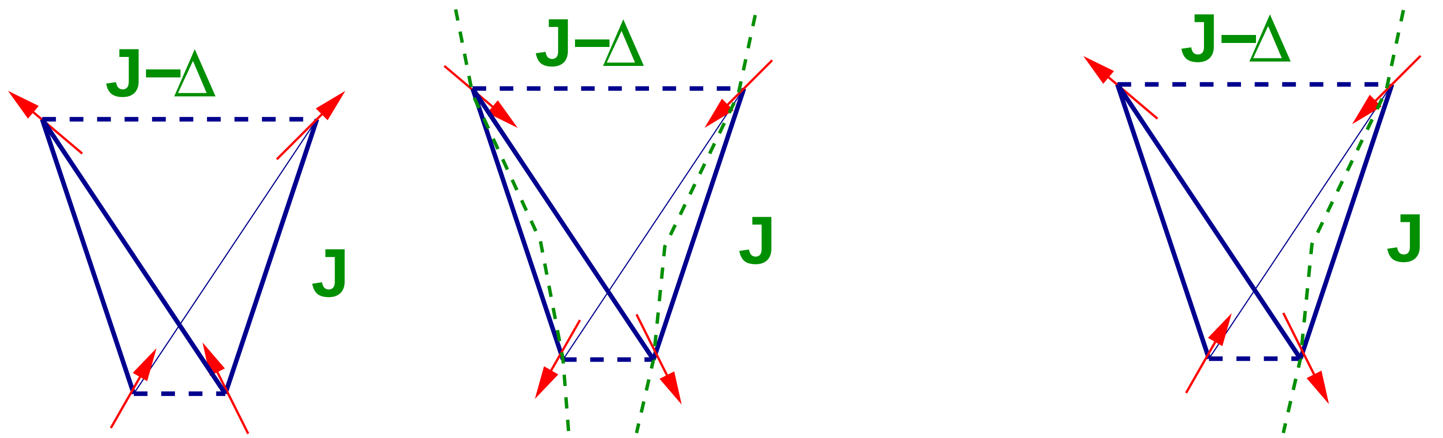
Data for $Dy_2Ti_2O_7$ at 1.8K

(Fukazawa *et al.* 2002)

$$k_{\rm B}T \simeq 1.6 J_{\rm eff}$$

Ferromagnetic ordering in strained spin ice

Classical-quantum mapping: ordering as reorientation of quantum spins



Low energy configurations **High energy configuration**

$$\mathcal{H} = -\mathcal{J}\sum_{\langle ij\rangle}[S_i^+S_j^- + S_i^-S_j^+] - \mathcal{D}\sum_{\langle ij\rangle}S_i^zS_j^z$$

$\mathcal{D} < \mathcal{J}$ **quantum spins in xy plane** $\mathcal{D} \equiv \Delta/T$

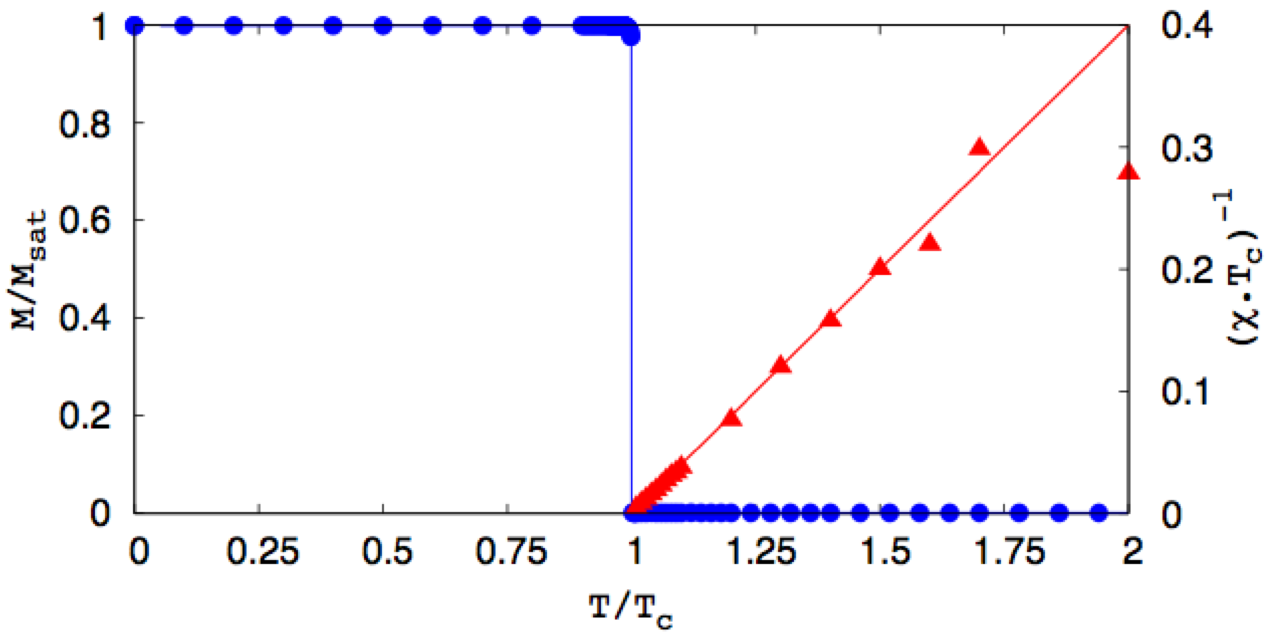
$\mathcal{D} > \mathcal{J}$ **quantum spins along z**

