Quantum Theory of Condensed Matter: Problem Set 2

Qu.1 In this question you are invited to demonstrate your understanding of Landau theory for Fermi liquids by deriving the relation (which follows from Galilean invariance)

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3}$$

between the bare mass, m, the effective mass, m^*, and the Landau parameter, F_1^s.

First, recall some definitions. Quasiparticle states are labelled by their momentum $\mathbf{p}$ and spin σ. Let $\delta n_{\mathbf{p},\sigma}$ be the change in quasiparticle occupation of the state $\mathbf{p}, \sigma$ relative to its occupation number in the ground state. The Landau expansion for the resulting change δF in free energy is

$$\delta F = \sum_{\mathbf{p},\sigma} [\epsilon(p) - \mu]\delta n_{\mathbf{p},\sigma} + \frac{1}{2}\sum_{\mathbf{pq},\sigma\lambda} f_{\mathbf{pq}}^{\sigma\lambda}\delta n_{\mathbf{p},\sigma}\delta n_{\mathbf{q},\lambda}\,.$$

The effective mass is defined by $p/m^* = d\epsilon(p)/dp$ at $p = p_F$, the Fermi momentum. The Landau parameters are separated into spin-symmetric and antisymmetric parts according to

$$f^{\uparrow\uparrow} = f^{\downarrow\downarrow} = f^s + f^a$$

and

$$f^{\uparrow\downarrow} = f^{\downarrow\uparrow} = f^s - f^a$$

(where momentum labels have been suppressed). They are also separated into spherical harmonics according to

$$f_{\mathbf{pq}} = \sum f_l P_l(\cos(\theta))$$

(where spin labels have been suppressed), in which $\mathbf{p}\cdot\mathbf{q} = p_F^2\cos(\theta)$ and $P_l(\cos(\theta))$ is the lth Legendre polynomial. Finally, they are expressed in dimensionless form by

$$f = \frac{F}{\nu}$$

(in which both spin and momentum labels have been suppressed), where ν is the density of states in energy at the Fermi surface.

Now consider the energy change that results when the system is set in uniform motion with a momentum per particle of magnitude p, for $p \ll p_F$. On elementary grounds, for a system of N particles this is

$$\delta F = N\frac{p^2}{2m}.$$

Use the Landau expansion to calculate the same quantity.

(i) Show that the change in quasiparticle number (induced when the system is set in motion) is

$$\int \delta n_{\mathbf{p}\sigma} = \frac{3Np}{2p_F}\cos(\theta)$$

where θ is the angle between $\mathbf{p}$ and the direction of the uniform motion, and the integral is over momenta in the radial direction.

(ii) Show that the resulting change in energy is

$$\delta F = N\frac{p^2}{2m^*} + \frac{1}{2\nu}F_1^s(\frac{Np}{p_F})^2$$

and that $\nu = 3Nm^*/p_F^2$. Hence derive the relation given between m, m^* and F_1^a.

Qu.2 The BCS Hamiltonian is

$$H = \sum_{k\sigma}\epsilon(k)c^\dagger_{k\sigma}c_{k\sigma} - V{\sum_{k,q}}' c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}c_{-q\downarrow}c_{q\uparrow}$$

where $c^\dagger_{k\sigma}$ and $c_{k\sigma}$ denote fermion creation and annihilation operators for states labelled by wavevector, k, and spin, $\sigma =\uparrow,\downarrow$. The second summation, $\sum'_{k,q}$, is restricted to wavevectors k, q of states lying within the Debye energy, $\hbar\omega_D$, of the chemical potential, μ: $|\epsilon(k) - \mu|, |\epsilon(q) - \mu| \leq \hbar\omega_D$. The BCS wavefunction is

$$|BCS\rangle = \Pi_k(u_k + v_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow})|0\rangle\,,$$

where $|0\rangle$ is the vacuum state and $\{u_k, v_k\}$ are variational parameters, with $|u_k|^2+|v_k|^2 = 1$. The number operator is

$$N = \sum_{k,\sigma}c^\dagger_{k\sigma}c_{k\sigma}.$$

(i) Calculate the mean particle number, $\langle BCS|N|BCS\rangle$, and its variance,

$$\langle BCS|N^2|BCS\rangle - \langle BCS|N|BCS\rangle^2$$

in the BCS wavefunction.

(ii) Show that the free energy of the system in the BCS state is

$$\langle BCS|H - \mu N|BCS\rangle = 2\sum_k(\epsilon(k) - \mu)|v_k|^2 - \frac{|\Delta|^2}{V}$$

where

$$\Delta = V{\sum_k}' u_k v_k^*\,.$$

(iii) By minimising this free energy with respect to the variational parameters, u_k and v_k, derive the zero-temperature gap equation

$$1 = \frac{V}{2}{\sum_k}' \frac{1}{[|\Delta|^2 + (\epsilon(k) - \mu)^2]^{1/2}}\ .$$

(iv) Show that, in the weak-coupling limit in which $|\Delta| \ll \hbar\omega_D$, the gap equation has the solution

$$|\Delta| = 2\hbar\omega_D\exp(-1/\nu V)$$

where ν is the density of states at the chemical potential.

