Logarithmic CFTs as limits of ordinary CFTs and some physical applications

John Cardy

University of Oxford

IHP, Paris
October 2011

- mostly arXiv:cond-mat/9911024, 0111031 and some new stuff

Introduction

- this talk will be physical rather than mathematical: how can logCFTs arise in physics?
- viewpoint will be to regard logCFTs as limits of ordinary (albeit irrational) CFTs
- most of the discussion not restricted to two dimensions, although some detailed examples will be
- two main examples:
  - random systems using the $n \to 0$ 'replica trick'
  - self-avoiding walks as the $n \to 0$ limit of the $O(n)$ model
- both these have partition function $Z = 1$ ($c = 0$), and one of the interesting questions is how t, the logarithmic partner of the stress tensor T, emerges in this picture
- I will not address logCFTs with $c \neq 0$ in this talk

LogCFTs in general

- a logCFT is a scale-invariant QFT ($\Theta = T^{\mu}_{\mu} = 0$) where the generator $\hat{S}$ of scale transformations cannot be completely diagonalized, but only brought to Jordan form
- e.g. for a 2×2 cell

$$\hat{S} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 1 & \Delta \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

- this has the consequence that (for scalar fields)

$$\begin{aligned} \langle D(r_1) D(r_2) \rangle &\sim \frac{-2\alpha \log |r_1 - r_2| + O(1)}{|r_1 - r_2|^{2\Delta}} \\ \langle C(r_1) D(r_2) \rangle &\sim \frac{\alpha}{|r_1 - r_2|^{2\Delta}} \\ \langle C(r_1) C(r_2) \rangle &= 0 \end{aligned}$$

where α depends on how D is normalised.

Quenched random systems and replicas

- classical stat mech system, energy $E[\{\phi\},\{h\}]$: $\{\phi(r)\}$ are fluctuating degrees of freedom; $\{h(r)\}$ are *quenched* random fields drawn from some distribution, e.g.

$$E[\{\phi\},\{h\}] = E_{\text{pure}}[\{\phi\}] + \lambda \int h(r)\Phi(r)d^d r$$

where Φ is some local field and $\overline{h(r)} = 0$, $\overline{h(r)h(r')} = \delta^{(d)}(r-r')$.

- we want to compute quenched averages of correlators of local fields, e.g.

$$\overline{\langle\Phi(r_1)\Phi(r_2)\rangle} = \overline{\left(\frac{\text{Tr}_\phi\,\Phi(r_1)\Phi(r_2)e^{-E[\{\phi\},\{h\}]}}{\text{Tr}_\phi\,e^{-E[\{\phi\},\{h\}]}}\right)}$$

- this is difficult because of the $\{h\}$-dependence in the denominator $Z[\{h\}]$

Two ways around this

- (a) find some other degrees of freedom $\{\psi\}$ such that $\mathrm{Tr}_{\psi}\, e^{-E[\{\psi\},\{h\}]} = Z[\{h\}]^{-1}$, then

$$\overline{\langle\Phi(r_1)\Phi(r_2)\rangle} = \overline{\mathrm{Tr}}_{\phi,\psi}\, \Phi(r_1)\Phi(r_2)e^{-E[\{\phi\},\{h\}]-E[\{\psi\},\{h\}]}$$

  The quenched average is now easy. In some cases this leads to a *supersymmetry* between $\{\phi\}$ and $\{\psi\}$.

- (b) consider n copies of the fields $\{\phi_a\}$, $a = 1,\ldots,n$, and note that

$$\overline{\langle\Phi(r_1)\Phi(r_2)\rangle} = \overline{\left(\frac{\mathrm{Tr}_{\phi_a}\, \Phi_1(r_1)\Phi_1(r_2)e^{-\sum_{a=1}^{n} E[\{\phi_a\},\{h\}]}}{Z[\{h\}]^n}\right)}$$

  for all integer $n \geq 1$.