$\mathcal{D} = \mathcal{J}$ **emergent SU(2) symmetry at critical point**

Exotic features of ordering in strained spin ice

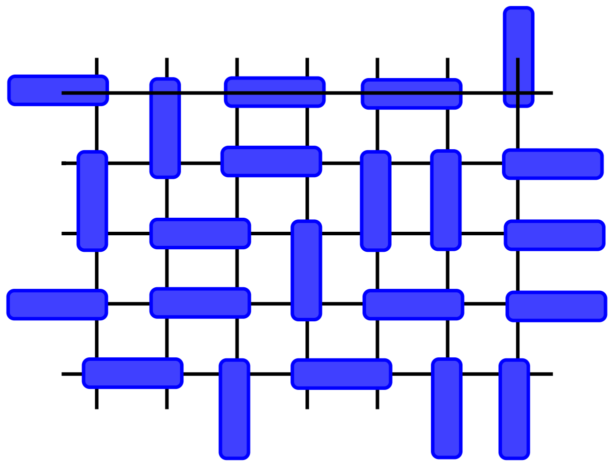
Transition is 'infinite order' multicritical point

- Magnetisation (maximally) discontinuous
- Susceptibility divergent as $T \longrightarrow T_c^+$
- Domain wall width divergent as $T \longrightarrow T_c^-$



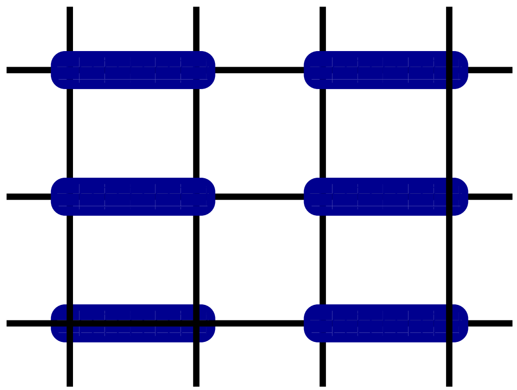
Ordering from the Coulomb phase of dimer models

Allowed states of close-packed dimer models



Dimer crystallisation

favour parallel pairs



$$\mathcal{H} = -J(n_{||} + n_{//} + n_{=})$$

Crystal for $T \ll J$

Coulomb phase for $T \gg J$

Simulations:

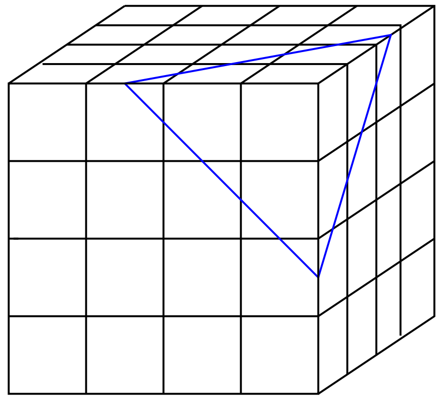
continuous transition possible

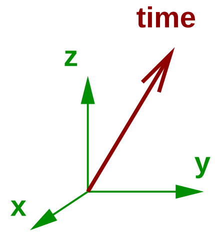
classical non-LGW critical point

Alet et al: 2006, 2010

Classical dimer ordering in 3d and bosons in 2d

From 3d classical to (2+1)d quantum





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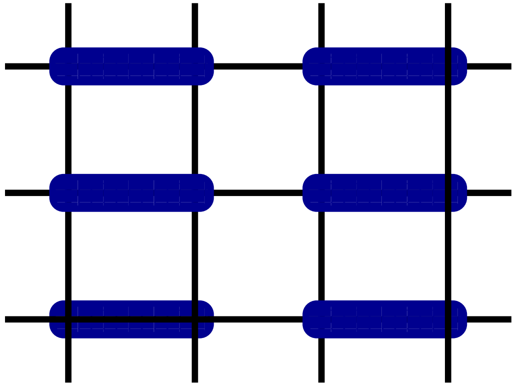
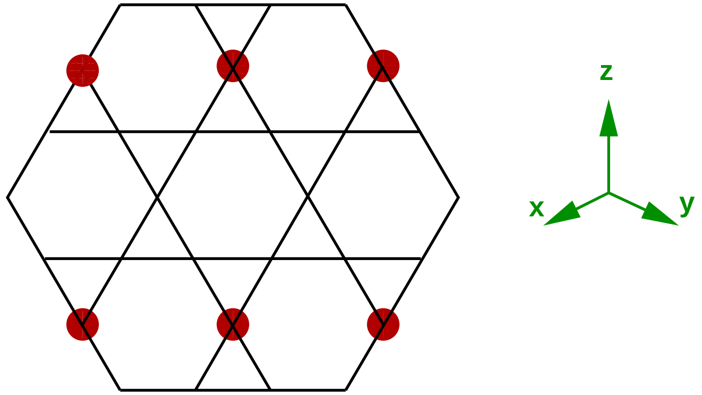