Qu.3 A one-dimensional model for localisation, which is exactly solvable, is defined as follows. The system consists of a chain of sites labelled by n. Waves propagate along this chain in both directions. The amplitudes of the left- and right going probability currents at the site n are given by the complex numbers w_n and z_n respectively. Scattering of these waves by disorder is represented in the model by a 2×2 transfer matrix, T_l, associated with each link between successive sites, l and $l+1$. This transfer matrix can be written as

$$T_l = \begin{pmatrix} e^{i\alpha_l} & 0 \\ 0 & e^{-i\alpha_l} \end{pmatrix} \cdot \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \cdot \begin{pmatrix} e^{i\beta_l} & 0 \\ 0 & e^{-i\beta_l} \end{pmatrix}$$

where the two phases, α_l and β_l, and backscattering strength, θ, are all real. The amplitudes obey

$$\begin{pmatrix} w_{n+1} \\ z_{n+1} \end{pmatrix} = T_n \begin{pmatrix} w_n \\ z_n \end{pmatrix} .$$

(i) Verify that scattering is unitary, showing that the net current along the chain is conserved by proving

$$|w_{n+1}|^2 - |z_{n+1}|^2 = |w_n|^2 - |z_n|^2$$

for any w_n, z_n. Show also that, if $w_n = z_n^*$, then $w_{n+1} = z_{n+1}^*$, and obtain an expression in this case for

$$\frac{|z_{n+1}|^2}{|z_n|^2} .$$

(ii) Consider an ensemble of systems in which the scattering parameter, θ, is fixed, but the phases, α_l and β_l, are random variables, independently chosen for each link from a distribution uniform between 0 and 2π. Denoting the ensemble average by $\langle \ldots \rangle$, show that, if $w_n = z_n^*$, then

$$\langle \log(|z_{n+1}|^2/|z_n|^2) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi \log(e^{2\theta} \cos^2(\phi) + e^{-2\theta} \sin^2(\phi)) = 2 \log(\cosh(\theta))$$

(iii) Explain why this shows that states are localised in this model, with a localisation length

$$\xi = \frac{1}{\log(\cosh(\theta))} .$$

Qu.4 This question is about wavefunctions in the lowest Landau level. Let x and y be the two position coordinates in the plane, and define the complex coordinate $z = x + iy$. Take the magnetic length to be the unit of length. Then, ignoring spin, single-particle basis functions in the lowest Landau level are $\psi_n(z) = (2^{n+1}\pi n!)^{-1/2} z^n \exp(-|z|^2/4)$ with $n = 0, 1, 2 \ldots$.

Consider the N-particle wavefunction

$$\Psi_0(z_1, z_2, \ldots, z_N) = \mathcal{N} \prod_{i<j}^{N} (z_i - z_j) \exp(-\sum_{k=1}^{N} |z_k|^2/4)$$

where $\mathcal{N}$ is a normalisation factor.

(i) Explain how this may be re-written as a Slater determinant.

(ii) The number density operator is $\rho(z) = \sum_{k=1}^{N} \delta(z - z_k)$. Describe the behaviour of $\langle\Psi_0|\rho(z)|\Psi_0\rangle$ as a function of z for $N \gg 1$.

(iii) Consider the N-particle wavefunction for a state with a quasi-hole at $z_{\rm H}$:

$$\Psi_1(z_1, z_2, \ldots, z_N) = \mathcal{N}' \prod_{l=1}^{N} (z_l - z_{\rm H}) \psi_0(z_1, z_2, \ldots, z_N)$$

where $\mathcal{N}'$ is a further normalisation factor. Calculate $\langle\Psi_1|\rho(z)|\Psi_1\rangle$ in the limit $N \to \infty$ (consider $z_{\rm H} = 0$ first, and then use translational invariance).

(iv) Now introduce spin. A trial wavefunction for a system with a skyrmion of radius $|\lambda|$ at the origin is

$$\Psi_2 = \begin{pmatrix} z_1 \\ \lambda \end{pmatrix}_1 \otimes \begin{pmatrix} z_2 \\ \lambda \end{pmatrix}_2 \otimes \ldots \begin{pmatrix} z_N \\ \lambda \end{pmatrix}_N \otimes \Psi_0$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix}_l$$

denotes a spinor for the l-th particle in the usual way. The spin density operator is

$$\begin{aligned} \vec{\sigma}(z) &= \delta(z_1 - z)\vec{\sigma}_1 \otimes \mathbf{1}_2 \ldots \otimes \mathbf{1}_N \\ &+ \delta(z_2 - z)\mathbf{1}_1 \otimes \vec{\sigma}_2 \ldots \otimes \mathbf{1}_N \\ &+ \ldots \\ &+ \delta(z_N - z)\mathbf{1}_1 \otimes \mathbf{1}_2 \ldots \otimes \vec{\sigma}_N \end{aligned}$$

where $\vec{\sigma}_l$ is the vector of Pauli matrices acting in the space of the l-th particle spinor, and $\mathbf{1}_l$ is the unit matrix acting in the same space. With the same notation, the number density operator is now $\rho(z) = \sum_{k=1}^{N} \delta(z - z_k)\mathbf{1}_1 \otimes \ldots \otimes \mathbf{1}_N$.

Calculate

$$\frac{\langle\Psi_2|\vec{\sigma}(z)|\Psi_2\rangle}{\langle\Psi_2|\rho(z)|\Psi_2\rangle}$$

and discuss how your results match what you expect for the behaviour of spin polarisation in the presence of a skyrmion.

Use the relation

$$\log(|\Psi_2(z_1, \ldots z_N)|^2) = \sum_l \log(|z_l|^2 + |\lambda|^2) + 2\sum_{i<j} \log|z_i - z_j| - \frac{1}{2}\sum_k |z_k|^2 + \text{constant}$$

and Laughlin's plasma analogy to discuss the behaviour of $\langle\Psi_2|\rho(z)|\Psi_2\rangle$.