- if this can be continued to $n = 0$, we can set the denominator $= 1$, the quenched average is again easy, but now the replicas interact.

Replica group theory

- the symmetry group is S_n
- for $\lambda = 0$, this acts trivially on the non-interacting replicas
- for $\lambda \neq 0$, assume that under the RG the theory flows to a CFT, which, for $n \neq 0$, is generically non-logarithmic
- eigenstates of $\hat{D}$ transform according to *irreducible* representations of S_n
- in particular, the multiplet $(\Phi_1, \ldots, \Phi_n)$ decomposes into

$$\begin{aligned} \Phi &\equiv \sum_{a=1}^{n} \Phi_n \\ \widetilde{\Phi}_a &\equiv \Phi_a - (1/n) \sum_{a=1}^{n} \Phi_a \end{aligned}$$

Operator product expansions

- in the non-interacting theory we have the OPE (suppressing indices)

$$\Phi_a(r_1)\cdot\Phi_b(r_2) = \delta_{ab}|r_{12}|^{-2\Delta_{\rm pure}}\left(1+\cdots+\frac{\Delta_{\rm pure}}{c_{\rm pure}}r_{12}^d T_a(r_1)+\cdots\right)$$

- the coefficient of T is fixed by global conformal invariance and is valid in any number of dimensions, if c is defined as the coefficient of the 2-point function $\langle T(r_1)T(r_2)\rangle \sim c/r_{12}^{2d}$ (suppressing indices).

These are equivalent to

$$\begin{aligned}
\widetilde{\Phi}_a \cdot \widetilde{\Phi}_a &= (1-\frac{1}{n})|r_{12}|^{-2\Delta_{\text{pure}}}\left(1+\frac{\Delta_{\text{pure}}}{nc_{\text{pure}}}r_{12}^d T + \frac{\Delta_{\text{pure}}}{c}r_{12}^d \widetilde{T}_a + \cdots\right) \\
\Phi \cdot \Phi &= n|r_{12}|^{-2\Delta_{\text{pure}}}\left(1+\frac{\Delta_{\text{pure}}}{nc_{\text{pure}}}r_{12}^d T + \cdots\right)
\end{aligned}$$

where T and $\widetilde{T}_a$ are the corresponding irreducible linear combinations of the T_a.

In the interacting theory these deform to

$$\begin{aligned}
\widetilde{\Phi}_a \cdot \widetilde{\Phi}_a &= (1-\frac{1}{n})|r_{12}|^{-2\tilde{\Delta}(n)}\left(1+\frac{\tilde{\Delta}(n)}{c(n)}r_{12}^d T + B(n) r_{12}^{d+\delta(n)} \widetilde{T}_a + \cdots\right) \\
\Phi \cdot \Phi &= n|r_{12}|^{-2\Delta(n)}\left(1+\frac{\Delta(n)}{c(n)}r_{12}^d T + \cdots\right)
\end{aligned}$$

where $(\Delta(n), \tilde{\Delta}(n), d, d+\delta(n))$ are the dimensions of $(\Phi, \widetilde{\Phi}_a, T, \widetilde{T}_a)$ respectively and $c(n)$ is the central charge of the interacting theory.

If we look at the 2-point functions we have

$$
\begin{aligned}
\langle \Phi_1(r_1)\Phi_1(r_2)\rangle &= \langle \widetilde{\Phi}_1\widetilde{\Phi}_1\rangle + \frac{1}{n^2}\langle \Phi\Phi\rangle = (1-\frac{1}{n})\frac{1}{|r_{12}|^{2\tilde{\Delta}(n)}} + \frac{1}{n}\frac{1}{|r_{12}|^{2\Delta(n)}} \\
\langle \Phi_1(r_1)\Phi(r_2)\rangle &= \frac{1}{n}\langle \Phi\Phi\rangle = \frac{1}{|r_{12}|^{2\Delta(n)}} \\
\langle \Phi(r_1)\Phi(r_2)\rangle &= \frac{n}{|r_{12}|^{2\Delta(n)}}
\end{aligned}
$$