Expect non-LGW critical point

Map to bosons on kagome lattice



1/6 filling with hard-core repulsion

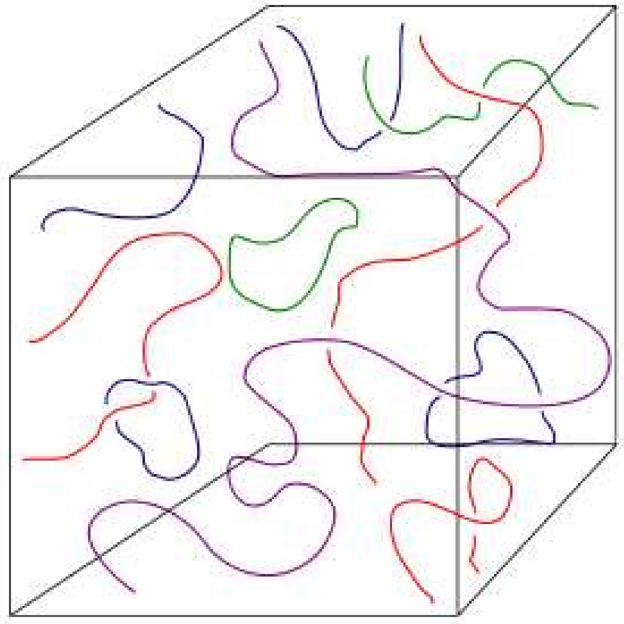
Dimer liquid maps to superfluid

Dimer crystal maps to boson crystal

Powell + JTC, 2009

Loop models

Continuum problem



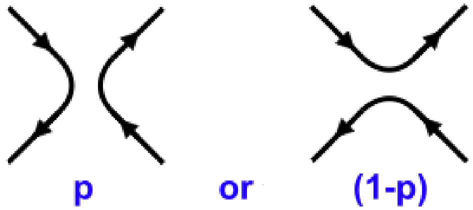
Lattice formulation

Close-packed loops with n colours

on lattice of (directed) links

Nahum, JTC, Serna, Ortuño, and Somoza, arXiv:1104.4096

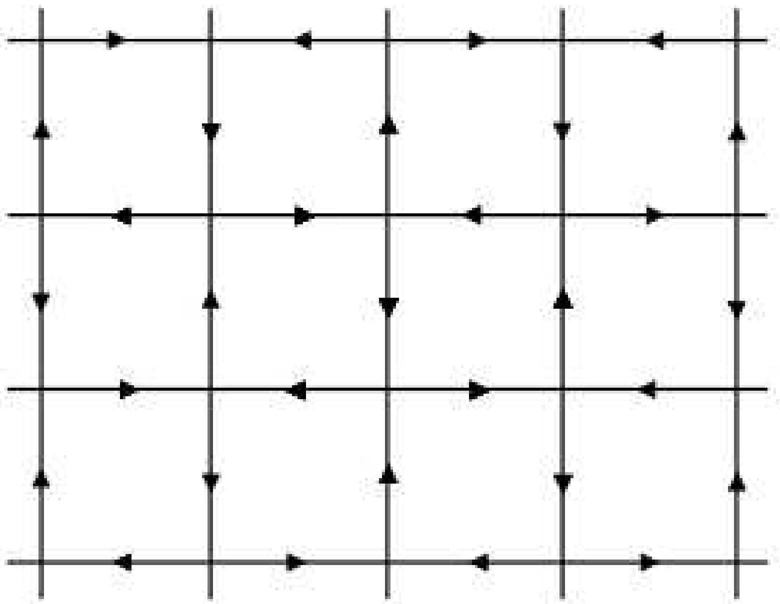
Phase transitions in loop models



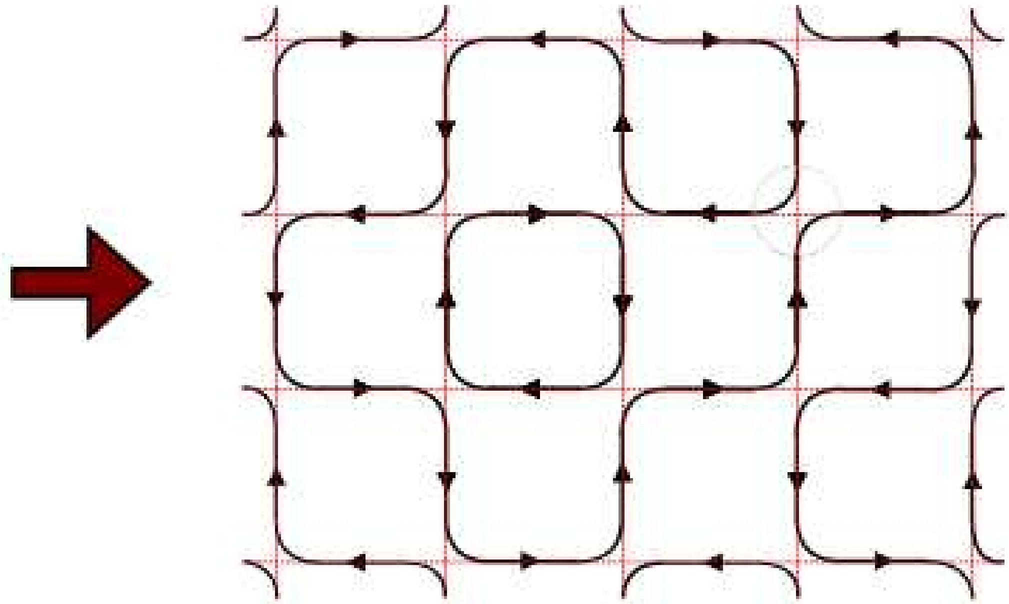
$$Z = \sum_{\text{configs}} p^{n_{\text{p}}}(1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