But

$$
\begin{aligned}
\lim_{n\to 0}\langle \Phi_1(r_1)\Phi_1(r_2)\rangle &= \overline{\langle \Phi(r_1)\rangle\langle \Phi(r_2)\rangle} \\
\lim_{n\to 0}\langle \Phi_1(r_1)\Phi(r_2)\rangle &= \overline{\langle \Phi(r_1)\Phi(r_2)\rangle - \langle \Phi(r_1)\rangle\langle \Phi(r_2)\rangle}
\end{aligned}
$$

and these had better be finite!

- as $n \to 0$, $\tilde{\Delta}(n) - \Delta(n) \to 0$, and

$$
\langle \Phi_1(r_1)\Phi_1(r_2)\rangle \sim \frac{-2\alpha \log |r_{12}|}{|r_{12}|^{2\Delta(0)}} \quad \text{where} \quad \alpha = \tilde{\Delta}'(0) - \Delta'(0)
$$

- (Φ_1, Φ) are an example of a *logarithmic pair*.

The '$c \to 0$ catastrophe'

$$
\begin{aligned}
\widetilde{\Phi}_a \cdot \widetilde{\Phi}_a &= (1 - \frac{1}{n})|r_{12}|^{-2\tilde{\Delta}(n)} \left(1 + \frac{\tilde{\Delta}(n)}{c(n)} r_{12}^d T + B(n) r_{12}^{d+\delta(n)} \widetilde{T}_a + \cdots \right) \\
\Phi \cdot \Phi &= n|r_{12}|^{-2\Delta(n)} \left(1 + \frac{\Delta(n)}{c(n)} r_{12}^d T + \cdots \right)
\end{aligned}
$$

- since $\lim_{n \to 0} c(n) = 0$, there is an apparent problem with the coefficient of T. This can be resolved in several different ways:
  a) the normalization of the physical fields vanishes as $c \to 0$: this is what happens for $\Phi \cdot \Phi$ above (and for many examples in percolation): no logs
  b) the scaling dimension $\Delta \to 0$: this happens for the Kac $(1, 2)$ operator in 2d percolation whose 4-pt function gives the crossing formula: no logs
  c) T collides with another operator as $c \to 0$, and the leading singularities cancel: this is what happens for $\widetilde{\Phi}_a \cdot \widetilde{\Phi}_a$ above, as long as $\delta(n) \to 0$ and $B(n) \sim 2\Delta(n)/c(n)$. In this case we are left with a term $\propto \delta'(0) r_{12}^d \log |r_{12}|$.

In this case T_1 and T form a logarithmic pair: in 2d

$$
\begin{aligned}
\langle T_1(z_1)T_1(z_2)\rangle &= \overline{\langle T(z_1)\rangle\langle T(z_2)\rangle} = \frac{2c'(0)\delta'(0)\log(z_{12}\bar{z}_{12})}{2z_{12}^4} \\
\langle T_1(z_1)T(z_2)\rangle &= \overline{\langle T(z_1)T(z_2)\rangle - \langle T(z_1)\rangle\langle T(z_2)\rangle} = \frac{c'(0)}{2z_{12}^4} \\
\langle T(z_1)T(z_2)\rangle &= 0
\end{aligned}
$$

In 2d logCFT this is usually written, with $t \propto T_1$,

$$
\begin{aligned}
\langle t(z_1)t(z_2)\rangle &= -\frac{b\log(z_{12}\bar{z}_{12})}{z_{12}^4} \\
\langle t(z_1)T(z_2)\rangle &= \frac{b}{2z_{12}^4}
\end{aligned}
$$

so that $b = -c'(0)/2\delta'(0)$.