To define model: specify lattice, link directions and nodes

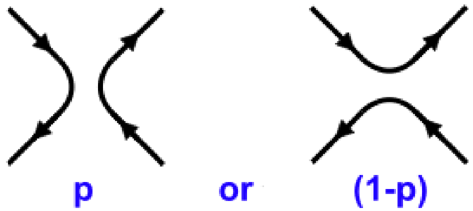
2D model



Sample configuration



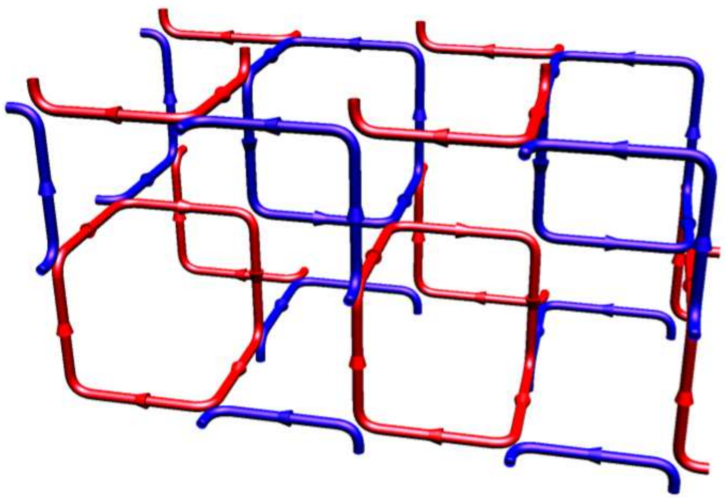
Phase transitions in loop models



$$Z = \sum_{\text{configs}} p^{n_{\rm p}} (1-p)^{n_{1-{\rm p}}} n^{n_{\rm loops}}$$

To define model: specify lattice, link directions and nodes

Configuration of 3D model



Lattice designed so that:

$p = 0$ only short loops

$p = 1$ all curves extended

(Alternative has symmetry

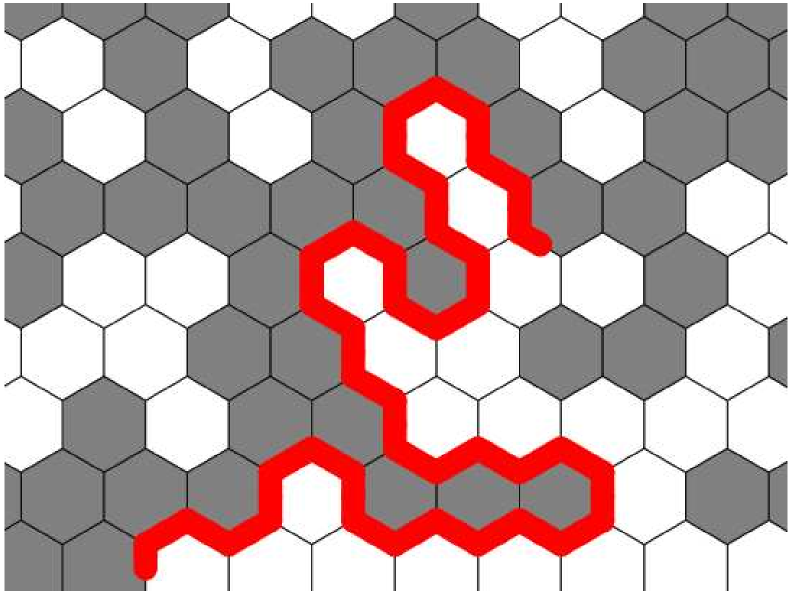
under $p \leftrightarrow [1-p]$)

Loop models and non-intersecting random curves in 3D

Random curves appear in many contexts

2D random curves

– zero-lines of random scalar field



Lattice version – percolation hulls

Loop models and non-intersecting random curves in 3D

Random curves appear in many contexts

2D random curves

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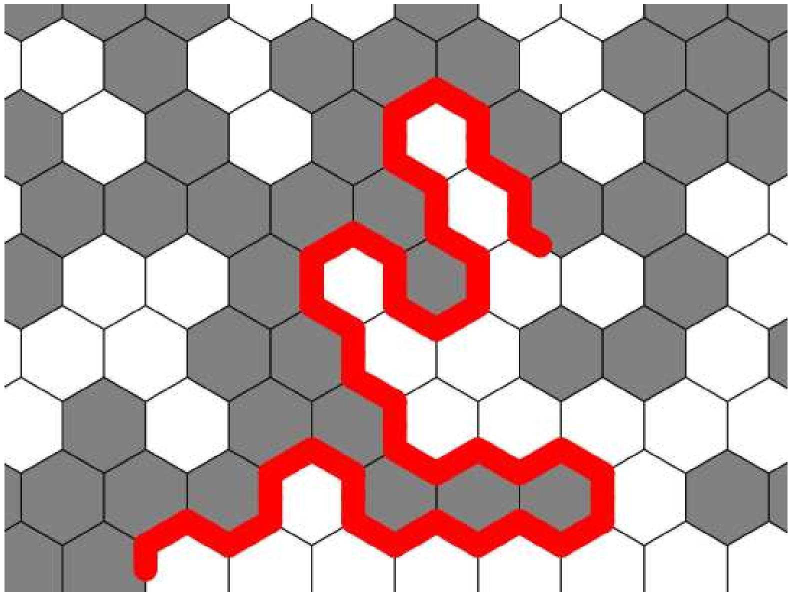
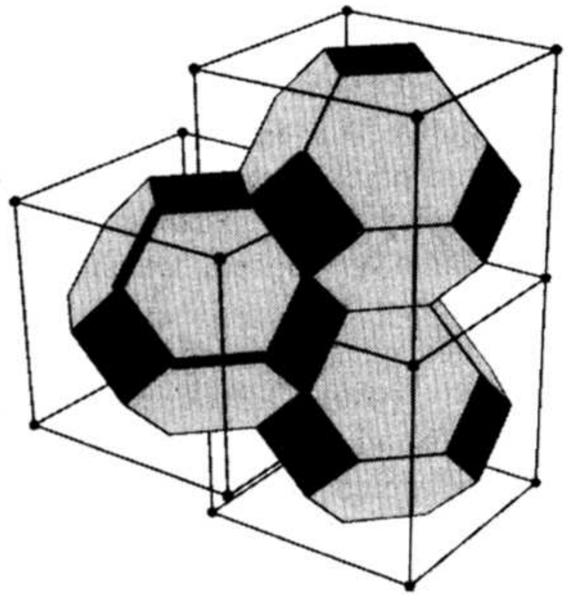


Lattice version – percolation hulls

3D random curves

– zero-lines of random 2-cpt field



Lattice version – tricolour percolation

Scaling properties match

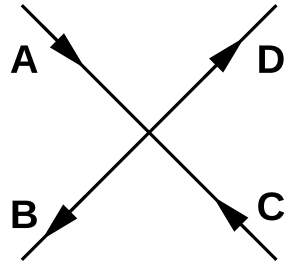
$n = 1$ **loop model**

Local Description and Continuum Theory

$$Z = \sum_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

Introduce n component complex

unit vector $\vec{z}_l$ on each link l



Calculate $\mathcal{Z} = \mathcal{N} \prod_l \int \mathrm{d}\vec{z}_l \; \mathrm{e}^{-\mathcal{S}}$

with $\mathrm{e}^{-\mathcal{S}} = \prod_{\text{nodes}} \left[p(\vec{z}_A^{\dagger} \cdot \vec{z}_B)(\vec{z}_C^{\dagger} \cdot \vec{z}_D) + (1-p)(\vec{z}_A^{\dagger} \cdot \vec{z}_D)(\vec{z}_C^{\dagger} \cdot \vec{z}_B) \right]$

Expand $\prod_{\text{nodes}} [\ldots]$ **Loops contribute factors**

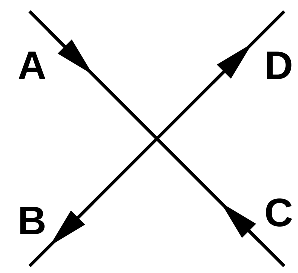
$$\sum_{\alpha,\beta,\ldots\gamma} \int \mathrm{d}\vec{z}_1 \ldots \int \mathrm{d}\vec{z}_L \; z_1^{*\alpha} z_2^{\alpha} z_2^{*\beta} \ldots z_L^{*\gamma} z_1^{\gamma}$$

Hence: (i) factor of n per loop (ii) invariance under $\vec{z}_l \to \mathrm{e}^{\mathrm{i}\varphi_l}$

Local Description and Continuum Theory

$$Z = \sum_{\text{configs}} p^{n_\text{p}} (1-p)^{n_{1-\text{p}}} n^{n_\text{loops}}$$

Introduce n **component complex unit vector** $\vec{z}_l$ **on each link** l



Calculate $\mathcal{Z} = \mathcal{N} \prod_l \int \mathrm{d}\vec{z}_l \, \mathrm{e}^{-\mathcal{S}}$

with $\mathrm{e}^{-\mathcal{S}} = \prod_{\text{nodes}} \left[p(\vec{z}_A^\dagger \cdot \vec{z}_B)(\vec{z}_C^\dagger \cdot \vec{z}_D) + (1-p)(\vec{z}_A^\dagger \cdot \vec{z}_D)(\vec{z}_C^\dagger \cdot \vec{z}_B) \right]$