- however in this case t is not a holomorphic operator, because $\widetilde{T}_a$ has dimensions $(2+\delta(n), \delta(n))$
- b is defined within the logCFT *at* $c = 0$, but the physical quantities $c'(0)$ and $\delta'(0)$ are not

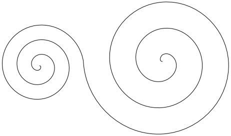
Example II. The $O(n \to 0)$ model and self-avoiding walks

- the $O(n)$ field theory with integer n is based on a multiplet of fields $(\phi_1, \ldots, \phi_n)$ and action

$$S = \int \left[\sum_{a=1}^{n} (\partial\phi_a)^2 + m_0^2 \sum_{a=1}^{n} \phi_a^2 + \lambda_0 \Big(\sum_{a=1}^{n} \phi_a^2 \Big)^2 \right] d^d r$$

- on a lattice, the free theory represents a sum over loops, weighted by $e^{-m_0^2(\text{length})}$ and with a factor n for each loop. In this way the model already makes sense for non-integer n.
- the interaction λ_0 provides a repulsion between different loops and different parts of the same loop: as $\lambda_0 \to \infty$ we get an ensemble of self-avoiding loops. For m_0^2 large, these are all small, at a critical value the typical size diverges and we have a CFT in the scaling limit.
- if we compute $\langle \phi_a(r_1)\phi_a(r_2) \rangle$ we get the weighted number of open walks from r_1 to r_2: at $n = 0$ these are SAWs.

Smirnov's observable ($d = 2$)



- Smirnov instead considered the quantity $\langle\psi_a(z_0)\psi_a(z)\rangle$ in which the open walks from z_0 to z are also weighted by $e^{-is\theta_{z_0 z}}$ where $\theta_{z_0 z}$ is the winding angle
- at the critical point and for the correct value of s ($= h_{2,1}(n)$ in the Kac table!), this is a *discretely holomorphic* function of z, i.e. obeys a discrete version of the Cauchy-Riemann relations
- *if* this continues to hold in the scaling limit, then
    a) this can be used to prove that the curves are SLE_κ
    b) we get a holomorphic conformal field (parafermion) $\psi_a(z)$ with dimensions $(h_{2,1}(n), 0)$

Consider now the OPE

$$\psi_a(z_1)\cdot\psi_b(z_2) = z_{12}^{-2h_{2,1}(n)}\left(\delta_{ab} + \frac{2h_{2,1}(n)}{c(n)}z_{12}^2\delta_{ab}T(z_1) + \cdots\right)$$

- as $n \to 0$ there is a potential catastrophe, which is avoided in this case by the collision of T with another operator
- since ψ_a is a Kac $(2,1)$ operator, this can only be in the list allowed by the fusion rules, and the only candidate is a Kac $(3,1)$ operator, since $\lim_{n\to 0} h_{3,1}(n) = 2$
- this can be identified physically as the deformation of the operator $T_{ab} \propto (\partial_z\phi_a)(\partial_z\phi_b)$ in the free theory: the trace deforms to T and the traceless part to $\widetilde{T}_{ab}$
- we can then identify $t \propto T_{11}$, and, since we know both $c(n)$ and $h_{3,1}(n)$, we can compute $b = -c'(0)/2h'_{3,1}(0) = \frac{5}{6}$
- the correlators $\langle tt\rangle$ and $\langle tT\rangle$ can be physically interpreted in terms of SA loops

Summary

- we can understand the appearance of logarithmic behaviour in some CFTs which are limits of regular CFTs
- this involves the presence of a global symmetry under which the operators transform irreducibly in general: however these representations become singular in the limit: logarithms then appear due to collisions of irreducible operators and cancellations in the OPE
- comparison with a non-interacting limit allows the physical identification of these operators
- the $c \to 0$ catastrophe may be avoided in various ways: when this happens due to the collision of another operator with T there is a logarithmic partner t (which however is not necessarily holomorphic in 2d)
- the parameter $b = -\frac{1}{2}(dc/d\Delta)$ where Δ is the dimension of the operator which collides with T

- there are important physical quantities (like $c'(0)$) which are not in the logCFT
- most of this still holds in general dimension d