Continuum limit CP(n-1) model

$S = \frac{1}{g} \int \mathrm{d}^d\mathbf{r} \, |(\nabla - \mathrm{i}A)\vec{z}|^2$ **with** $A = \frac{\mathrm{i}}{2}(z^{*\alpha} \nabla z^{\alpha} - z^{\alpha} \nabla z^{*\alpha})$

with $|\vec{z}|^2 = 1$ **and invariance under** $\vec{z} \longrightarrow \mathrm{e}^{\mathrm{i}\varphi(\mathrm{r})} \vec{z}$

see also: Candu, Jacobsen, Read and Saleur (2009)

Phase transitions in CP^{n-1} model

Gauge-invariant degrees of freedom: 'spins' $Q \equiv zz^\dagger - 1/n$

(**Mapping to Heisenberg model for** $n = 2$ **via** $S^\alpha = z^\dagger \sigma^\alpha z$)

Correlations

$\langle \mathrm{tr}\, Q(\mathbf{0}) Q(\mathbf{r}) \rangle \propto G(r)$ **– prob. points $\mathbf{0}$ and $\mathbf{r}$ lie on same loop**

Paramagnetic phase
— only finite loops

$$G(r) \sim \frac{1}{r} \mathrm{e}^{-r/\xi}$$

Critical point
— fractal loops

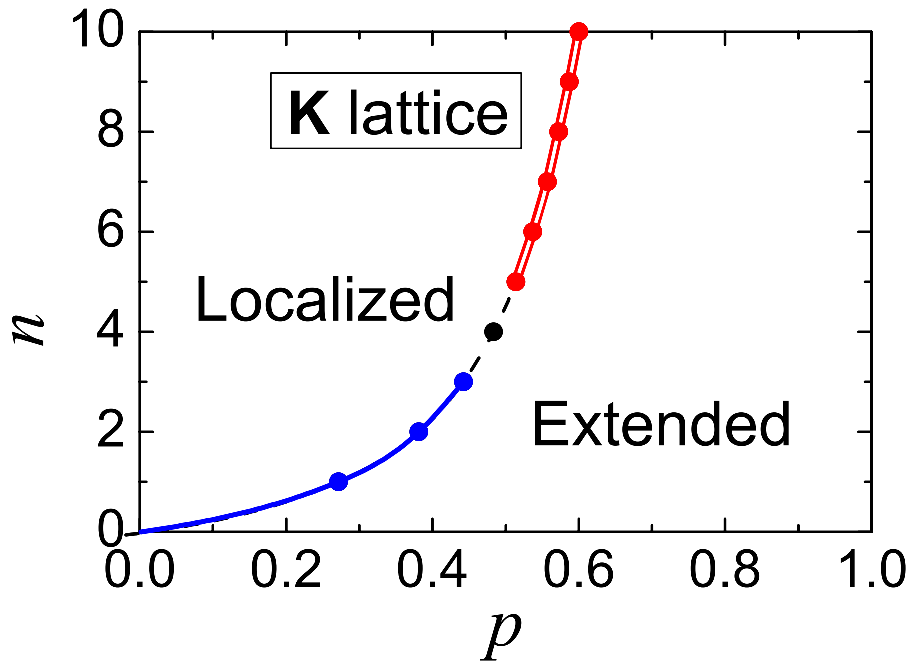
$$G(r) \sim r^{-(1+\eta)} \qquad d_f = \frac{5-\eta}{2}$$

Ordered phase
— Brownian loops escape to infinity

$$G(r) \sim r^{-2}$$

Results from simulations

Phase diagram



Critical exponents

n	ν	γ
1	0.9985(15)	2.065(18)
2	0.708(6)	1.39(1)
3	0.50(2)	1.01(2)

$n \geq 5:$ **1st order**

— consistent at $n = 2$ **with best estimates for classical**

Heisenberg model: $\nu = 0.7112(5)$ $\gamma = 1.3960(9)$

Summary

Two classes of system having non-local degrees of freedom:

- **Coulomb phases in spin Ice + dimer models**
- **Loop models**

Exotic critical behaviour at ordering transitions:

- **Symmetry-sustaining:**

  one-sided Kasteleyn transition

- **Symmetry-breaking:**

  non-standard critical behaviour at Curie transition

  non-LGW critical point for dimer ordering transition

- **Symmetry-breaking:**

  loops as representation of CP^{n-1} model